EE2210 Lecture 8A:
Second-Order Systems

Chapter 12 and Appendix C of textbook

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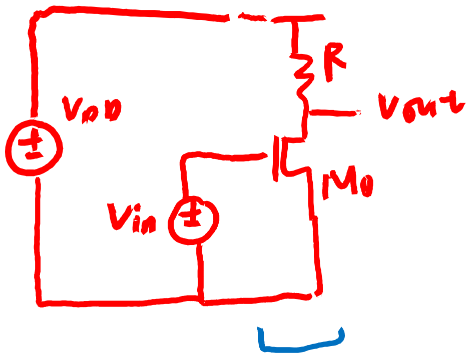
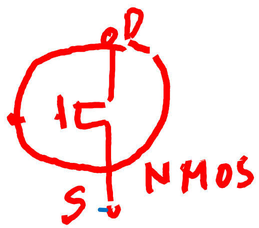
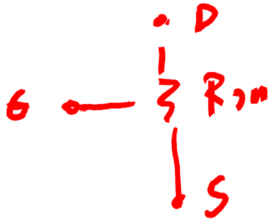
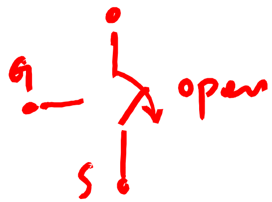


inverter

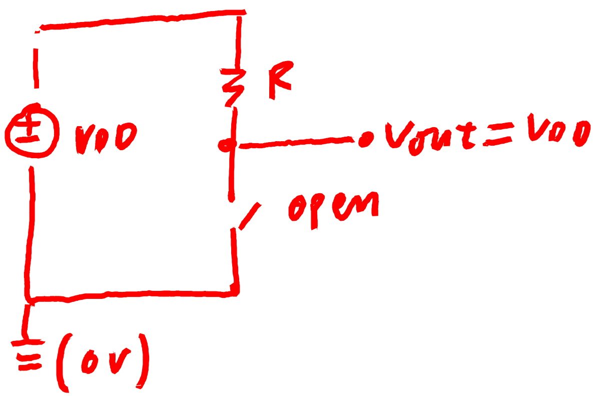




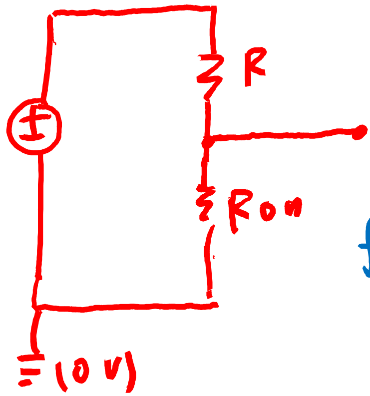


$V_{GS} > V_{th}$ $\quad$ $V_{GS} < V_{th}$

#1 $V_{in} < V_{th}$ M0 OFF



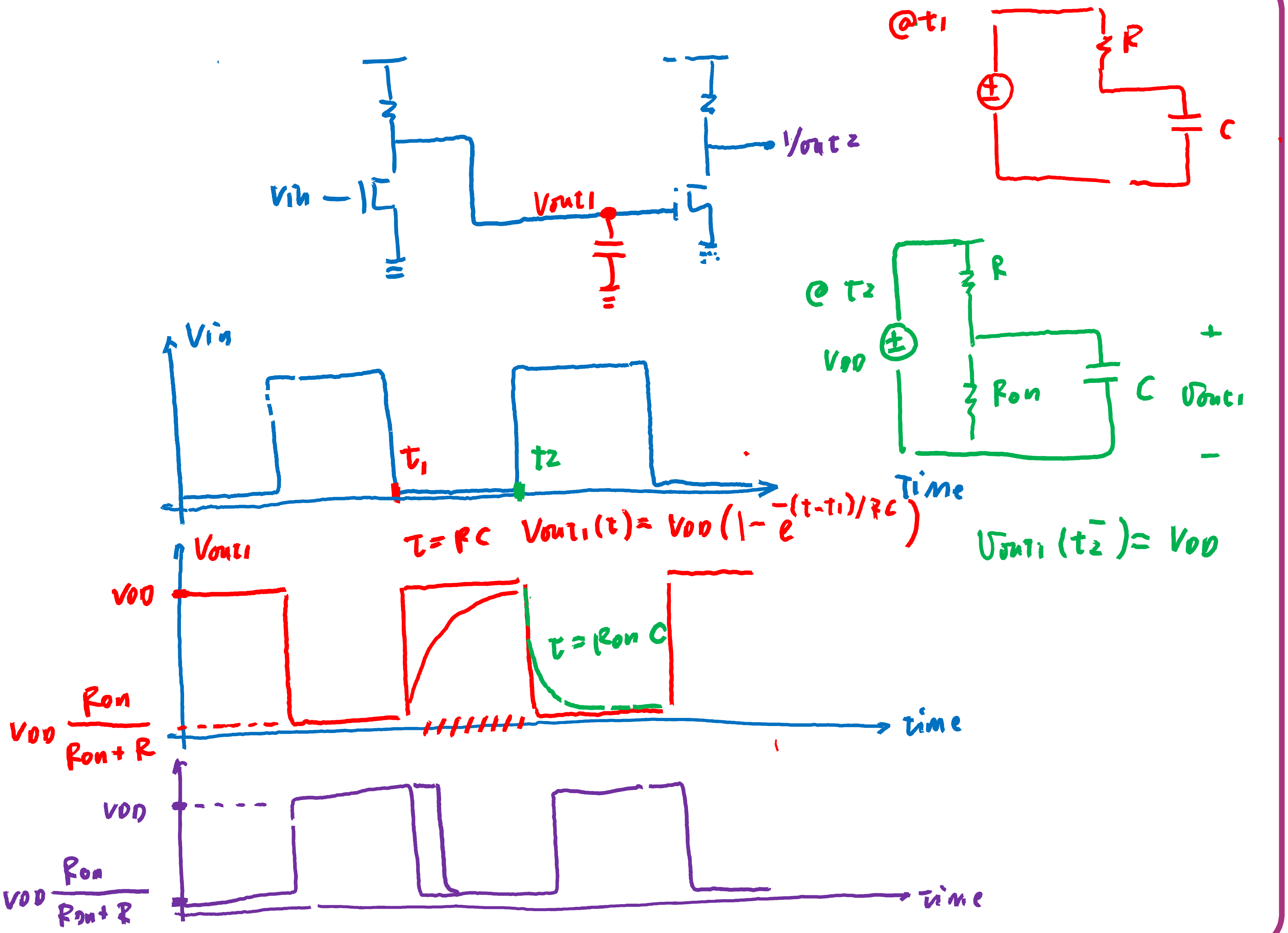
#2 $V_{in} > V_{th}$



$$V_{out} = V_{DD} \frac{R_{on}}{R + R_{on}}$$

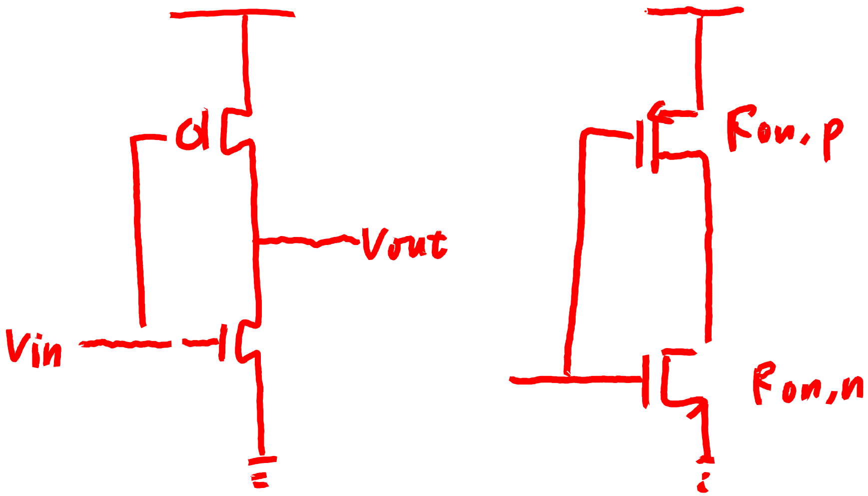
for vout to be logical low

$R \gg R_{on}$

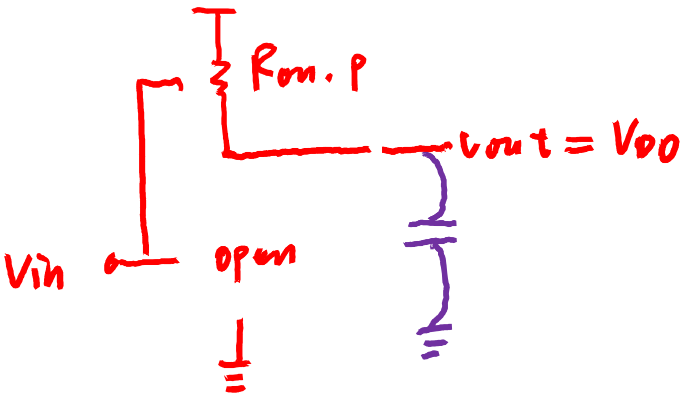


Vin
Vout1
Vout2
@t1
R
C
@ t2
VDD
R
Ron
C
Vout1
+
−
Vin
t1
t2
Time
Vout1
τ = RC
Vout1(t) = VDD (1 − e^{−(t−t1)/RC})
Vout1(t2−) = VDD
VDD
τ = Ron C
VDD Ron / (Ron + R)
time
VDD
VDD Ron / (Ron + R)
time

CMOS

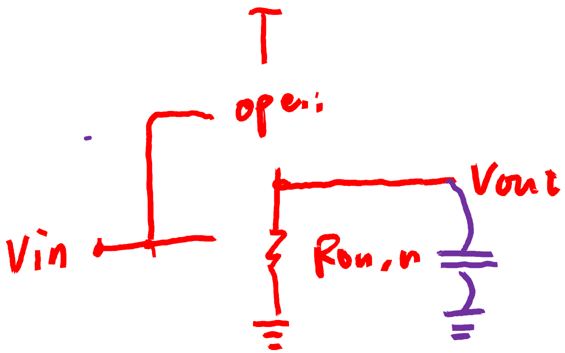


#1 $V_{in} < V_{th,n}$



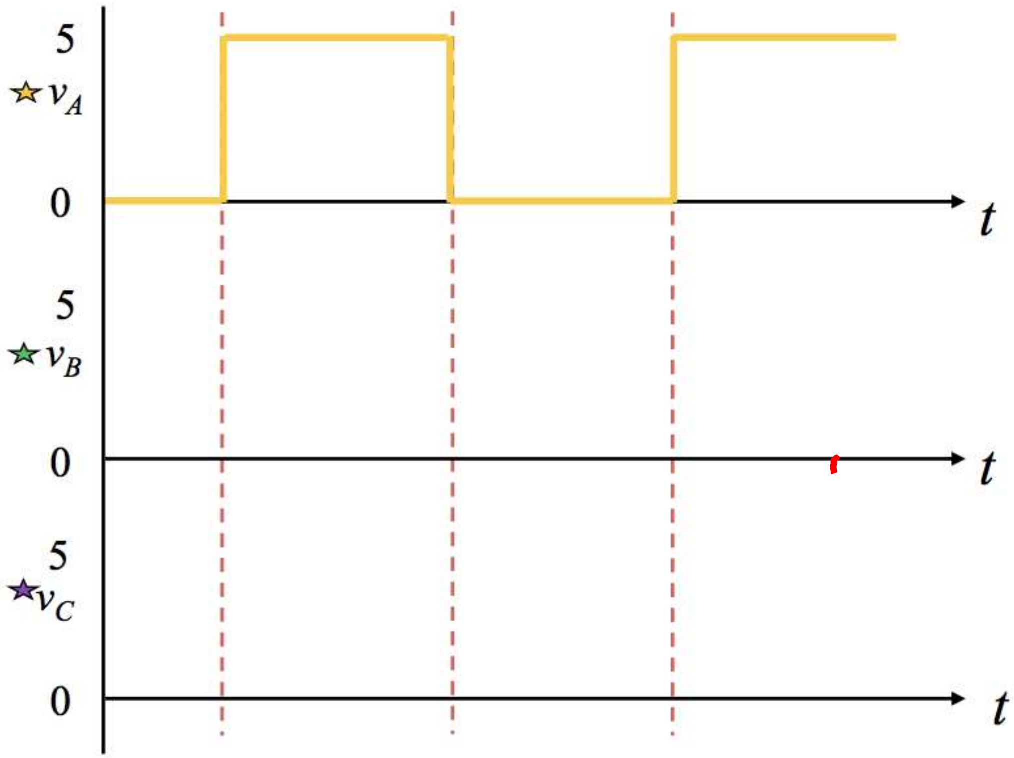
#2 $V_{in} > V_{th,n}$

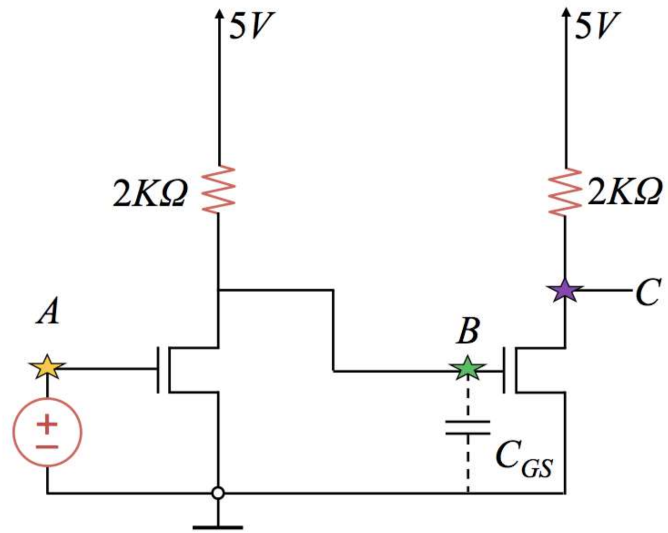
$(V_{DD} - V_{in} < |V_{th,p}|)$



An Inverter Chain Example

- **With C_{GS} of NMOS on node B**



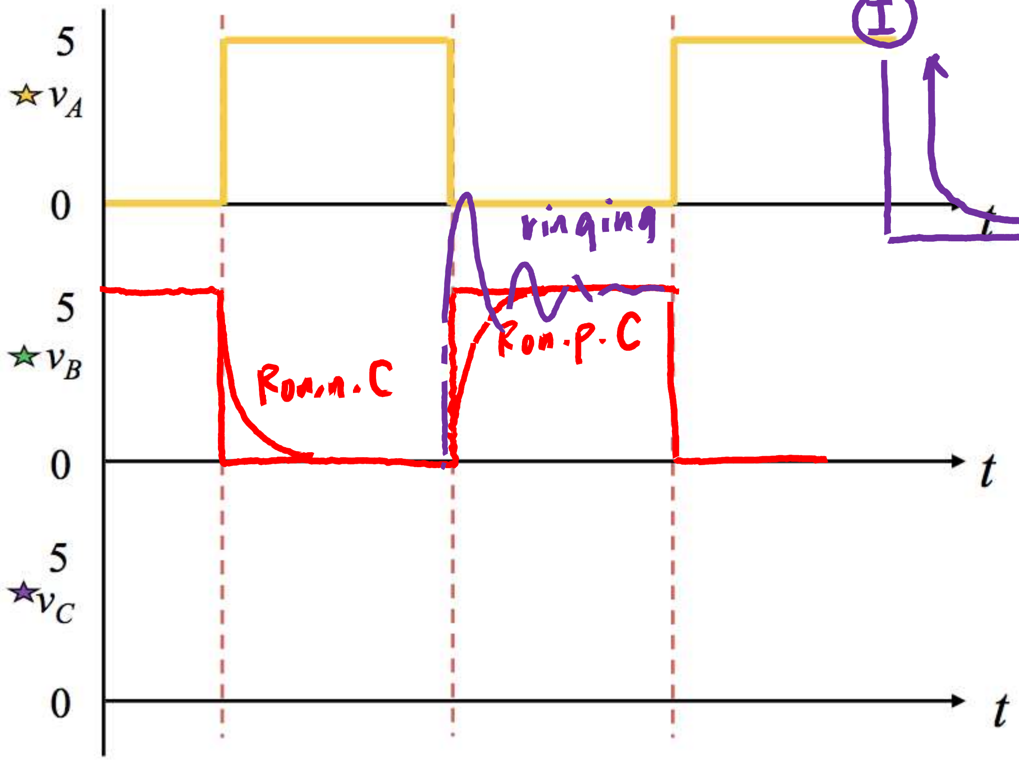


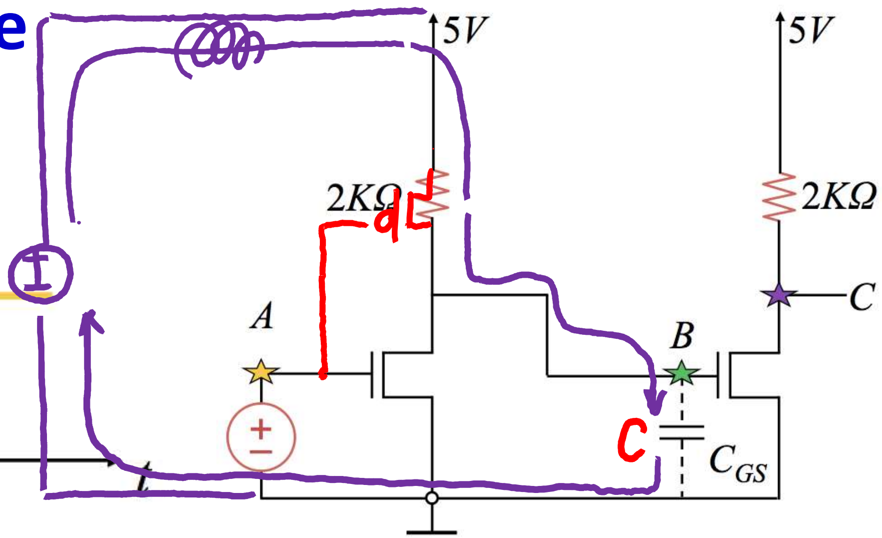
– Slow rise time and delay in signal

– We can try to increase the speed with a smaller R_L (from 2 kΩ to 50 Ω)

Observed Output – Fast Case

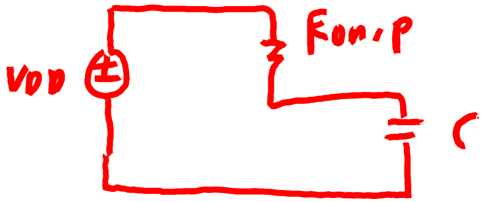
- **With C_{GS} of NMOS on node B**

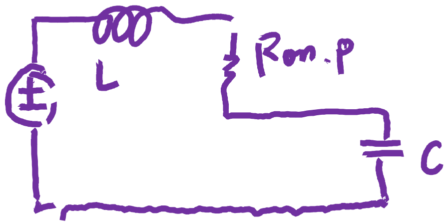




– Ringing behavior observed!!

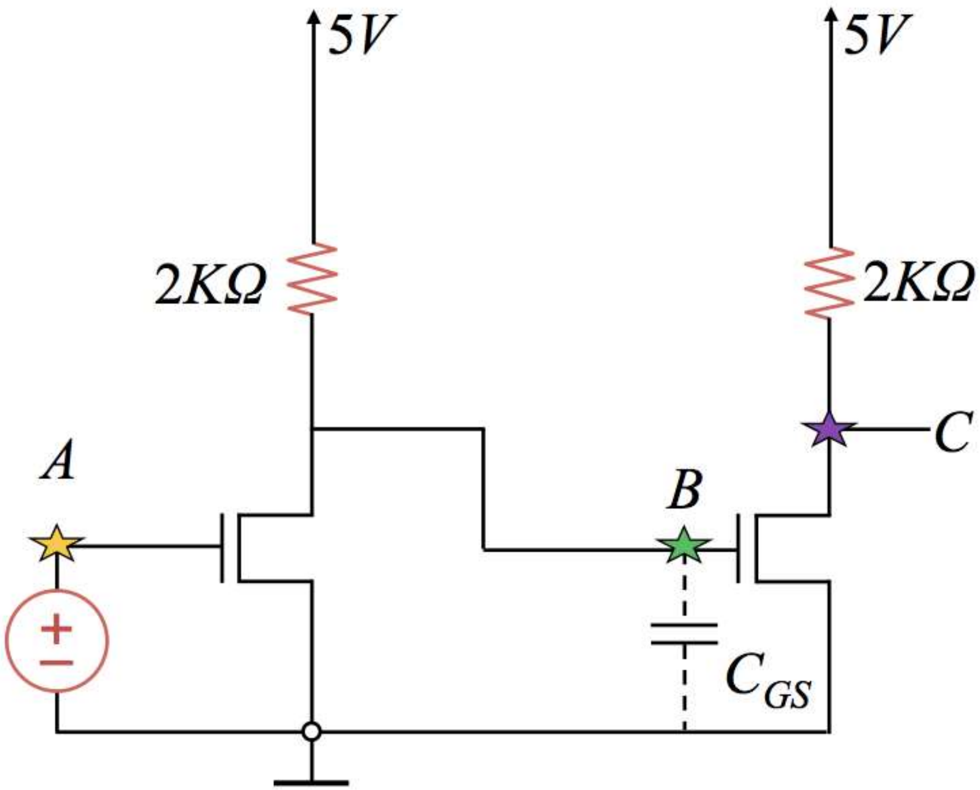
CMOS inverter





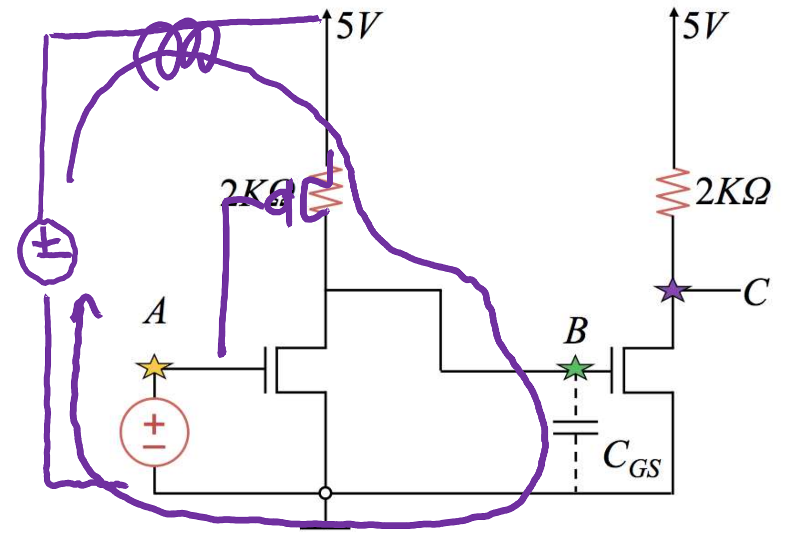
Fast Case – What's Really Going On

- **The parasitic inductance of the wire is included**

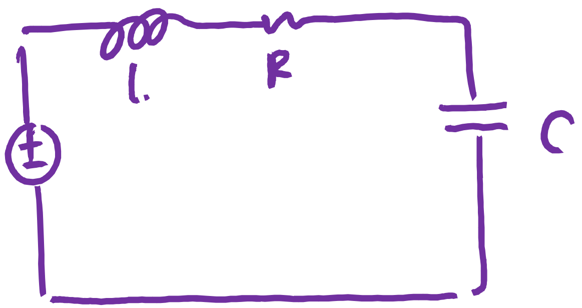


Second-Order Systems

- **Involving R, L, C circuit elements**



- **Relevant circuit model**

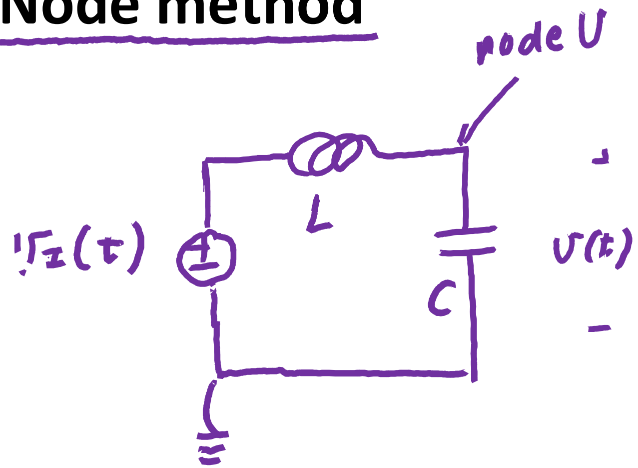


  – Two energy-storage elements → second-order system

First, Let's Analyze the LC Network

- We will introduce R into the circuit later

- Node method



① reference point

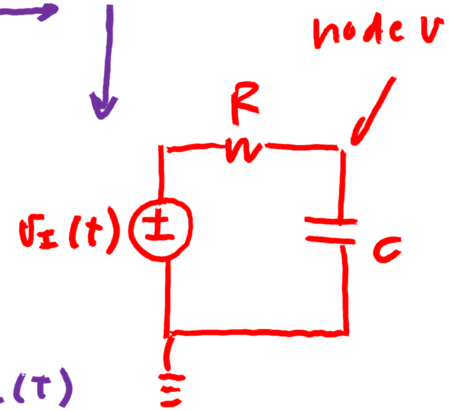
② KCL @ node v

for C:

$$i_C(t) = C\frac{dv(t)}{dt}$$

for L:

$$(v_I(t) - v(t)) = L\frac{di_L(t)}{dt}$$



$$\Rightarrow \text{KCL}: \frac{1}{L}\int_{-\infty}^{t}(v_I(t)-v(t))\,dt = C\frac{dv(t)}{dt}$$

$$LC\frac{d^2v(t)}{dt^2} + v(t) = v_I(t)$$

LC: 單位 s^2

$$RC\frac{dv(t)}{dt} + v(t) = v_I(t)$$

RC: 單位 s

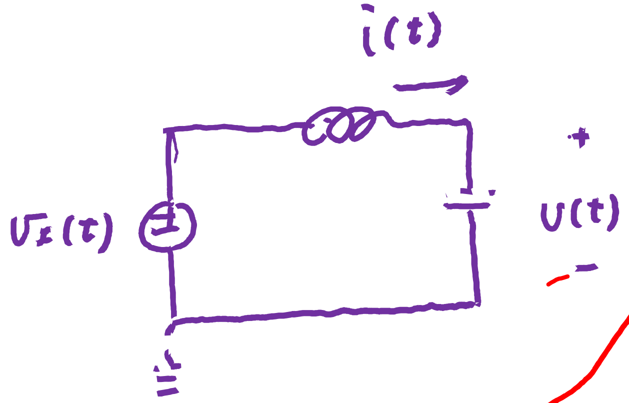
Method of Particular and Homogeneous Solutions

- **Four-step procedure**

$$LC\frac{\mathrm{d}^2 v}{\mathrm{d}t} + v = v_I$$

1. **Find the particular solution** $v_P(t)$
2. **Find the homogeneous solution** $v_H(t)$
   - Four-step procedure
3. **The total solution is the sum of the particular solution and homogeneous solution**
4. **Use initial condition to solve for the remaining constraints**

Let's Solve

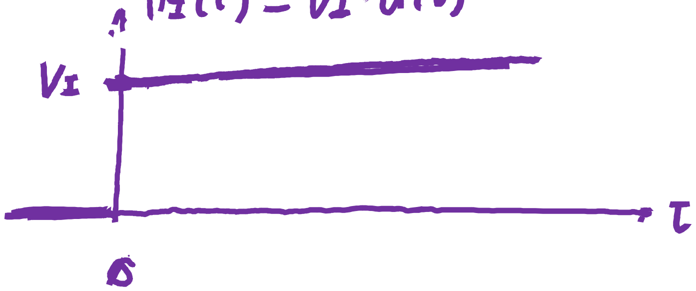


$$LC\frac{d^2v(t)}{dt^2} + v(t) = v_I(t)$$

$\Rightarrow$ 求 $v(t) = f(L, C, v_I(t))$

For step input

$v_I(t) = V_I \cdot u(t)$



and for initial conditions

$$\text{(zero-state)} \quad \begin{cases} v(0) = 0 \\ i(0) = 0 \end{cases}$$

求其 response $v(t)$ for $t \geq 0$

1. Particular Solution $v_p(t)$

$\left(v(t) = v_p(t) + v_H(t) \right)$

For $t \geq 0$

$$LC\frac{d^2 v_p(t)}{dt^2} + v_p(t) = V_I$$

猜 $v_p(t) = V_I$

代回去有合, okay!!

2. Homogeneous Solution

- Look for solution to $LC\cdot\frac{d^2 v_H(t)}{dt^2}+v_H(t)=0$
- Four-step method:

(2A) 猜 $v_H(t)=Ae^{st}$ (A, s 待求) $A\neq 0$

(2B) 代進去 $LC\cdot\cancel{A}\, s^2\cancel{e^{st}}+\cancel{A}\cdot\cancel{e^{st}}=0$

$\Rightarrow LC\cdot s^2+1=0$ (char. eq.) $RC\cdot s+1=0$

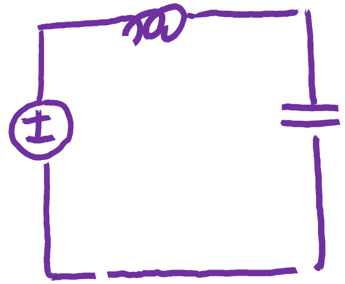
(2C) 求 s 解 $\therefore s=\pm j\sqrt{\frac{1}{LC}}$ ($j=\sqrt{-1}$, Appendix C)

令 $\sqrt{\frac{1}{LC}}=\omega_0 \Rightarrow s=\pm j\omega_0$

(2D) general solution $v_H(t)=A_1\cdot e^{j\omega_0 t}+A_2\cdot e^{-j\omega_0 t}$

3. Total Solution

$$v(t) = v_P(t) + v_H(t)$$
$$= V_I + A_1 \cdot e^{j\omega_0 t} + A_2 \cdot e^{-j\omega_0 t}$$



4. Find unknowns from initial conditions

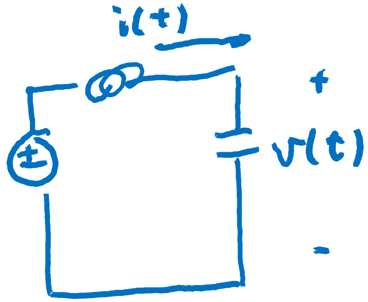
$$v(0) = 0 \quad \therefore \quad 0 = V_I + A_1 + A_2$$
$$i(t) = C\frac{dv(t)}{dt}$$
$$i(0) = 0$$
$$= C \cdot A_1 \cdot (j\omega_0) - C A_2 (j\omega_0) = 0$$

$$A_1 = A_2 = -\frac{V_I}{2}$$

$$\boxed{v(t) = V_I - \frac{V_I}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)}$$

3. Total Solution

- **Remember Euler relation:**



$$e^{jx} = \cos x + j\sin x$$

$$V_I - \frac{V_I}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$= V_I - \frac{V_I}{2}\left(\cos\omega_0 t + \cancel{j\sin\omega_0 t} + \cos\omega_0 t - \cancel{j\sin\omega_0 t}\right)$$

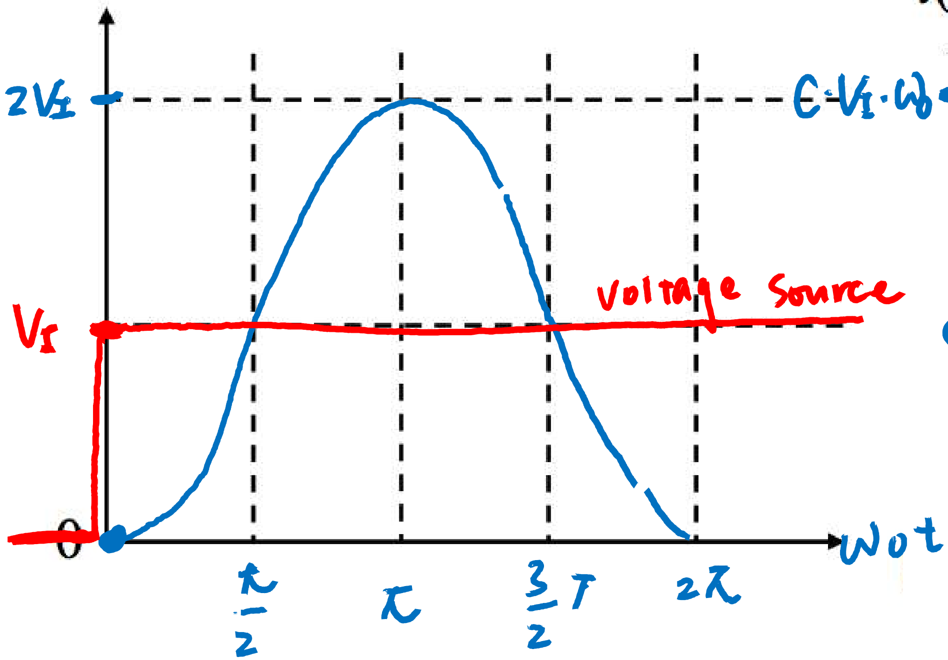
$$\boxed{v(t) = \underbrace{V_I}_{\text{particular solution}} - \underbrace{V_I \cdot \cos\omega_0 t}_{\text{homogeneous solution}}}$$

the output looks sinusoidal

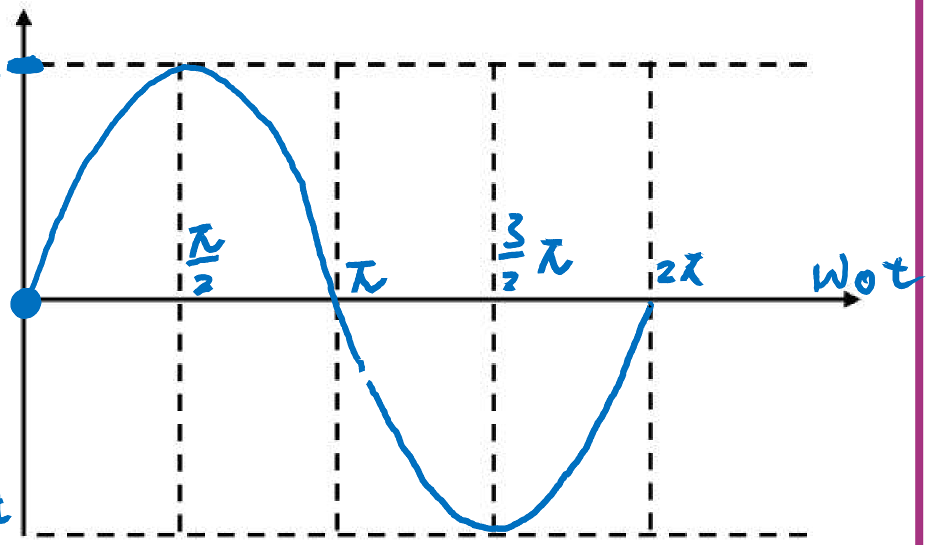
$$i(t) = C\frac{dv(t)}{dt} = C \cdot V_I \cdot \omega_0 \cdot \sin(\omega_0 t)$$

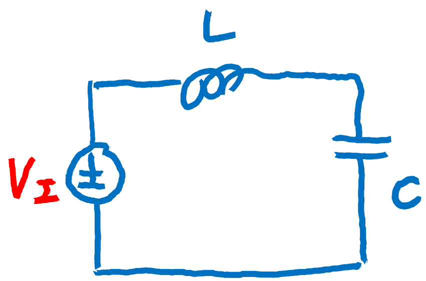
Plotting the Total Solution

$v(t) = V_I - V_I \cdot \cos w_0 t$



$i(t) = C \cdot V_I \cdot w_0 \cdot \sin(w_0 t)$





$v(0) = 0$

$i(0) = 0$

$Q = CV$

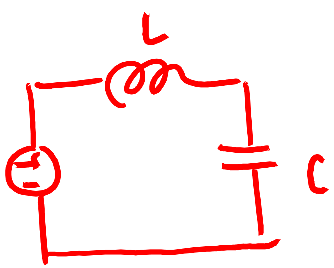
$i_c(t) = C \frac{d v_c(t)}{dt}$

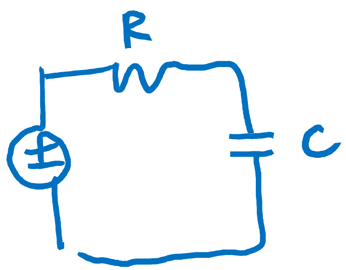
$v_L(t) = L \frac{d i_L(t)}{dt}$

Summary of Method

1. **Write DE for circuit by applying node method**
2. **Find particular solution v_P by guessing and trial & error**
3. **Find homogeneous solution v_H**
   A. Assume solution of the form Ae^{st}
   B. Obtain characteristic equation $LCs^2+1=\emptyset$ $RCs+1=\emptyset$
   C. Solve characteristic equation for roots s_i
   D. Find v_H by summing $A_i \exp(s_i t)$ terms
4. **Total solution is $v_P + v_H$, then solve for remaining constants using initial conditions**

$$LC\frac{d^2v(t)}{dt^2} + v(t) = V_I$$



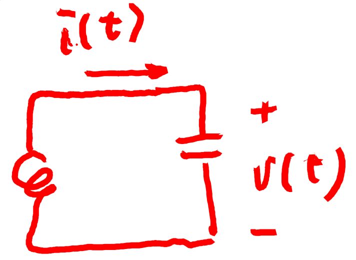


Example – Undriven LC Network Response

- What if we have: $V_I = 0$

  zero-input response

- We can obtain the answer directly from the homogeneous solution (with $v_I = 0$)



$V_I = 0$

$$\begin{cases} v(0) = V \\ i(0) = 0 \end{cases}$$

$$LC \frac{d^2 v(t)}{dt} + v(t) = 0$$

猜 $A \cdot e^{st}$ $A \neq 0$ 代進去 $s = \pm j\sqrt{\frac{1}{LC}} = \pm j\omega_0$

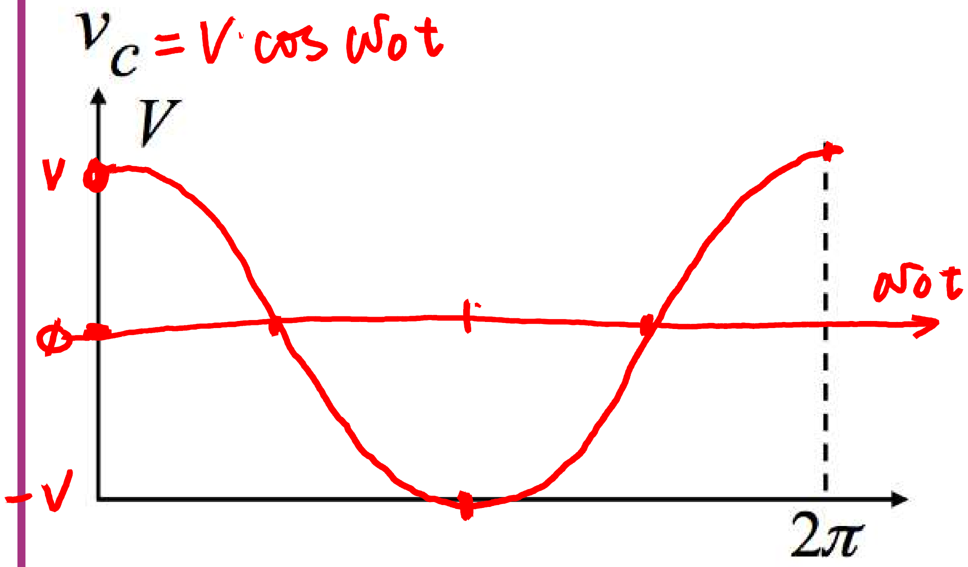
$$v(t) = A_1 \cdot e^{+j\omega_0 t} + A_2 \cdot e^{-j\omega_0 t}$$

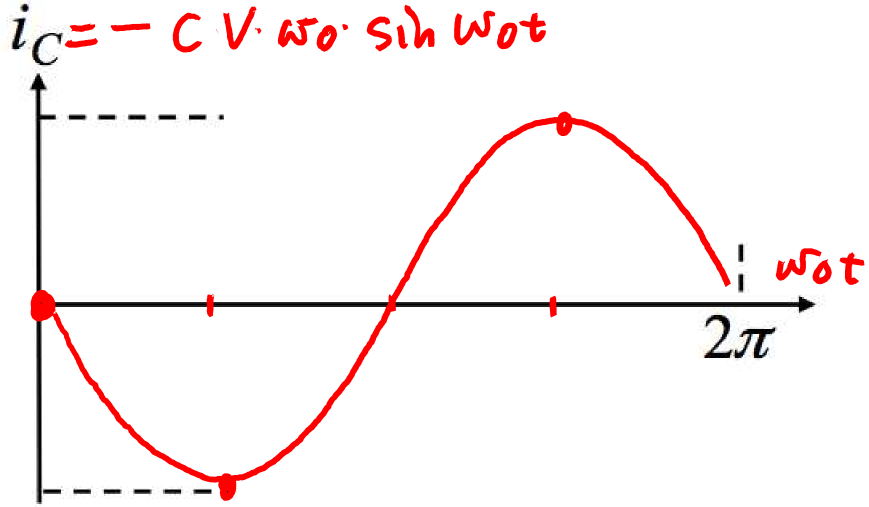
代 initial condition 1) $V = A_1 + A_2$

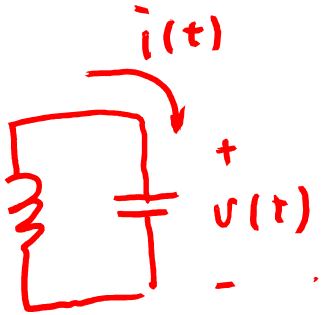
2) $i(t) = C \cdot \frac{dv(t)}{dt} = C \cdot (j\omega_0) A_1 e^{j\omega_0 t} - C(j\omega_0) \cdot A_2 e^{-j\omega_0 t}$

$i(0) = 0 = A_1 - A_2 \Rightarrow v(t) = \frac{V}{2} e^{j\omega_0 t} + \frac{V}{2} e^{-j\omega_0 t} = V \cdot \cos\omega_0 t$

Undriven LC Network Response





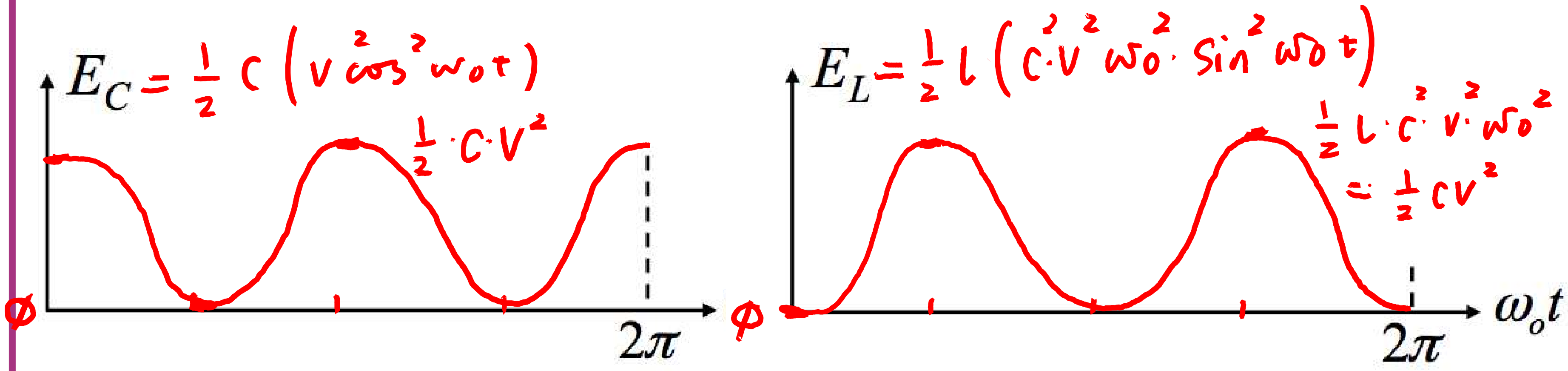


$v(0) = V$

$i(0) = 0$

C

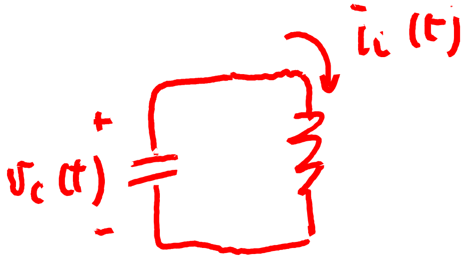
Energy



$E_c(t) + E_L(t) = \frac{1}{2} C V^2$

- **Notice:**

- **Total energy in the system is a constant, but it goes back and forth between the capacitor and the inductor.**



$$
\begin{cases}
-i_L(t) = C \cdot \dfrac{dv_c(t)}{dt} \\
v_c(t) = L \dfrac{di_L(t)}{dt}
\end{cases}
$$

猜 $i_L(t) = B \cdot \sin(\omega t)$

$v_c(t) = A \cos(\omega t)$