Electric Circuits
Lecture 10 Damped Second-Order Systems

EE2210, Fall 2022

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Lecture Outline

- Chapter 9 in the textbook

Review

- 2nd-order Circuits with L, C

1) Write 2nd-order differential equation

2) Particular solution (constant)

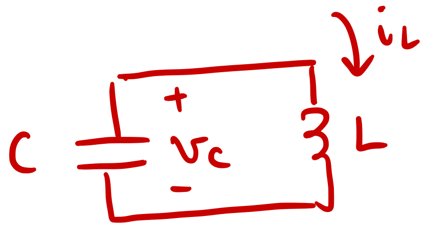
3) Homogeneous solution (Ae^{st}), s_1 and s_2

$\Rightarrow A_1 e^{s_1 t} + A_2 e^{s_2 t}$

4) Total = 2) + 3) use initial conditions $\Rightarrow A_1, A_2$

Review

Example



$C = 1\,\mu F, \quad L = 100\,\mu H$

$i_L(0) = 0, \quad v_C(0) = 1\,V.$

Find i_L and v_C.

$$\omega_0 = \frac{1}{\sqrt{L \cdot C}} = \frac{1}{\sqrt{10^{-4} \cdot 10^{-6}}} = 10^5 \text{ rad/s}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{10^5}{2\pi} \text{ Hz}, \quad T_0 = \frac{1}{f_0} = \frac{2\pi}{10^5} \text{ s}.$$

$$v_C(t) = v_C(0) \cdot \cos\omega_0 t = 1 \cdot \cos 10^5 t \quad V$$

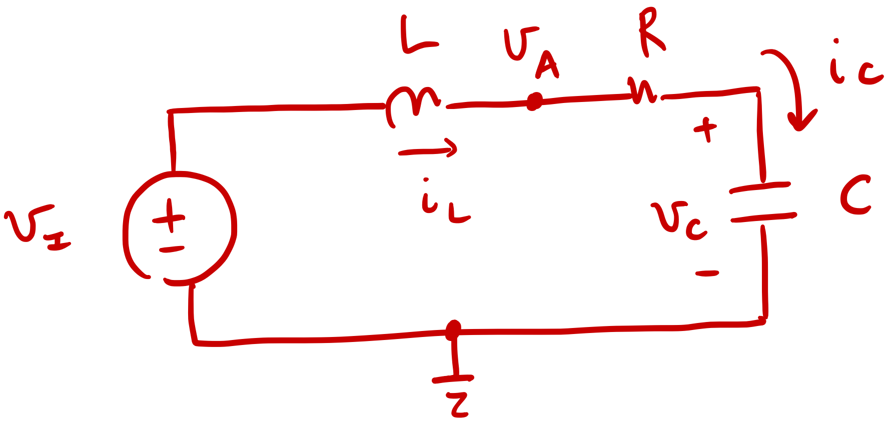
$$i_L(t) = -i_C(t) = -C \cdot \frac{dv_C}{dt} = 0.1 \cdot \sin 10^5 t \quad A$$

Review

Review

R LC

Let's Analyze the RLC Network (Damped Oscillator)



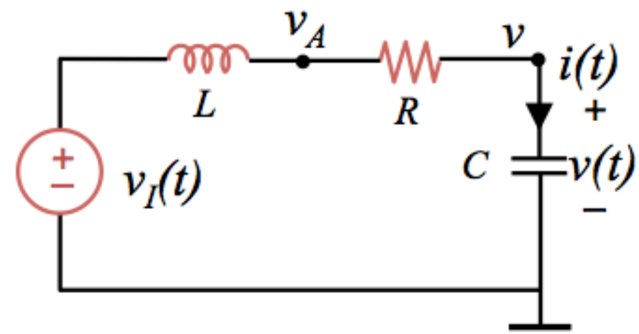
Variables : v_C , i_L

KCL @ v_C : $\dfrac{v_A - v_C}{R} = i_C = C\,\dfrac{dv_C}{dt}$ ①

@ v_A : $\dfrac{1}{L}\displaystyle\int_{-\infty}^{t} (v_I - v_A)\,dt = \dfrac{v_A - v_C}{R}$ ②

Setup the Differential Equation

- Need to get rid of v_{A}.



From ①, $\quad v_A = RC \cdot \dfrac{dv_c}{dt} + v_c \quad$ ③

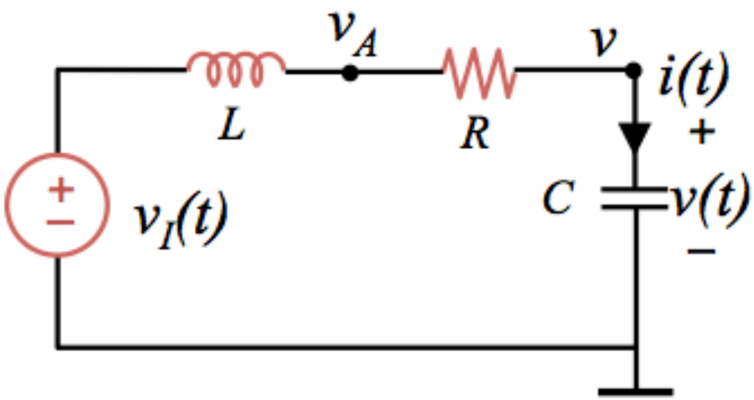
Plug ③ into ②, $\quad \dfrac{1}{L}\displaystyle\int_{-\infty}^{t}\left(v_I - RC\dfrac{dv_c}{dt} - v_c\right)dt = C\dfrac{dv_c}{dt}$

$\overset{\frac{d}{dt}}{\Rightarrow} \quad \dfrac{1}{L}\left(v_I - RC\dfrac{dv_c}{dt} - v_c\right) = C\dfrac{d^2 v_c}{dt^2}$

with only LC (lec9)

$\Rightarrow \quad \dfrac{d^2 v_c}{dt^2} + \dfrac{R}{L}\dfrac{dv_c}{dt} + \dfrac{1}{LC}v_c = \dfrac{1}{LC}v_I \quad \longleftrightarrow \quad \dfrac{d^2 v_c}{dt^2} + \dfrac{1}{LC}v_c = \dfrac{1}{LC}v_I$

Setup the Differential Equation Differently



KVL: $v_I = v_c + v_R + v_L = v_c + R \cdot i + L \cdot \frac{di}{dt}$ ①

$i = C \frac{dv_c}{dt}$ ②

use ② into ① $\Rightarrow$ $v_I = LC \frac{d^2 v_c}{dt^2} + RC \frac{dv_c}{dt} + v_c$

Method of Particular and Homogeneous Solutions

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

- Four-step procedure

1. Find the particular solution $v_P(t)$
2. Find the homogeneous solution $v_H(t)$ Ae^{st} , s_1, s_2
   - Four-step procedure

   $A_1 e^{s_1 t} + A_2 e^{s_2 t}$
3. The total solution is the sum of the particular solution and homogeneous solution.
4. Use initial conditions to solve for the remaining constraints.

   (A_1, A_2)

Let's Solve

$$\boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I}$$

- Given $v_I(t) = V_I \cdot u(t)$, $\cancel{v(t) = 0, i(t) = 0}$. Find $v(t)$ for t > 0.

  $v(t=0)=0$, $i(t=0)=0$

1. Particular Solution

$$\boxed{\cancel{\frac{d^2v}{dt^2}} + \frac{R}{L}\cancel{\frac{dv}{dt}} + \frac{1}{LC}v = \frac{1}{LC}v_I}$$

Assume $v_{cp} = K$

$$\Rightarrow \quad \frac{1}{LC}\cdot K = \frac{1}{LC}\cdot V_I$$

$$\Rightarrow \quad v_{cp} = V_I$$

2. Homogeneous Solution

$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

① Assume $v_{CH}=Ae^{st}$

② $A\cdot s^2\cdot e^{st}+\frac{R}{L}\cdot A\cdot s\cdot e^{st}+\frac{1}{LC}\cdot Ae^{st}=0$

$\Rightarrow Ae^{st}\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)=0$

$\Rightarrow s^2+\frac{R}{L}s+\frac{1}{LC}=0$ characteristic equation

2. Homogeneous Solution

$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

Assume $\omega_0=\frac{1}{\sqrt{LC}}$, $\alpha=\frac{R}{2L}$, rewrite $s^2+2\alpha s+\omega_0^2=0$

(3) Solve the characteristic equation, two roots s_1 and s_2.

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_0^2}$$

$$s_2=-\alpha-\sqrt{\alpha^2-\omega_0^2}$$

(4) $v_{CH}=A_1e^{s_1t}+A_2e^{s_2t}=A_1e^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)}+A_2e^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)}$

3. Total Solution

$$v_C = V_I + A_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)t}$$

4. Find unknowns from initial conditions

$$v_C(t=0) = 0 = V_I + A_1 + A_2$$

$$i(t=0) = 0 = C \cdot \frac{dv_C}{dt}(t=0) = C \cdot A_1 \cdot \left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right) + C \cdot A_2 \cdot \left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)$$

$$\Rightarrow \quad A_1, \quad A_2$$

Let's Stare at the Total Solution for a While Longer

$$\boxed{v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\underbrace{\sqrt{\alpha^2-\omega_0^2}}_{>0}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}}$$

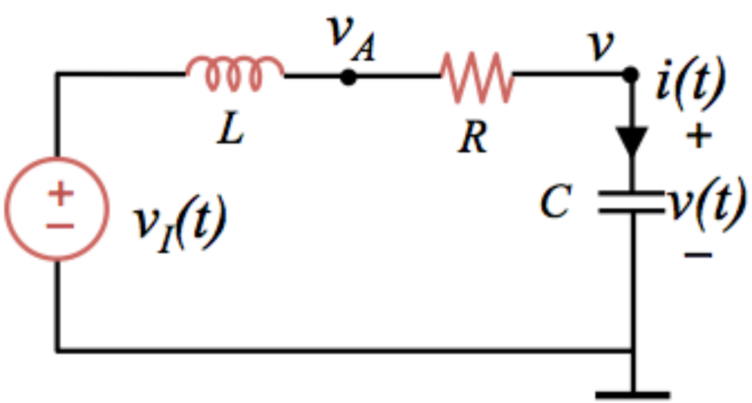
- There are 3 possible cases:

1) $\alpha > \omega_0$, s_1, s_2: real, overdamped

2) $\alpha < \omega_0$, s_1, s_2: complex conjugate $(-\alpha + j\beta, -\alpha - j\beta)$, under-damped

3) $\alpha = \omega_0$, $s_1 = s_2$, critically damped

Over-Damped $\left(\alpha > \omega_0 , \quad \frac{R}{2L} > \frac{1}{\sqrt{LC}} \Rightarrow R > 2\sqrt{\frac{L}{C}} \right)$



$v_c(t) = V_I + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

- Considering $\alpha \gg \omega_0$

$v_c(t) = V_I + A_1 e^{-\alpha_1 t} = V_I + A_1 e^{-\frac{\omega_0^2}{2\alpha} t}$, $i = C\frac{dv_c}{dt} = C \cdot A_1 \cdot \left(\frac{-1}{2}\frac{\omega_0^2}{\alpha}\right) \cdot e^{-\frac{\omega_0^2}{2\alpha} t}$

- Use Taylor series ($\alpha \gg \omega_0$), $\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots$ when $|x^2| \le 1$

$$\alpha_1 = -s_1 = \alpha - \sqrt{\alpha^2 - \omega_0^2} = \alpha \cdot \left(1 - \sqrt{1 - \left(\frac{\omega_0}{\alpha}\right)^2}\right)$$

$$\approx \alpha \cdot \left(1 - 1 + \frac{1}{2}\left(\frac{\omega_0}{\alpha}\right)^2 + \underline{\frac{1}{8}\left(\frac{\omega_0}{\alpha}\right)^4} - \cdots\right)$$

$$\simeq \alpha \cdot \frac{1}{2}\left(\frac{\omega_0}{\alpha}\right)^2 = \frac{\omega_0^2}{2\alpha}$$

$$\alpha_2 = -s_2 = \alpha + \sqrt{\alpha^2 - \omega_0^2} = \alpha\left(1 + \sqrt{1 - \left(\frac{\omega_0}{\alpha}\right)^2}\right)$$

$$\approx \alpha\left(1 + 1 - \frac{1}{2}\left(\frac{\omega_0}{\alpha}\right)^2 - \frac{1}{8}\left(\frac{\omega_0}{\alpha}\right)^4 + \cdots\right)$$

$$\simeq \alpha\left(2 - \frac{1}{2}\left(\frac{\omega_0}{\alpha}\right)^2\right) \simeq \alpha \cdot 2$$

$\alpha_2 \gg \alpha_1$, $v_{CH} = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \simeq A_1 e^{-\alpha_1 t}$