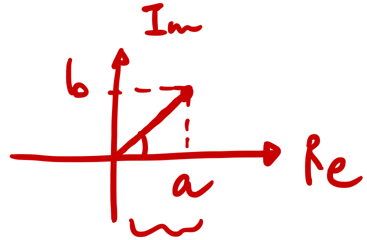
Complex Numbers

phasor



- $x = a + jb = |x| \angle x = \sqrt{a^2+b^2} \cdot \angle \tan^{-1}\frac{b}{a}$

$$= \underbrace{|x| \cdot \cos\angle x}_{a} + j \cdot \underbrace{|x| \cdot \sin\angle x}_{b}$$

- x, y : complex numbers

$$A = x \cdot y \Rightarrow |A| = |x| \cdot |y| , \quad \angle A = \angle x + \angle y$$

$$B = \frac{x}{y} \Rightarrow |B| = \frac{|x|}{|y|} , \quad \angle B = \angle x - \angle y$$

phasor representation: represent sinewave w. respect to its mag & phase

- $v = A \cdot \cos(\omega t + \theta) \longrightarrow A\angle\theta = a + jb = A \cdot e^{j\theta}$

polar form | rectangular form | exponential form

- $1 = 1 \cdot \angle 0^\circ$, $j = 1 \cdot \angle 90^\circ$, $-1 = 1 \cdot \angle \pm 180^\circ$, $-j = 1 \cdot \angle 270^\circ = 1 \cdot \angle -90^\circ$

- Example.

$$x(t) = 120 \cdot \underline{\underline{\sin}}\left(400t - \frac{\pi}{4}\right) \quad , \text{phasor} = ?$$

$$x(t) = 120 \cdot \cos\left(400t - \frac{\pi}{4} - \frac{\pi}{2}\right) = 120 \cos\left(400t - \frac{3}{4}\pi\right)$$

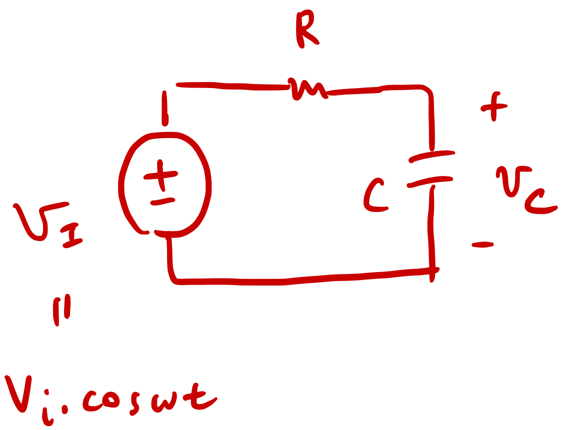
$$= 120 \cdot \angle \frac{-3}{4}\pi$$

Practice : rectangular form $= a + jb = ?$

exponential form $= ?$

Sinusoidal Response of RC Network

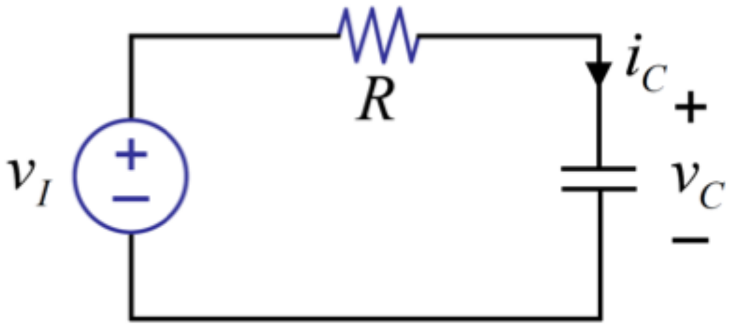
- Find $v_C(t)$ for t ≥ 0. Assume $v_I(t) = V_i \cos(\omega t)$ for t ≥ 0 and $v_I(t) = 0$ for t < 0. Initial condition $v_C(0) = 0$.



Our Usual Approach

1. Write DE for circuit by applying node method.
2. Find particular solution v_P by guessing and trial & error.
3. Find homogeneous solution v_H.
4. Total solution is $v_P + v_H$, then solve for remaining constants using initial conditions .

Step 1: Set Up the Differential Equation



$$\frac{v_I - v_C}{R} = C \cdot \frac{dv_c}{dt}$$

$$\Rightarrow RC\frac{dv_c}{dt} + v_c = v_I = V_i \cdot \cos\omega t$$

Step 2: Find the Particular Solution

$$RC\frac{dv_P}{dt} + v_P = V_i \cos(\omega t)$$

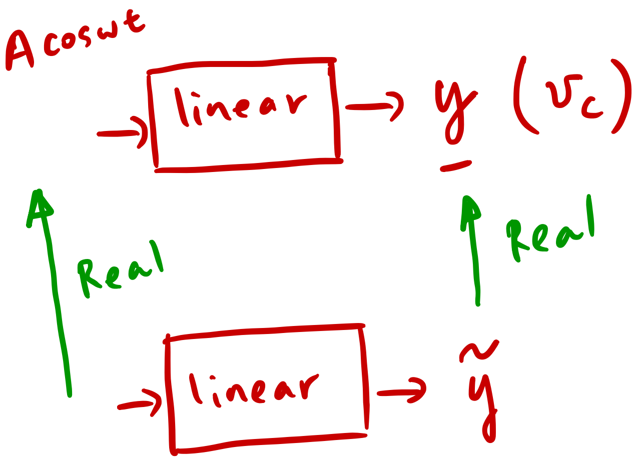
Assume $v_p = A\cos\omega t + B\sin\omega t$

Sneaky Approach

$$\boxed{RC\frac{dv_P}{dt}+v_P=V_i\cos(\omega t)}$$

- Instead of input $v_I(t)=V_i\cos\omega t$

→ Find particular solution for another input $v_{IS}(t)=V_i\exp(st)$



$Ae^{j\omega t}$

$\parallel$

$A\cos\omega t + jA\sin\omega t$

$s=j\omega$

$$RC\cdot\frac{d\tilde{v}_p}{dt}+\tilde{v}_p=V_i\cdot e^{st}$$

Assume $\tilde{v}_p=V_p\cdot e^{st}$

$$RC\cdot s\cdot V_p e^{st}+V_p e^{st}=V_i\cdot e^{st}$$

$$\Rightarrow e^{st}(sRCV_p+V_p)=V_i\cdot e^{st}$$

$$\Rightarrow V_p=\frac{V_i}{1+sRC}\ ,\quad s\neq\frac{-1}{RC}$$

Sneaky Approach

- Assume $s = j\omega$, then particular solution for input $v_{IS}(t) = V_i \exp(st)$

$$\tilde{v}_p = \frac{V_i}{1+sRC} \cdot e^{st}$$

- Finding the particular solution to $v_{IS}(t) = V_i \exp(j\omega t)$ was easy.
- From Euler relation $v_{Is}(t) = V_i \exp(j\omega t) = V_i \cos(\omega t) + j \sin(\omega t)$

- An inverse superposition argument, assuming system is real and linear.

Sneaky Approach

v_P(t) particular response to $V_i \cos\omega t$

v_{PS}(t) particular response to $V_i \exp(j\omega t)$

$$v_p = \mathrm{Re}(\tilde{v}_p) = \mathrm{Re}\left(\frac{V_i}{1+sRC}e^{st}\right) = \mathrm{Re}\left(\underbrace{\frac{V_i}{1+j\omega RC}e^{j\omega t}}\right)$$

$$= \mathrm{Re}\left(\frac{V_i}{\sqrt{1+(\omega RC)^2}} \cdot e^{j\tan^{-1}(-\omega RC)} \cdot e^{j\omega t}\right)$$

$$= \frac{V_i}{\sqrt{1+(\omega RC)^2}} \cos\left(\omega t + \tan^{-1}(-\omega RC)\right)$$

Step 3: Homogeneous Solution

$RC\frac{dv_c}{dt} + v_c = 0$

- Recalled from Chapter 10, the homogeneous solution for RC circuit v_H:

$v_{c,H} = Ae^{st}$

$v_c(0) = 0$

- The total solution:

$$v_C(t) = v_P(t) + v_H(t) = \underbrace{\frac{V_i}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \tan^{-1}(\omega RC)}_{v_p} + \underbrace{A \cdot \exp(\frac{-t}{RC})}_{v_H}$$

$$A = -\frac{V_i}{\sqrt{1+(\omega RC)^2}} \cdot \cos\left(-\tan^{-1}(\omega RC)\right)$$

Sinusoidal Steady State

- The total solution

$$v_C(t) = \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos(\omega t+\phi) - \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos\phi \cdot e^{\frac{-t}{RC}}$$

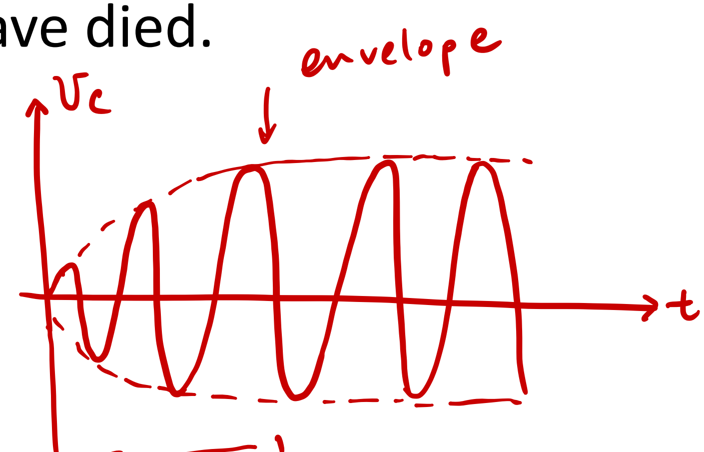
where $\phi = \tan^{-1}(-\omega RC)$

- We are usually interested only in the particular solution for sinusoids, i.e., after the transients have died.

暫態

Steady-state

穩態

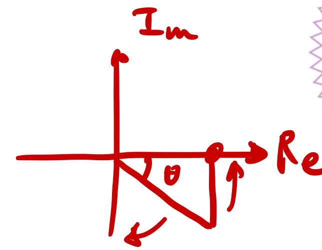


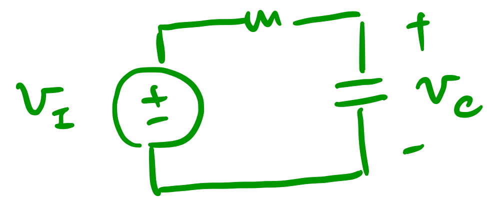
- When $t \to \infty$,

$$v_C(t) = \frac{V_i}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \tan^{-1}(\omega RC))$$

Sinusoidal Steady State

$$v_c = \frac{V_i}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \underline{\theta}), \quad \theta = \tan^{-1}(\underset{\sim}{-\omega RC})$$





1. when $\omega \to 0$, $\theta = \tan^{-1}(-\omega RC) = 0$

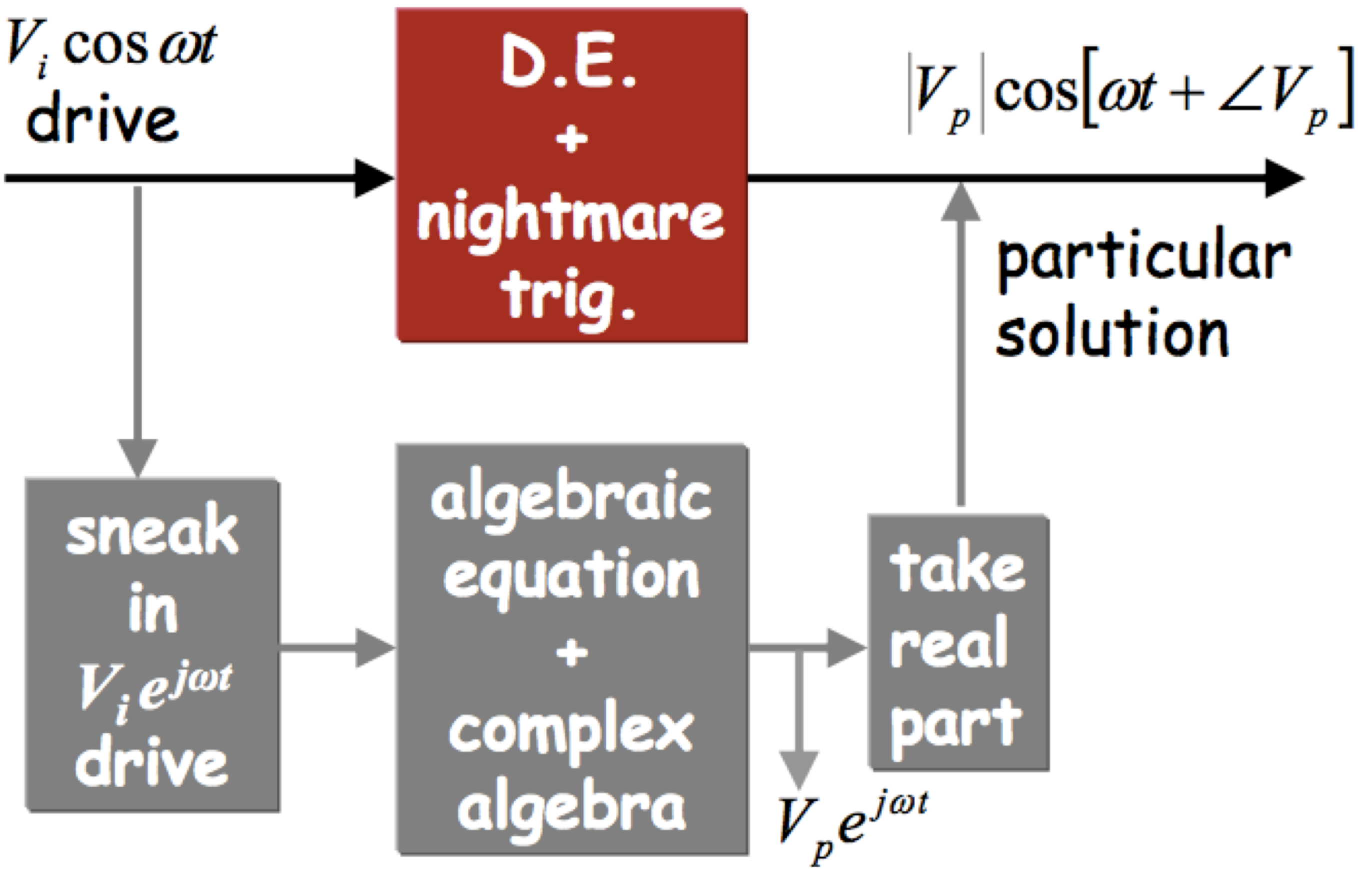
$$\underline{v_c} = \frac{V_i}{\sqrt{1+(\omega \cancel{RC})^2}} \cos(\omega t + \cancel{\theta}) = \frac{V_i}{1}\cos(\omega t) = \underline{V_I}$$

2). when $\omega \to \infty$, $v_c = \frac{V_i}{\sqrt{1+(\omega RC)^2}} \cos(\omega t + \theta) = \frac{V_i}{\underset{\sim}{\omega} RC} \cos(\omega t - 90^\circ) \simeq 0$

$\underline{\text{Low}}$-pass

Visualizing Sinusoidal Steady State

- The process of finding the particular solution v_P.



Summary of SSS Approach

1. Set up differential equation. (KCL, KVL, node method, …)
2. Apply sneaky input $V_i \cdot e^{j\omega t}$ $$RC\frac{dv_P}{dt}+v_P=V_i\cos(\omega t)$$
   then find particular solution

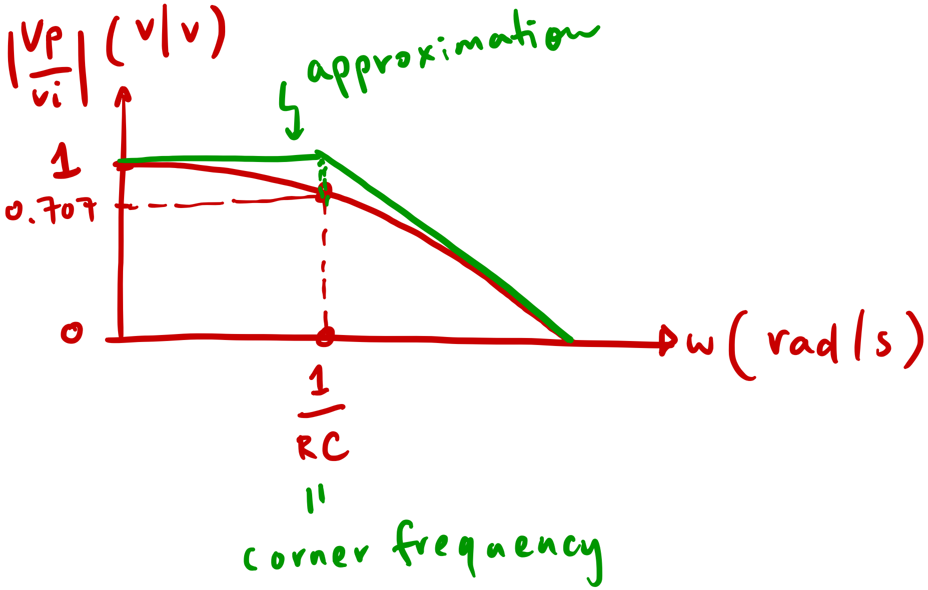
Assume

$$v_{PS}(t)=\underline{\underline{V_P}}\cdot\exp(j\omega t)$$

Magnitude Plot

$= \frac{v_i / \sqrt{1+(\omega RC)^2}}{v_i} = H$

- Transfer function $\frac{V_P}{V_i} = \frac{1}{1+j\omega RC} = \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot \exp(j \cdot \tan^{-1}(-\omega RC))$
  - Also known as a system function, is the ratio of the complex amplitude of the network output to the complex amplitude of the input.
- Magnitude plot: Low-pass



- At low freq, $(\omega RC)^2 \ll 1$,
  $|H| \sim 1$
- At high freq, $(\omega RC)^2 \gg 1$,
  $|H| \sim \frac{1}{\omega RC} \to 0$
- At $\omega = \frac{1}{RC}$, $|H| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$
  $= 0.707$

- Output voltage falls off at high frequencies!!

- Decibel : $\left|\frac{V_p}{V_i}\right|$ (V/V) $\longrightarrow$ $20\log\left|\frac{V_p}{V_i}\right|$ (dB)

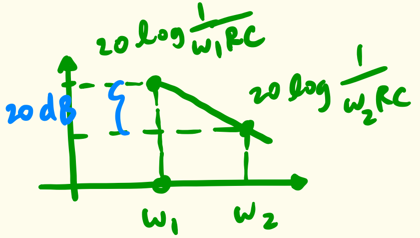
$\log \omega_1$

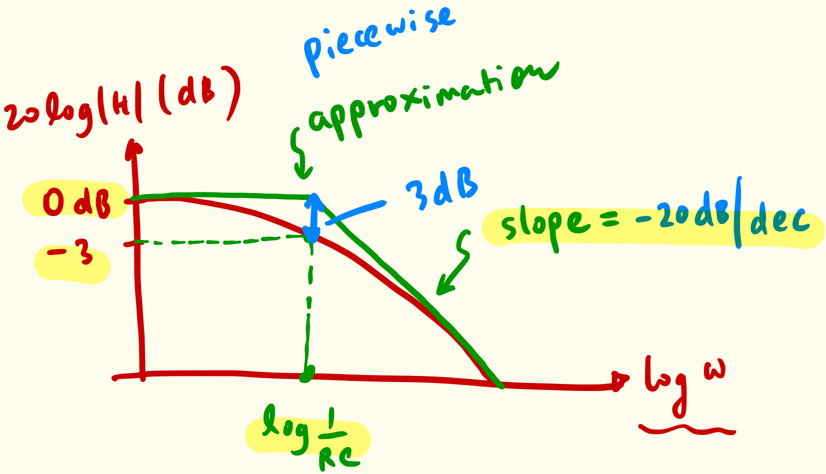
$\log(10\omega_1)$

$= \log 10 + \log \omega_1$

$= 1 + \log \omega_1$

- Low freq, $|H| \to 1$ $\quad 20\log 1 = \underline{0 \text{ dB}}$
- High freq, $|H| \sim \frac{1}{\omega RC}$ $\quad 20\log\frac{1}{\omega RC} = -20\log \omega RC$
- Corner freq, $|H| = \frac{1}{\sqrt{2}}$ $\quad 20\log\frac{1}{\sqrt{2}} = -3$ dB





Assume

$\omega_2 = 10\omega_1$ $\quad$ 10 x (decade)

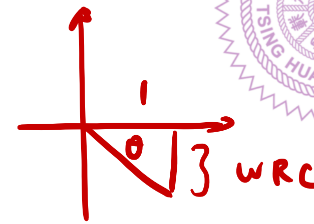
$$20\log\frac{1}{\omega_2 RC} = 20\log\frac{1}{10\omega_1 RC}$$

$$= 20\log\frac{1}{10} + 20\log\frac{1}{\omega_1 RC}$$

$$= -20 + 20\log\frac{1}{\omega_1 RC}$$

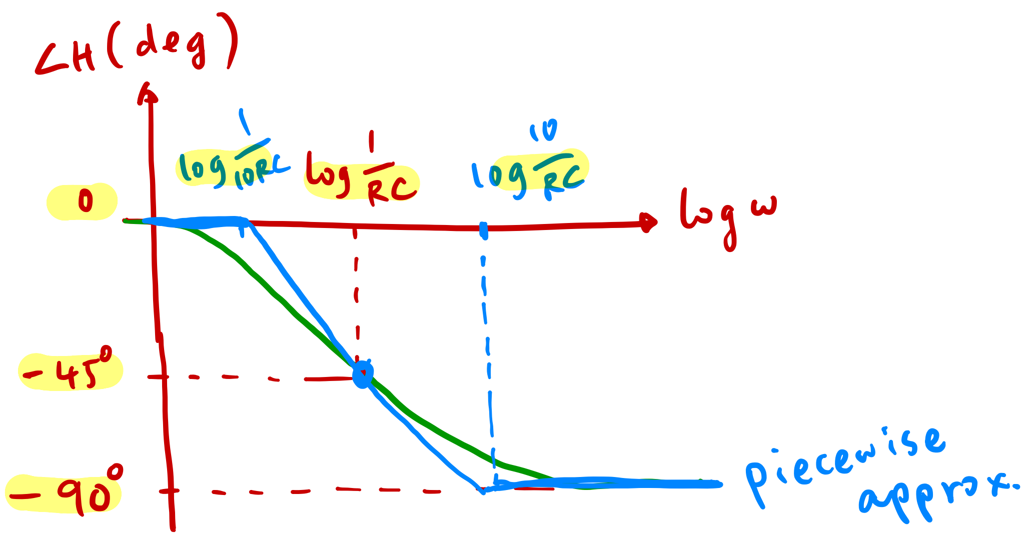
Phase Plot

- Transfer function $H(j\omega)=\dfrac{V_p}{V_i} = \dfrac{1}{1+j\omega RC}$



$\angle H = \tan^{-1}(-\omega RC)$

- At low freq, $\angle H \sim 0^\circ$
- At high freq, $\angle H \sim -90^\circ$
- At $\omega = \dfrac{1}{RC}$, $\angle H = -45^\circ$

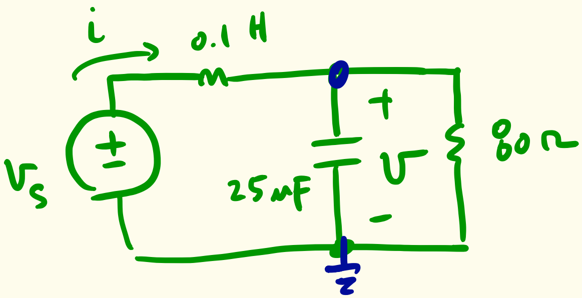


Example 1.

Practice.

$V_1 = -1.796 + j3.852$

Example 2.



$v_s = 48 \cdot \cos(500t + 75°)$

Find $v(t)$ and $i(t)$, SSS

Practice

$v_H = ?$

$$\begin{cases} i = 25\mu \cdot \dfrac{dv}{dt} + \dfrac{v}{80} & \text{①} \\ v_s - v = v_L = 0.1 \dfrac{di}{dt} & \text{②} \end{cases}$$

Use ① in ② $\Rightarrow$ $v_s - v = 0.1\left(25\mu \dfrac{d^2v}{dt^2} + \dfrac{1}{80}\dfrac{dv}{dt}\right)$

$$\Rightarrow v_s = 25\times10^{-7}\frac{d^2v}{dt} + \frac{0.1}{80}\frac{dv}{dt} + v$$

$$v_s \rightarrow \tilde{v}_s = 40 \cdot e^{j(500t+75°)}$$

Assume $\tilde{v} = Ae^{st}$

$$\tilde{v}_s = 25\times10^{-7}\frac{d^2\tilde{v}}{dt} + \frac{0.1}{80}\frac{d\tilde{v}}{dt} + \tilde{v}$$

$$40 \cdot e^{j(500t+75°)} = 25\times10^{-7}\cdot A \cdot s^2 \cdot e^{st} + \frac{0.1}{80}Ase^{st} + Ae^{st}$$

$$= Ae^{j500t}\left(25\times10^{-7}\times\omega^2 + \frac{0.1}{80}\cdot j500 + 1\right)$$

$s = j\omega$
$= j \cdot 500$

$\Rightarrow A = 63.4 + j\,18.2$

$$\tilde{v} = (63.4+j18.2)e^{j500t} \xrightarrow{\text{Real}} v = \sqrt{(63.4)^2+(18.2)^2}\cos\left(500t + \tan^{-1}\frac{18.2}{63.4}\right)$$

$$= 66\cos(500t+16°)$$

$$i = 25\mu \frac{dv}{dt} + \frac{v}{80} \longrightarrow \tilde{i} = 25\mu \cdot \frac{d\tilde{v}}{dt} + \frac{\tilde{v}}{80}$$

$$\tilde{v} = 66 e^{j16^\circ} e^{j500t}$$

$$\Rightarrow \tilde{i} = 25\mu \cdot 66 \cdot e^{j16^\circ} \cdot j500 \cdot e^{j500t} + \frac{1}{80} \cdot 66 e^{j16^\circ} e^{j500t}$$

$$= j0.825 e^{j16^\circ} e^{j500t} + 0.825 e^{j16^\circ} e^{j500t}$$

$$= 0.825 e^{j500t} \left(e^{j90^\circ} e^{j16^\circ} + e^{j16^\circ} \right)$$

$$= 0.825 e^{j500t} \left(e^{j106^\circ} + e^{j16^\circ} \right)$$

Real

$$\longrightarrow i = 0.825 \cos(500t + 106^\circ) + 0.825 \cos(500t + 16^\circ)$$