★ Quiz 3 : 12/26 (Wed) 8 am - 9:50 am

Under-Damped

$\alpha < \omega_0 \quad \left(R < 2\sqrt{\frac{L}{C}} \right)$, s_1, s_2: complex conjugate

$$v_C(t) = V_I + A_1 e^{-\alpha t} e^{j\sqrt{\omega_0^2-\alpha^2}\,t} + A_2 e^{-\alpha t} e^{-j\sqrt{\omega_0^2-\alpha^2}\,t}$$

Assume $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, ω_d: natural frequency, $\omega_d \le \omega_0$

$$v_C(t) = V_I + A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$= V_I + A_1 e^{-\alpha t} \left(\cos\omega_d t + j \sin\omega_d t \right) + A_2 e^{-\alpha t} \left(\cos\omega_d t - j \sin\omega_d t \right)$$

use
$K_1 = A_1 + A_2$
$K_2 = jA_1 - jA_2$

$$= V_I + \underbrace{e^{-\alpha t}}_{\text{exp. decay}} \underbrace{\left(K_1 \cos\omega_d t + K_2 \sin\omega_d t \right)}_{\text{sinusoids, frequency } \omega_d/2\pi \text{ (Hz)}}$$

Under-damped

1 0

$$v_c(t) = V_I + K_1 e^{-\alpha t}\cos\omega_d t + K_2 e^{-\alpha t}\sin\omega_d t$$

- Furthermore, $v(0) = 0$ and $i(0) = 0$

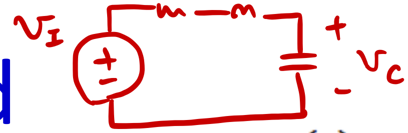
1) $v_c(0) = V_I + K_1 + K_2 \cdot 0 = 0 \quad \Rightarrow \quad K_1 = -V_I$

2) $i(0) = 0 \quad \Rightarrow \quad C \cdot \dfrac{dV_c(t=0)}{dt} = C \cdot K_1 \cdot (-\alpha) + C \cdot K_2 \cdot \omega_d = 0$

$$\Rightarrow K_2 = -\frac{\alpha}{\omega_d} \cdot V_I$$

Total solution $v_c(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - \dfrac{\alpha}{\omega_d} V_I \cdot e^{-\alpha t}\sin\omega_d t$

Under-damped



$$v(t) = V_I + K_1 e^{-\alpha t}\cos\omega_d t + K_2 e^{-\alpha t}\sin\omega_d t$$

- Scaled sum of sines (of the same frequency) are also sines (Appendix B.7).

Use $a_1 \cos\omega_d t + a_2 \sin\omega_d t$

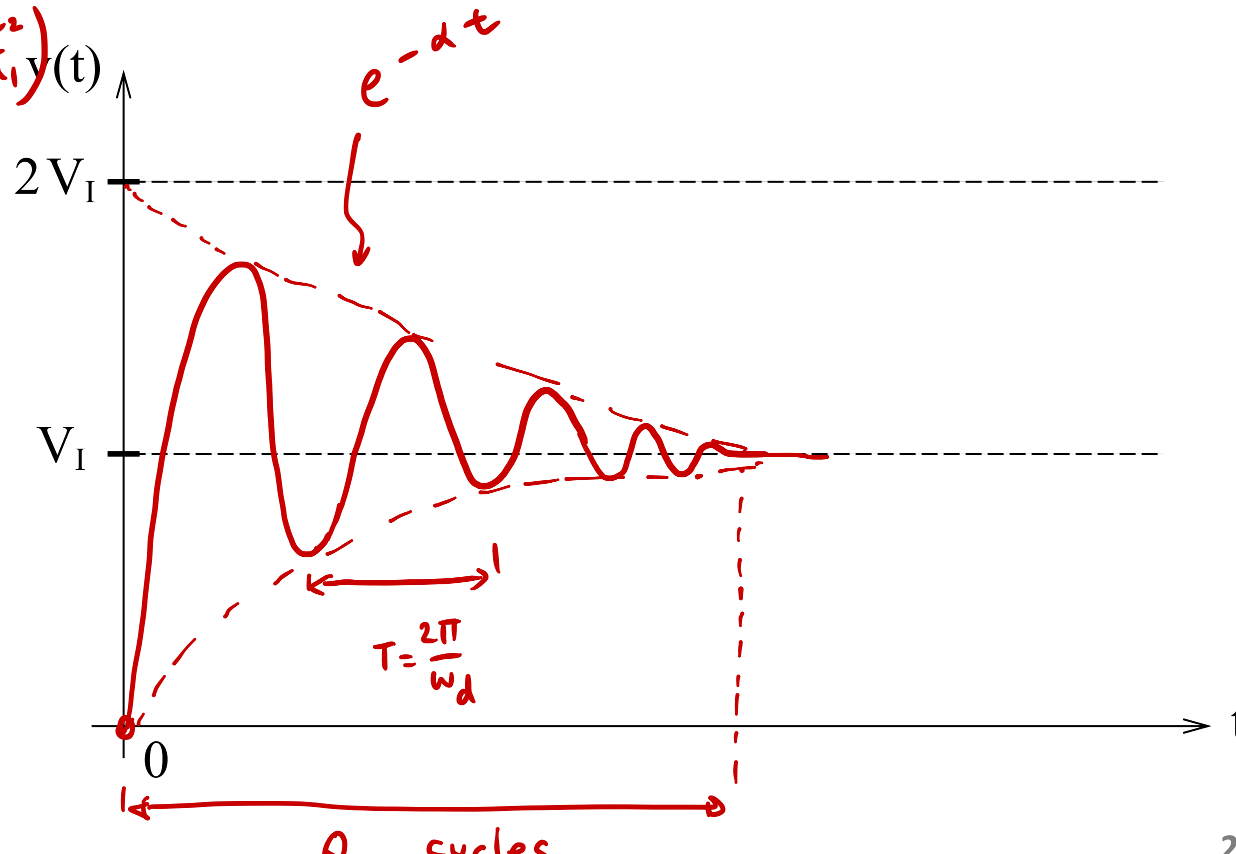
$$= \sqrt{a_1^2 + a_2^2}\,\cos\left(\omega_d t - \tan^{-1}\frac{a_2}{a_1}\right)$$

$$v_c(t) = V_I + e^{-\alpha t}\cdot V_I \cdot \frac{\omega_0}{\omega_d}\cdot \cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$

α: damping factor

Q : Q factor, quality factor

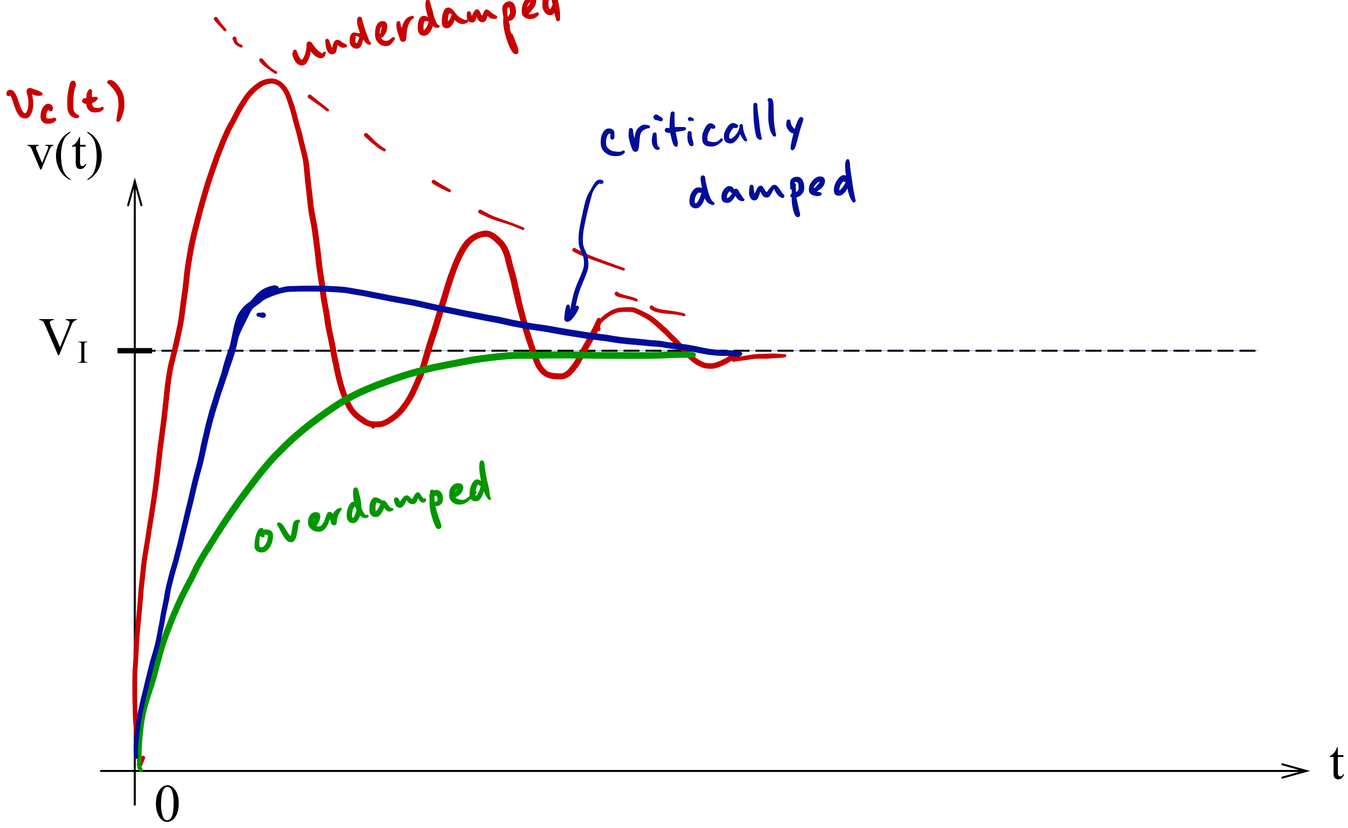
$$Q = \frac{\omega_0}{2\alpha} = \frac{1}{R}\sqrt{\frac{L}{C}}$$



Critically-Damped

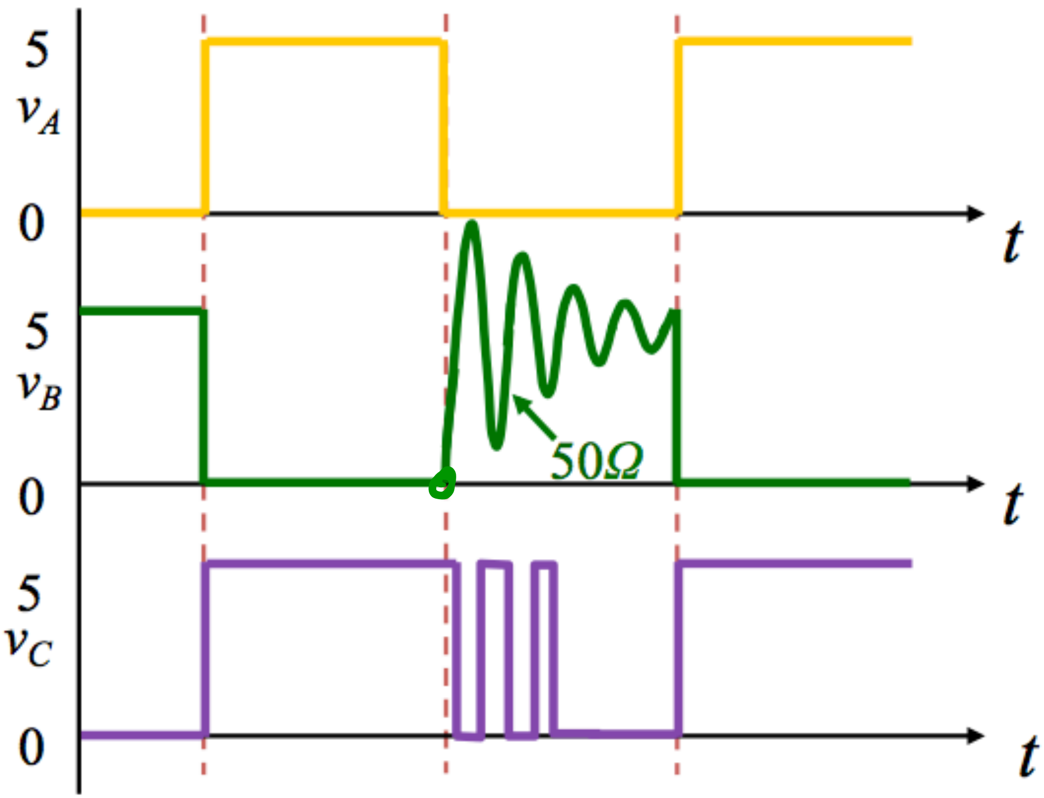
$\alpha = \omega_0, \quad s_1 = s_2 = -\alpha$

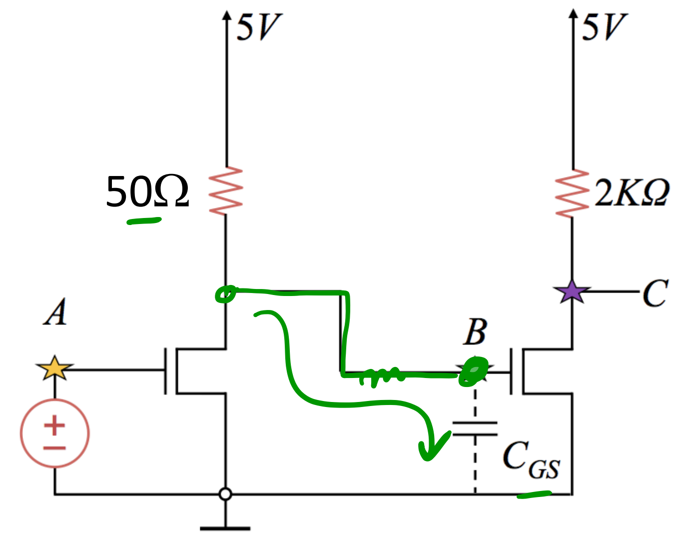
- Total solution will be in the form of $v_c = V_I + A_1 e^{-\alpha t} + A_2 e^{-\alpha t} + t =$

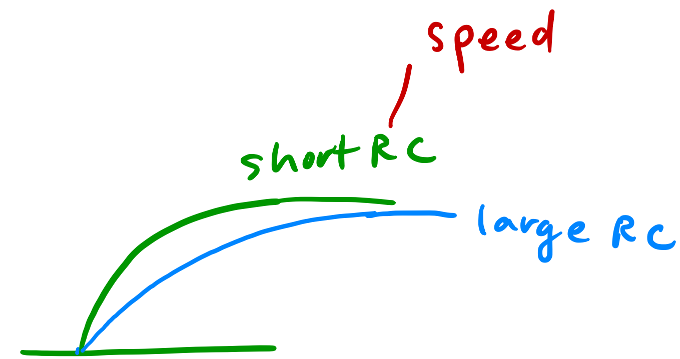


Remember This?

- With 50Ω load resistor, hoping to speed up the pull up.





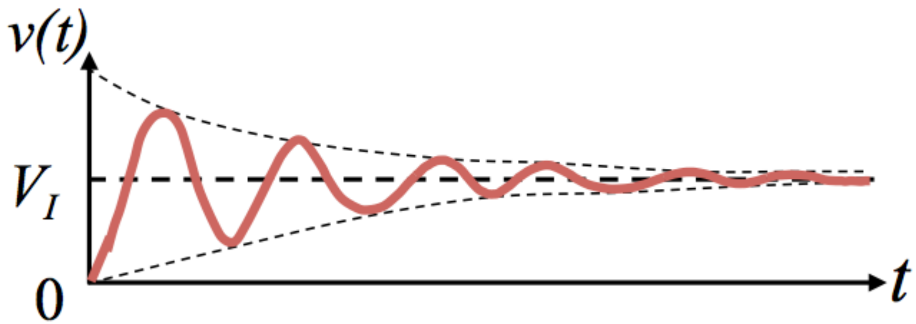


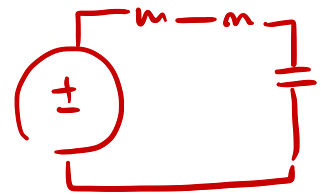
speed ⟷ stability
correctness

Particular solution

Excitation	Assumed response
K	A
$K \cdot t$	$At + B$
$K \cdot t^2$	$At^2 + Bt + C$
$K \cdot \sin wt$	$A\sin wt + B\cos wt$
$K \cdot e^{-\alpha t}$	$A \cdot e^{-\alpha t}$

Easy Way: Characteristic Equation Tells the Whole Story





- For series RLC circuit:

characteristic equation: $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Intuitive Analysis

$i = C\frac{dv}{dt} < 0$

- What if $v(0) > 0$ and $i(0) < 0$?

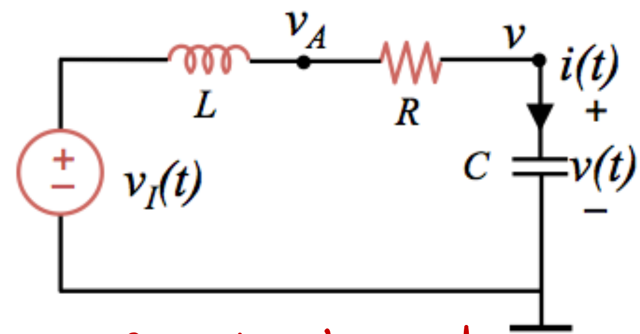
1) $v_c(0)$

v_{final}

2) α, ω_0, determine under-damped

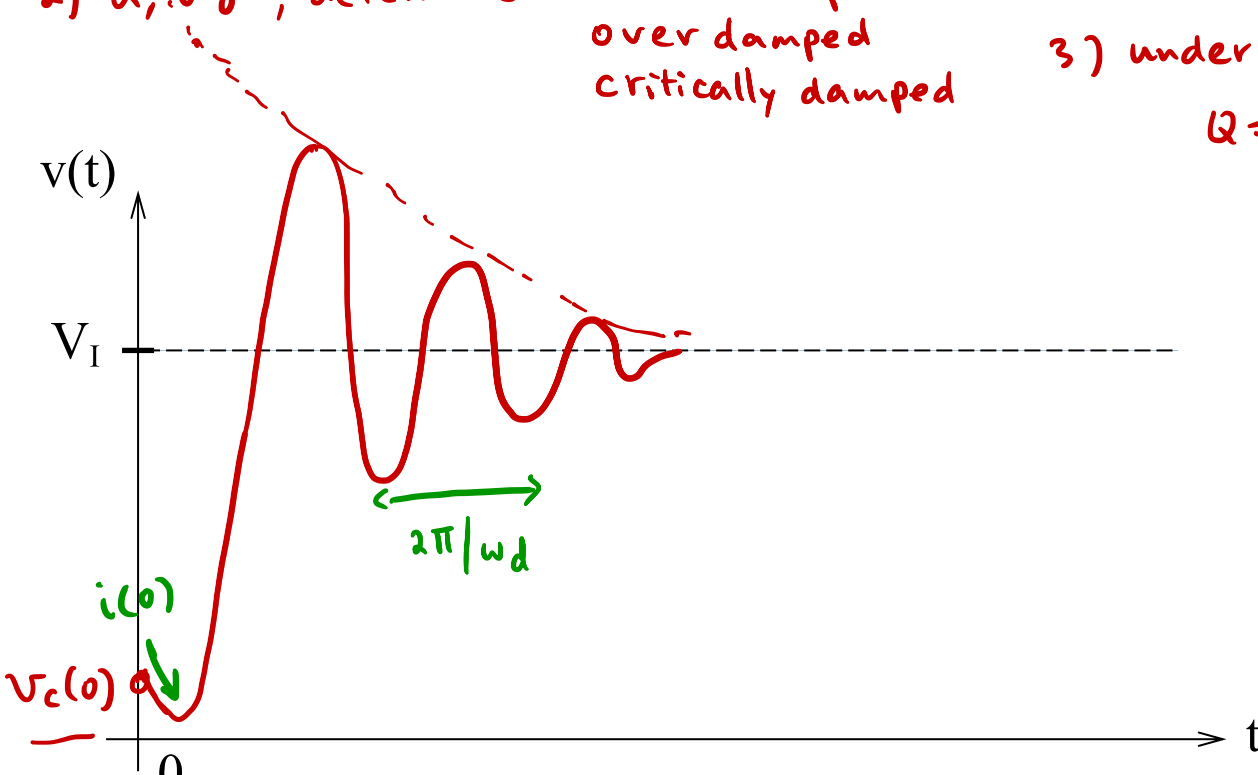
over damped

critically damped



3) under damped

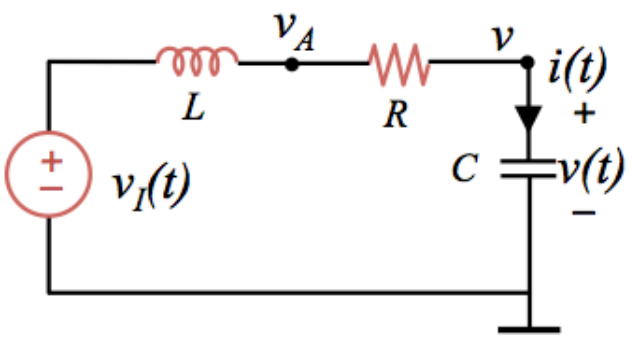
$Q = \frac{\omega_0}{2\alpha}$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$



Q cycles of ringing

$\left(Q \cdot \frac{2\pi}{\omega_d}\ s\right)$

What about Other Variables?



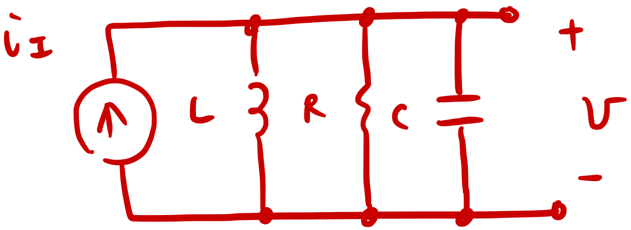
$$i = C\frac{dv}{dt}$$

$$v_R = i \cdot R$$

$$v_L = V_I - v_C - v_R = L \cdot \frac{di}{dt}$$

Parallel RLC – Characteristic Equation Says It All

Practice



From KCL: $i_I = C\dfrac{dv}{dt} + \dfrac{v}{R} + \dfrac{1}{L}\int v\,dt$

Characteristic equation: $s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC} = 0$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\Rightarrow \alpha,\ \omega_0$$

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Series RLC

characteristic equation

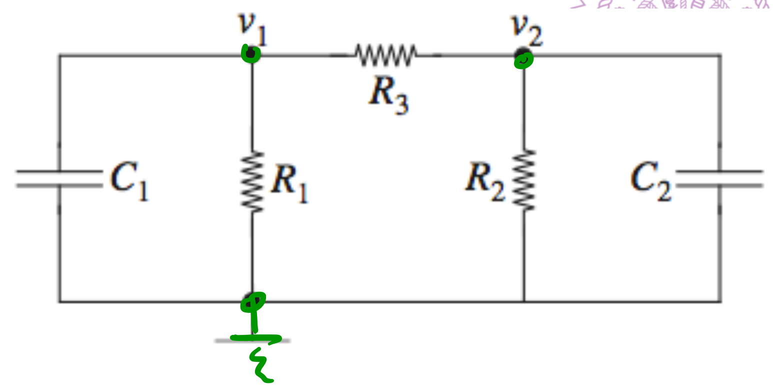
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Practice

Two-Capacitor Circuits



For Node #1 $C_1\frac{dv_1(t)}{dt}+\frac{1}{R_1}v_1(t)+\frac{1}{R_3}(v_1(t)-v_2(t))=0$

For Node #2 $C_2\frac{dv_2(t)}{dt}+\frac{1}{R_2}v_2(t)+\frac{1}{R_3}(v_2(t)-v_1(t))=0$

Express v_2(t) in terms f v_1(t) $v_2(t)=R_3C_1\frac{dv_1(t)}{dt}+\left(1+\frac{R_3}{R_1}\right)v_1(t)$

Differential equation of v_1(t) ✓

$$\frac{d^2v_1(t)}{dt^2}+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_3C_1}+\frac{1}{R_3C_2}\right)\frac{dv_1(t)}{dt}$$

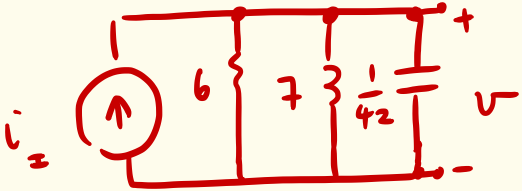
$$+\left(\frac{1}{R_1R_2C_1C_2}+\frac{1}{R_1R_3C_1C_2}+\frac{1}{R_2R_3C_1C_2}\right)v_1(t)=0.$$

$v_1=A_1e^{s_1t}+A_2e^{s_2t}$

$\Rightarrow s^2+2\alpha s+\omega_0^2=0$

- Circuits with only resistors and capacitors have characteristic equations with only real non-positive roots.

Example 1:



$i_I = 8 \cdot e^{-2t}$, Find $v(t)$

KCL: $i_I = i_R + i_L + i_C = \dfrac{v}{R} + \dfrac{1}{L}\displaystyle\int v\,dt + C\dfrac{dv}{dt}$

$$\xrightarrow{\frac{d}{dt}} \frac{di_I}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = -16 \cdot e^{-2t}$$

① Particular, $v_p = A \cdot e^{-2t}$

$$-16 \cdot e^{-2t} = \frac{1}{6}(-2) \cdot A \cdot e^{-2t} + \frac{1}{7} \cdot Ae^{-2t} + \frac{1}{42} A \cdot 4 \cdot e^{-2t}$$

$\Rightarrow A = 168$, $\Rightarrow v_p = 168\, e^{-2t}$

② Homogeneous. Assume $v_H = B \cdot e^{st}$

$$\frac{1}{6}\frac{dv_H}{dt} + \frac{v_H}{7} + \frac{1}{42}\frac{d^2 v_H}{dt^2} = 0 \Rightarrow \text{characteristic equation } s^2 + 7s + 6 = 0$$

$\alpha = \frac{7}{2}$, $\omega_0 = \sqrt{6}$

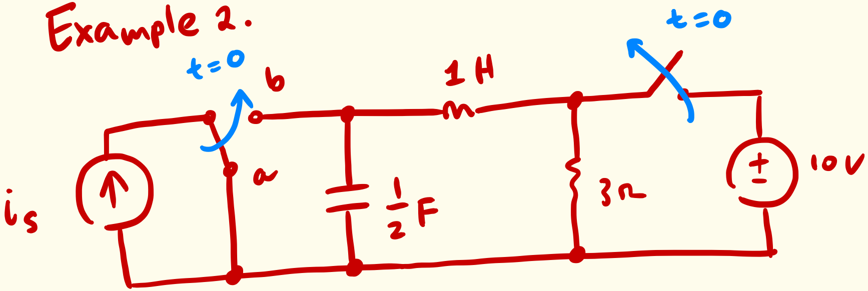
$s_1, s_2 = -1, -6$

over damped

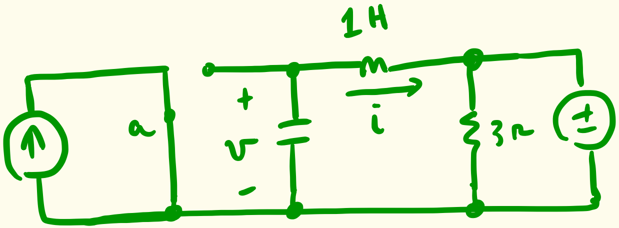
③ $v = 168 e^{-2t} + B_1 e^{-t} + B_2 e^{-6t}$

Example 2.

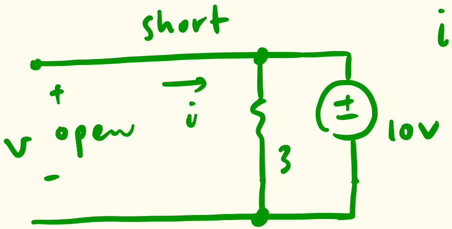
$i_s = 2e^{-3t}$. Find $v(t)$.



At t<0



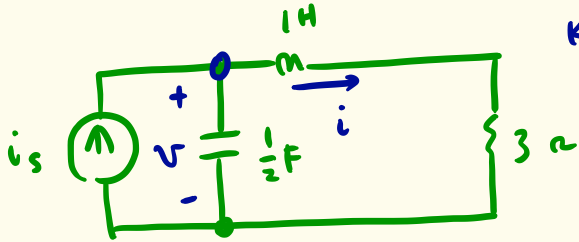
initial conditions



$i(0) = 0A$

$v(0) = 10\,V$

At $t \geq 0$



KCL: $i_s = i + \frac{1}{2}\frac{dv}{dt}$ ①

KVL: $v = L\frac{di}{dt} + 3i$ ②

$$\Rightarrow i_s = i + \frac{3}{2}\frac{di}{dt} + \frac{1}{2}\frac{d^2 i}{dt^2}$$

Practice

$$i = 2e^{-3t} - 14e^{-2t} + 12e^{-t} \quad A$$

$$v = \frac{di}{dt} + 3i$$

$=$