Networks Theorems

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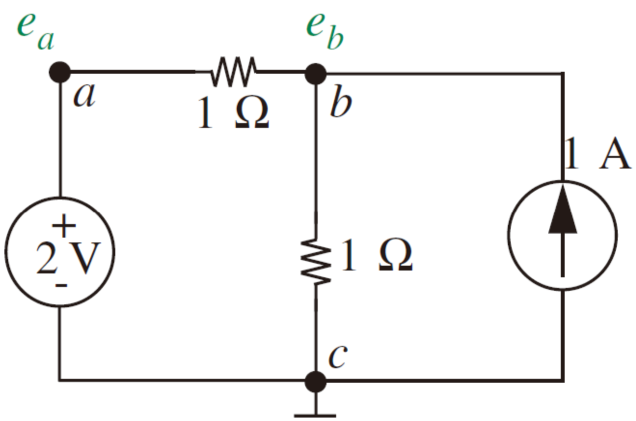
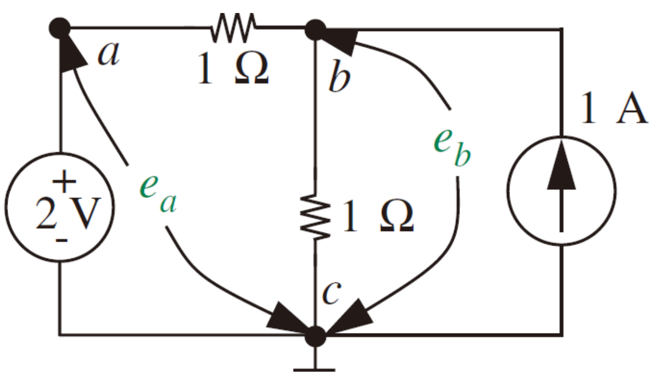
Node voltage

- A ***node voltage*** is the potential difference between the given node and some other node that has been chosen as a ***reference node***. The reference node is called the ***ground***.
- Node *c* has been chosen as ground. The upside down "T" symbol is the notation for the ground node. Nodes *a* and *b* are two other nodes of this circuit. Their node voltages e_a and e_b are marked.





- Although the choice of *reference node* is in fact arbitrary, it is most convenient to *choose the node that has the maximum number of circuit elements connected to it*. The potential at this node is defined to be zero V, or ground-zero potential.

Node voltages from branch variables

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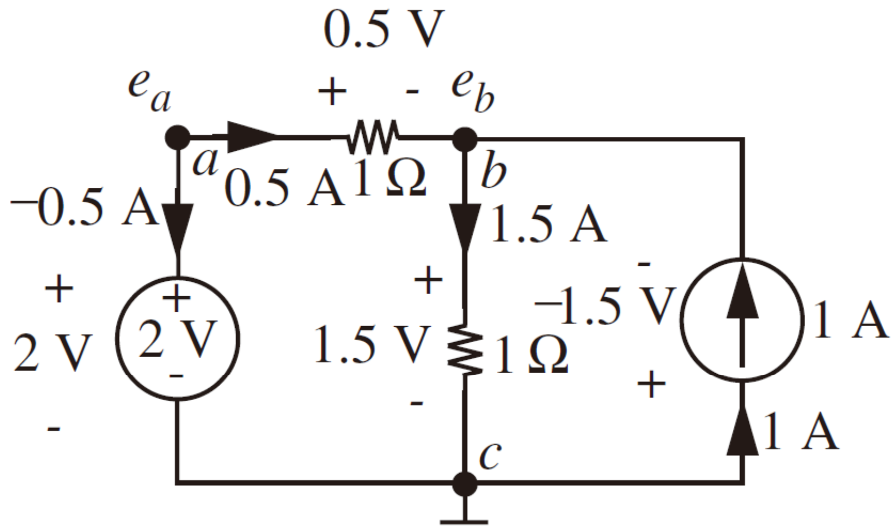
- Let us determine the node voltages from the known branch variables.
- Figure shows our circuit with a known set of branch voltages and currents. Let us determine the node voltages e_a and e_b.

$$e_a = 2\ \text{V}$$

$$e_b = 1.5\ \text{A} \times 1\ \Omega = 1.5\ \text{V}$$



Example

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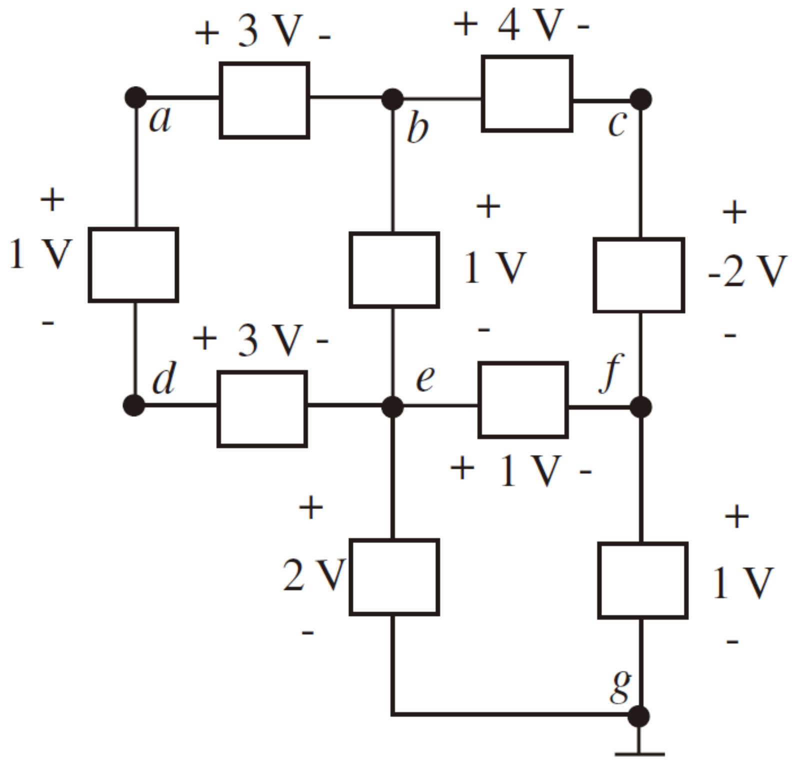
- Determine the node voltages corresponding to nodes e_b and e_c for the circuit. Assume that g is taken as the ground node.

$$e_b = v_{be} + v_{eg} = 1 + 2 = 3 \text{ V}$$

$$e_c = v_{cf} + v_{fg} = -2 + 1 = -1 \text{ V}$$



Branch variables from Node voltages

- Let us determine the values of the branch variables with a known set of node voltages.
- Figure shows our circuit with a known set of node voltages. Let us determine the branch variables v_0, i_0, v_1, i_1, v_2, i_2, v_3, and i_3.
- Branch voltage v_{ab} and node voltages e_a and e_b is related as:

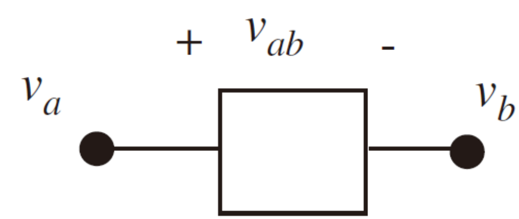
$$v_{ab} = e_a - e_b$$

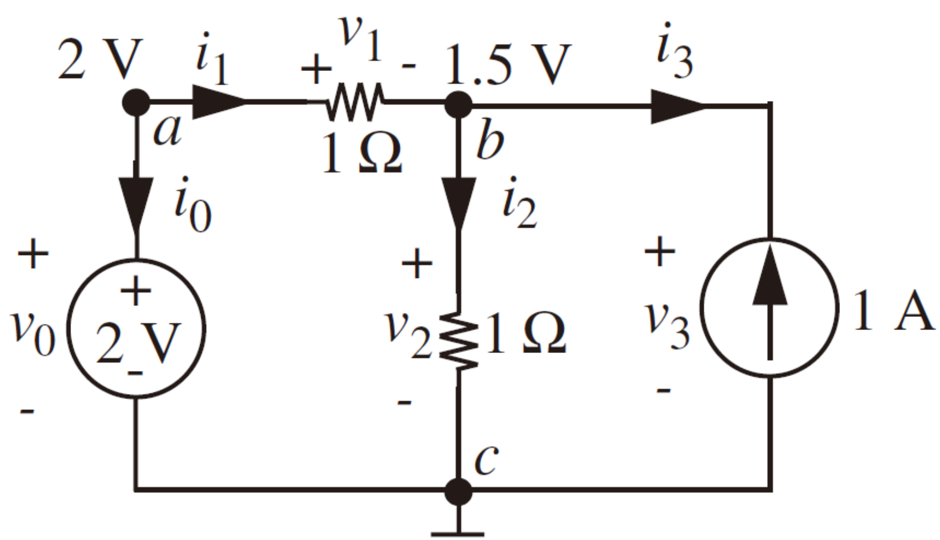
$$v_1 = e_a - e_b = 2 - 1.5 = 0.5 \text{ V}$$

$$i_1 = \frac{v_1}{1\,\Omega} = \frac{0.5 \text{ V}}{1\,\Omega} = 0.5 \text{ A}$$

$$v_2 = e_b = 1.5 \text{ V}$$

$$i_2 = \frac{v_2}{1\,\Omega} = \frac{1.5 \text{ V}}{1\,\Omega} = 1.5 \text{ A}$$





Example

- Determine all the branch voltages for the circuit in Figure when the node voltages are measured with respect to node *e*.

$$v_1 = e_a - e_b = 1 - 2 = -1\text{ V}$$

$$v_2 = e_b - e_e = 2 - 0 = 2\text{ V}$$

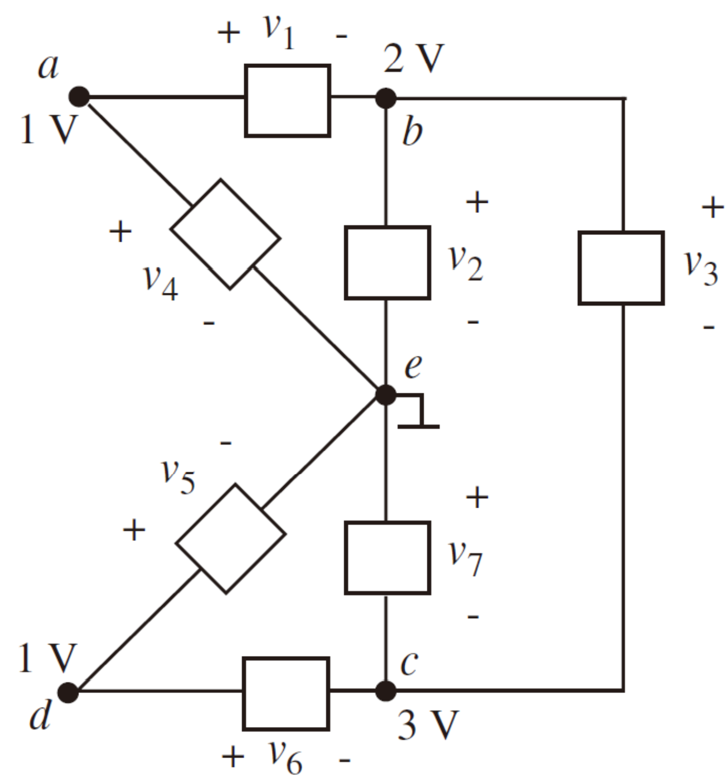
$$v_3 = e_b - e_c = 2 - 3 = -1\text{ V}$$

$$v_4 = e_a - e_e = 1 - 0 = 1\text{ V}$$

$$v_5 = e_d - e_e = 1 - 0 = 1\text{ V}$$

$$v_6 = e_d - e_c = 1 - 3 = -2\text{ V}$$

$$v_7 = e_e - e_c = 3 - 0 = -3\text{ V}$$

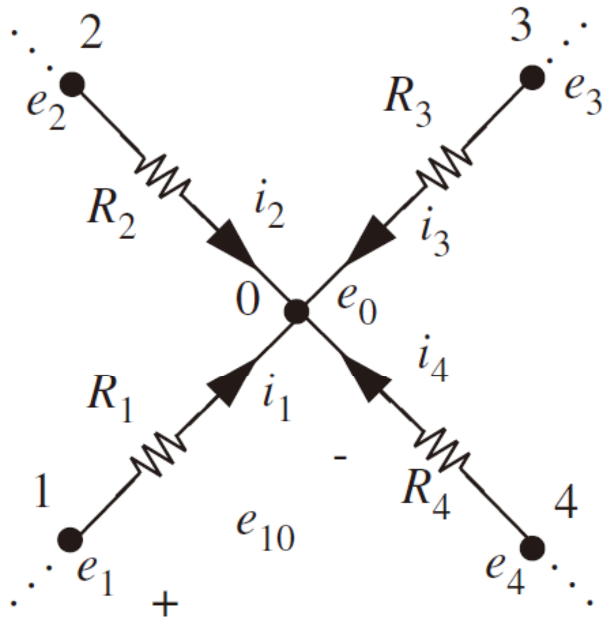


KCL

- Consider the subcircuit as shown, let us write KCL for Node 0 directly in terms of the node voltages e_0, e_1, e_2, e_3, and e_4,(defined with respect to some ground).

$$i_1 + i_2 + i_3 + i_4 = 0$$

$$\frac{e_1 - e_0}{R_1} + \frac{e_2 - e_0}{R_2} + \frac{e_3 - e_0}{R_3} + \frac{e_4 - e_0}{R_4} = 0$$



Method 3—Node analysis

- The most powerful approach of circuit analysis
- Node analysis is based on the combination of element laws, KCL, and KVL.
- It is a particular application of KVL, KCL method

1. Select reference node (ground) from which voltages are measured.
2. Label voltages of remaining nodes with respect to ground. These are the primary unknowns.
3. Write KCL for all but the groundnode, substituting device laws and KVL.
4. Solve for node voltages.
5. Back solve for branch voltages and currents (i.e. the secondary unknowns)

Example

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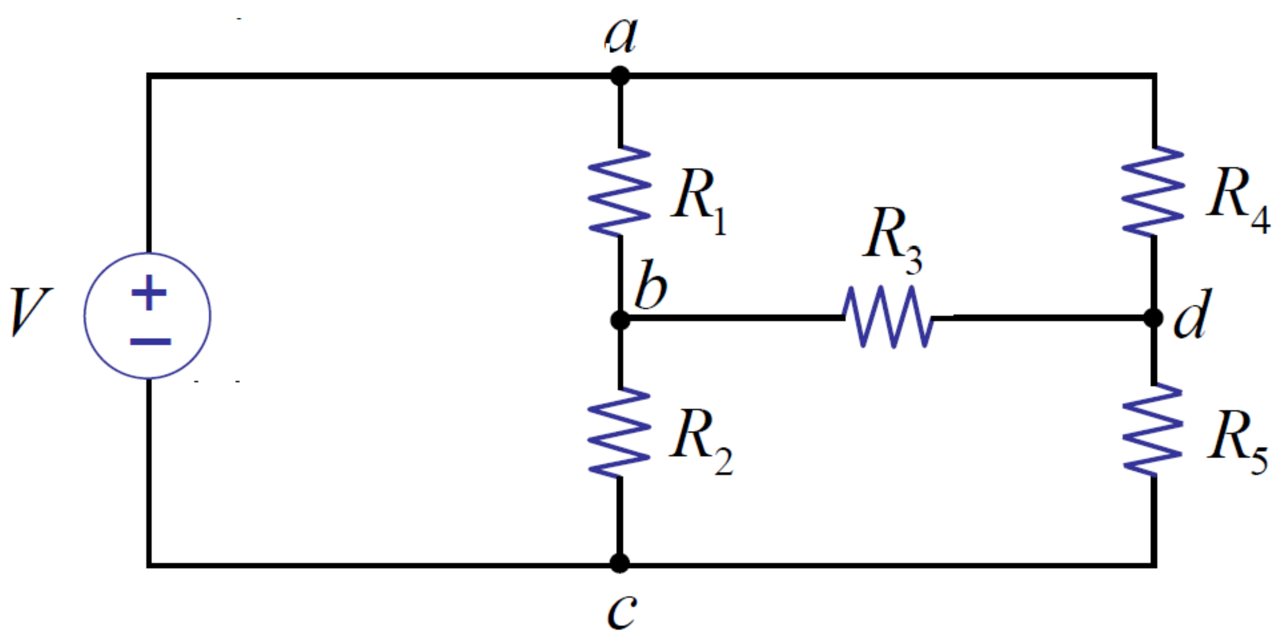
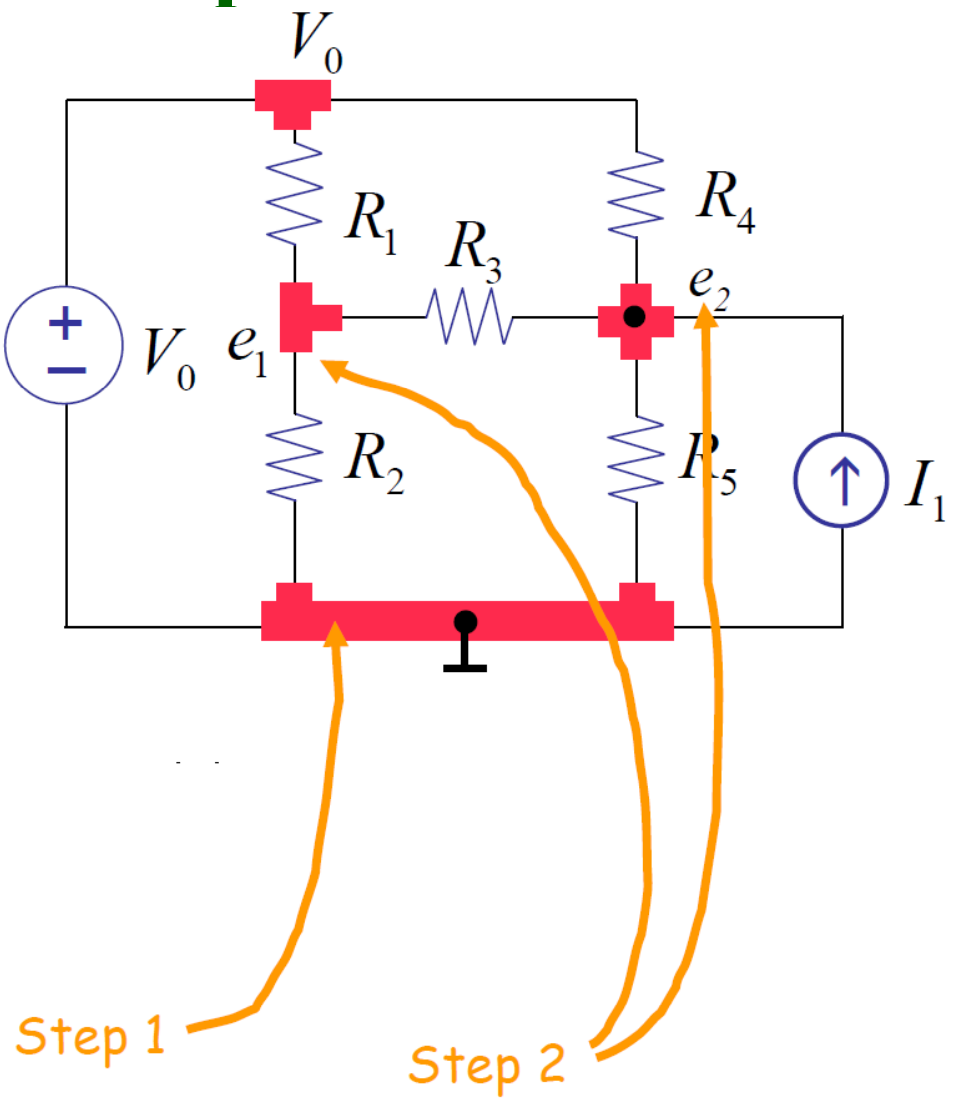
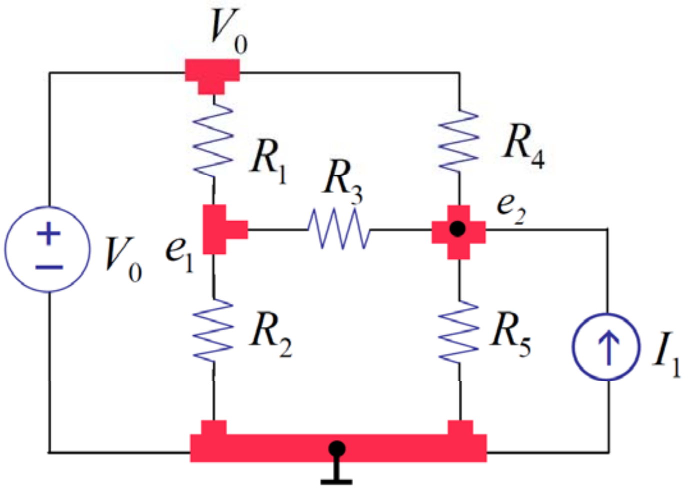


Example: Step1 and 2



Example: Step3



For convenience, let's use conductance

$$G_i = \frac{1}{R_i}$$

KCL at e_1: $(e_1 - V_0)G_1 + (e_1 - e_2)G_3 + (e_1 - 0)G_2 = 0$

KCL at e_2: $(e_2 - e_1)G_3 + (e_2 - V_0)G_4 + (e_2 - 0)G_5 - I_1 = 0$

Example: Step4

- Move constant terms to right-hand side and collect unknowns

$$e_1(G_1 + G_2 + G_3) + e_2(-G_3) = V_0(G_1)$$

$$e_1(-G_3) + e_2(G_3 + G_4 + G_5) = V_0(G_4) + I_1$$

Two equations and two unknowns $\Rightarrow$ Solve for e's

$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 + G_5 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} G_1 V_0 \\ G_4 V_0 + I_1 \end{bmatrix}$$

conductivity matrix

unknown node voltages

sources

Step 4:cont

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- Solve

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{\begin{bmatrix} G_3+G_4+G_5 & G_3 \\ G_3 & G_1+G_2+G_3 \end{bmatrix} \begin{bmatrix} G_1V_0 \\ G_4V_0+I_1 \end{bmatrix}}{(G_1+G_2+G_3)(G_3+G_4+G_5)-G_3^2}$$

$$e_1 = \frac{(G_3+G_4+G_5)(G_1V_0)+(G_3)(G_4V_0+I_1)}{G_1G_3+G_1G_4+G_1G_5+G_2G_3+G_2G_4+G_2G_5+G_3G_4+G_3G_5}$$

$$e_2 = \frac{(G_3)(G_1V_0)+(G_1+G_2+G_3)(G_4V_0+I_1)}{G_1G_3+G_1G_4+G_1G_5+G_2G_3+G_2G_4+G_2G_5+G_3G_4+G_3G_5}$$

Super Node

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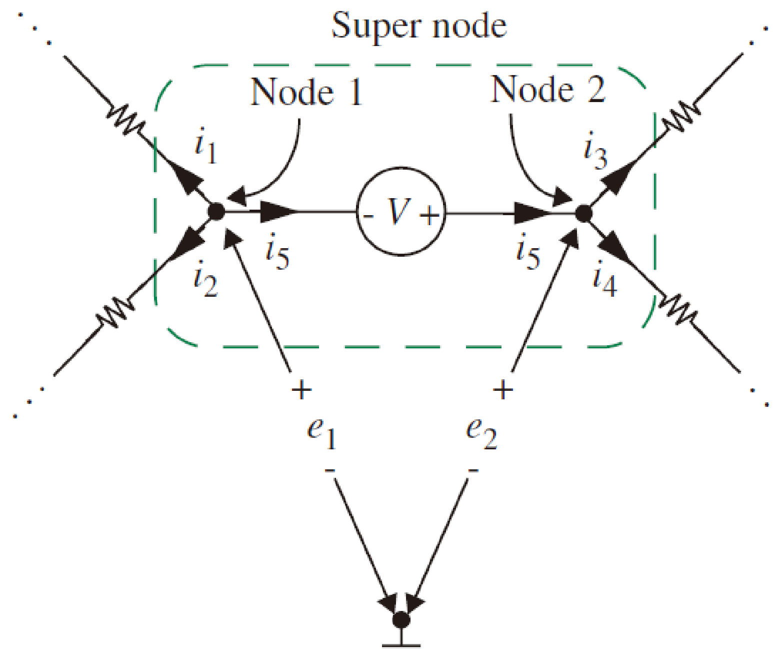
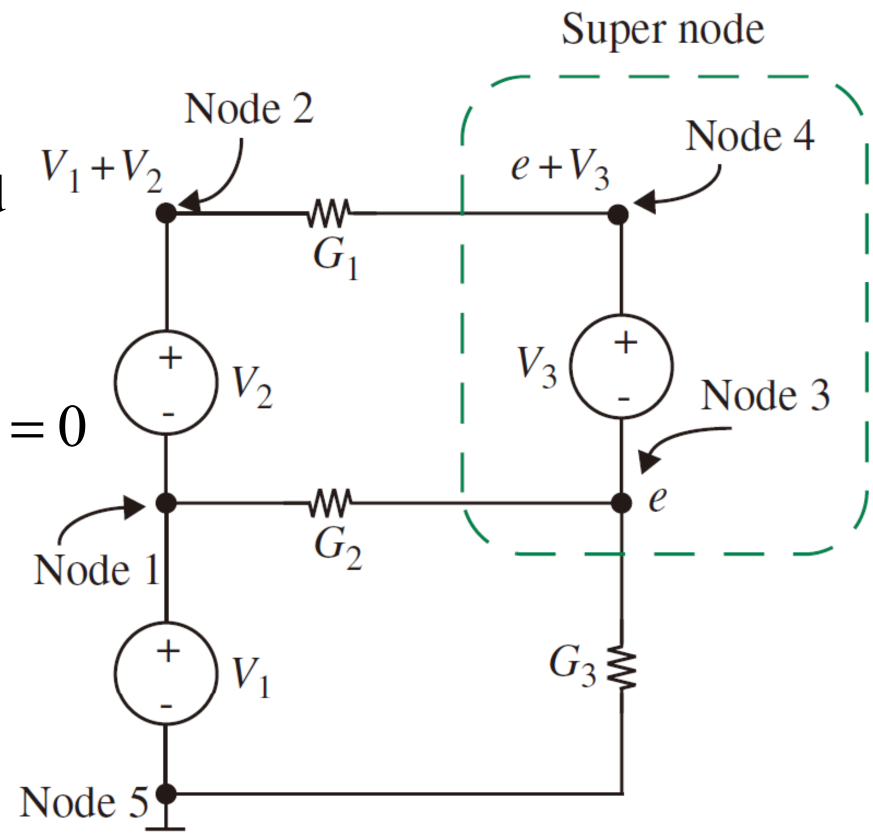
- A floating independent voltage source is a voltage source that has neither terminal connected to ground, neither directly nor through one or more other independent voltage sources.
- It is not possible to complete Step 3 of node analysis since i_5 is not known.
- To derive the desired statement of KCL, we draw a surface around both nodes. Nodes 1 and 2 form a super node.
- KCL for node 1 $i_1 + i_2 + i_5 = 0$
- KCL for node 2 $i_3 + i_4 - i_5 = 0$
- KCL for super node $i_1 + i_2 + i_3 + i_4 = 0$



Example

- In this circuit, the voltage source having value V_3 is the only floating independent voltage source.
- Nodes 3 and 4 form a super node.
- Node 3 is labeled with the unknown node voltage e, and so Node 4 is labeled with the node voltage $e + V_3$.

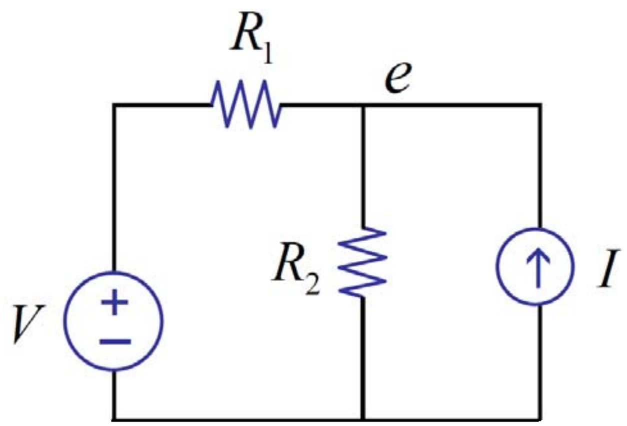


$$G_1[e+V_3-(V_1+V_2)]+G_2(e-V_1)+G_3e=0$$

$$e=\frac{(G_1+G_2)V_1+G_1V_2-G_1V_3}{G_1+G_2+G_3}$$

Linearity

- Consider



- Write down the node equation

$$G_1(e - V) + G_2(e) - I = 0$$

Linear in e, V, I.
No eV, V^2, VI....terms.

- Rearrange

$$(G_1 + G_2)e = G_1 V + I$$

- conductance matrix node voltages = linear sum of sources
  G e S

Linearity

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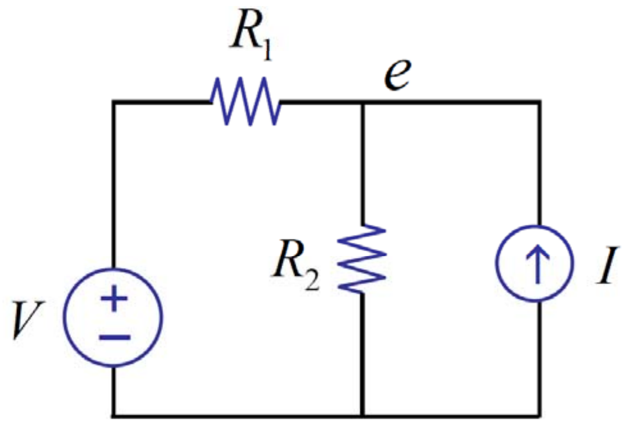
- Consider



- Solve

$$e = \frac{G_1}{G_1 + G_2} V + \frac{1}{G_1 + G_2} I$$

- Or

$$e = \frac{R_2}{R_1 + R_2} V + \frac{R_1 R_2}{R_1 + R_2} I$$

- In general, $e = a_1 V_1 + a_2 V_2 + \ldots\ldots + b_1 I_1 + b_2 I_2 + \ldots\ldots$

Linearity

- Now we can find the branch variables v_0, i_0, v_1, i_1, v_2, i_2, v_3, and i_3.

$$v_1 = V - e = \frac{G_2}{G_1+G_2}V - \frac{1}{G_1+G_2}I$$

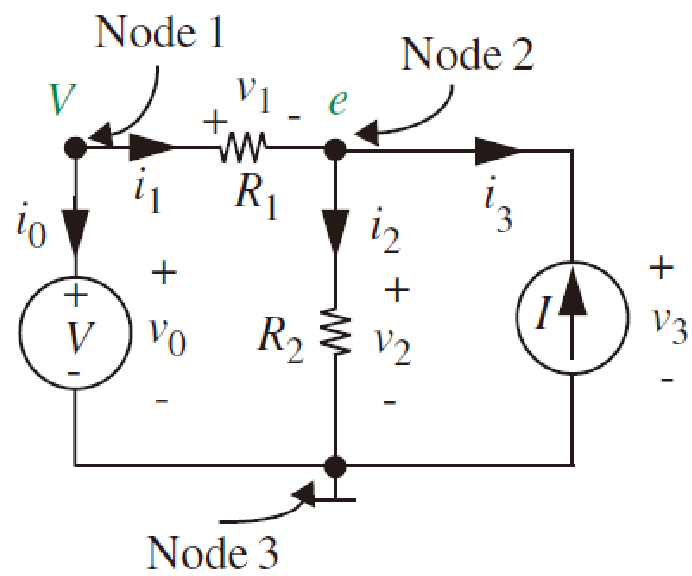
$$i_1 = (V-e)G_1 = \frac{G_1G_2}{G_1+G_2}V - \frac{G_1}{G_1+G_2}I$$

$$v_0 = V \quad i_0 = -i_1$$

$$v_2 = v_3 = e = \frac{G_1}{G_1+G_2}V + \frac{1}{G_1+G_2}I$$

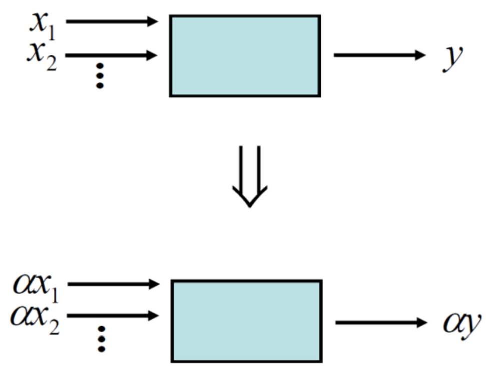
$$i_2 = eG_2 = \frac{G_1G_2}{G_1+G_2}V + \frac{G_2}{G_1+G_2}I$$

$$i_3 = -I$$

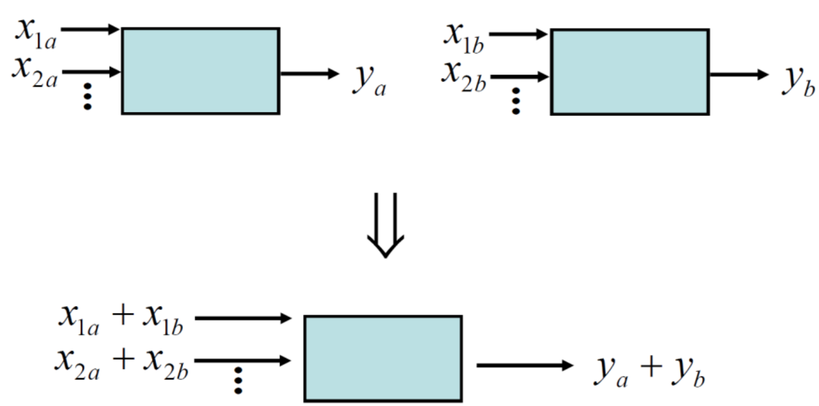


Linearity

- ***Homogeneity*** and ***Superposition***
- Homogeneity

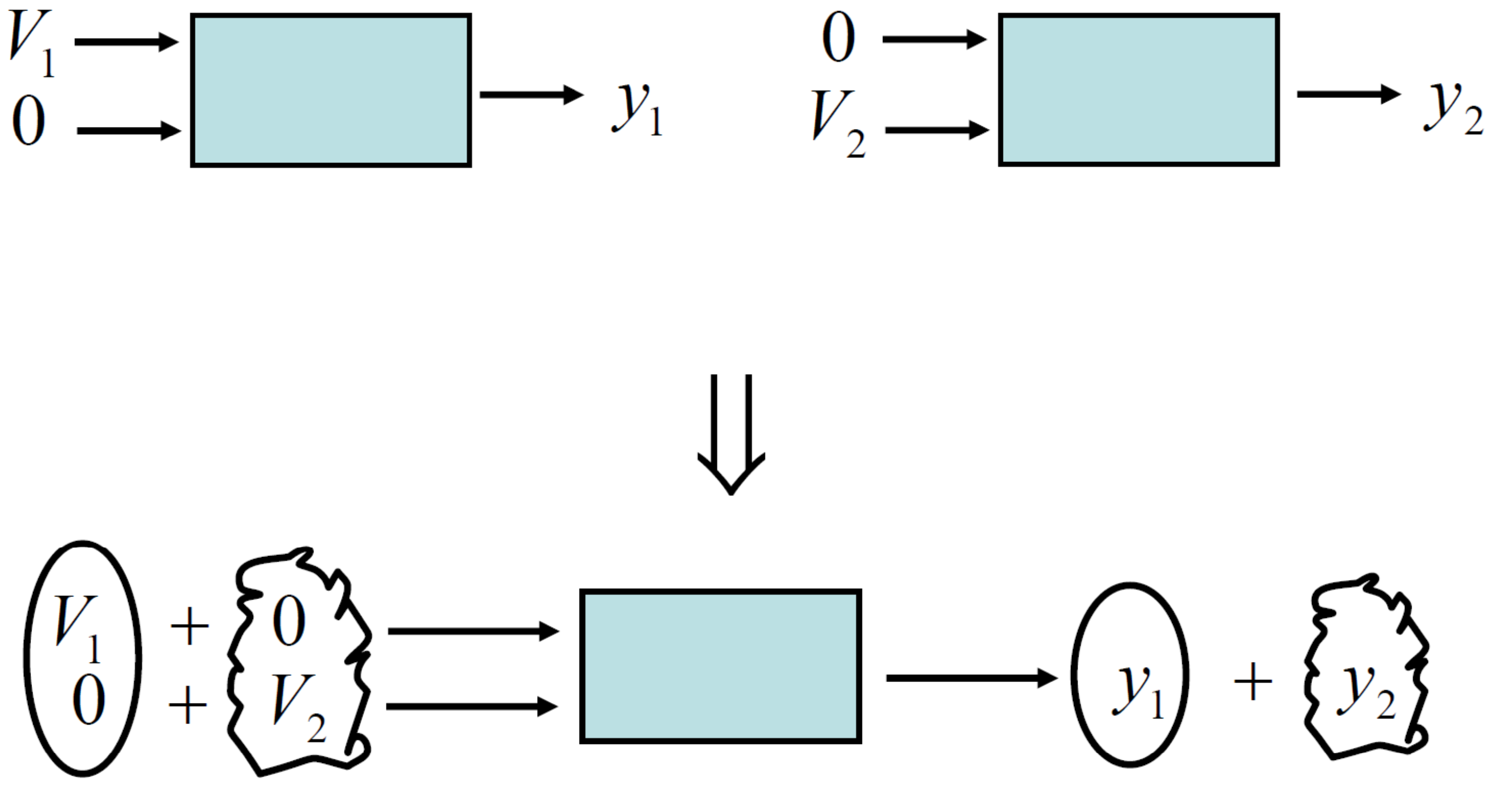


- Superposition



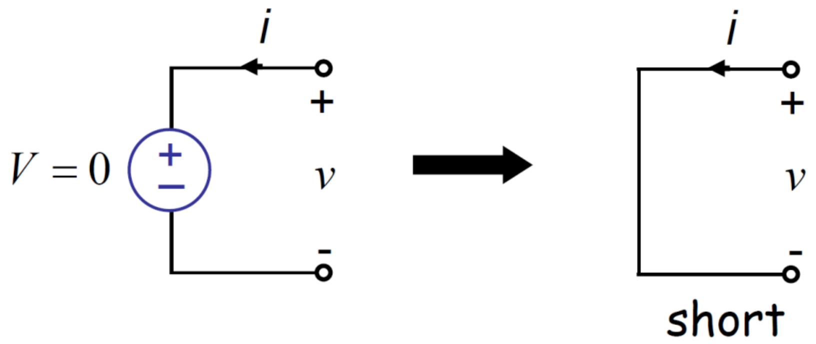
Linearity

- Specific superposition example:

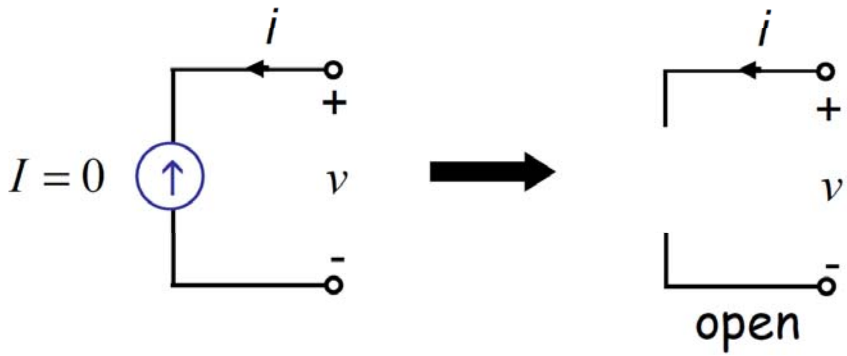


Method 4—Superposition method

- The output of a circuit is determined by summing the responses to each source acting alone.
- To set $V = 0$, replacing Independent voltage source by a short circuit

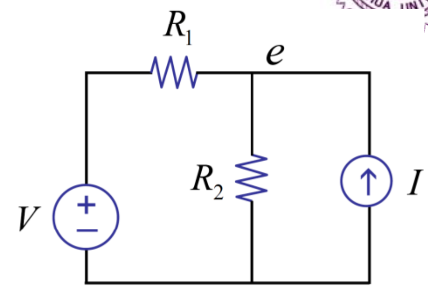


- To set $I = 0$, replacing Independent voltage source by an open circuit

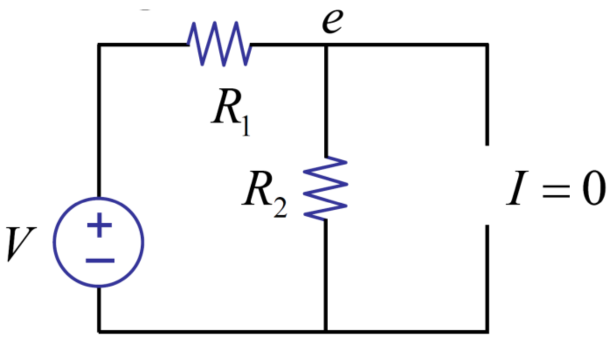


Example

- To find node voltage e of the circuit

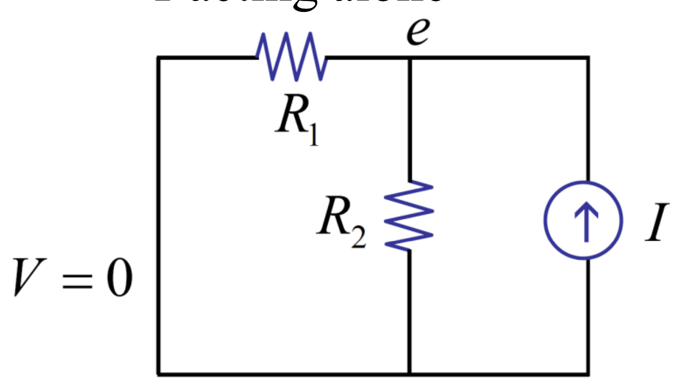


- V acting alone



$$e_V = \frac{R_2}{R_1 + R_2} V$$

I acting alone

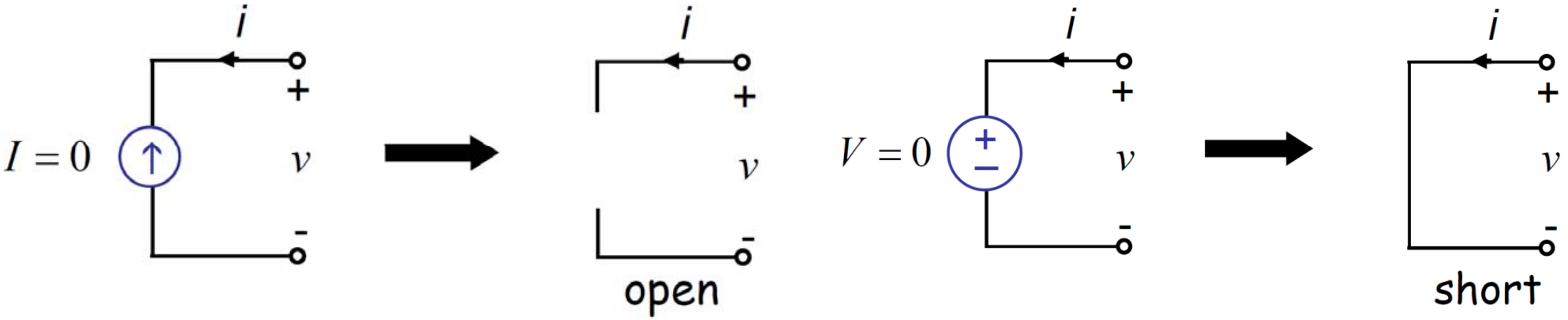


$$e_I = \frac{R_1 R_2}{R_1 + R_2} I$$

- Sum or Superposition $\Rightarrow$ $$e = e_V + e_I = \frac{R_2}{R_1 + R_2} V + \frac{R_1 R_2}{R_1 + R_2} I$$

Superposition method

- For each independent source, form a subcircuit with all other independent sources set to zero. Setting a voltage source to zero implies replacing the voltage source with a short circuit, and setting a current source to zero implies replacing the current source with an open circuit.
- From each subcircuit corresponding to a given independent source, find the response to that independent source acting alone. This step results in a set of individual responses.
- Obtain the total response by summing together each of the individual responses.



Example

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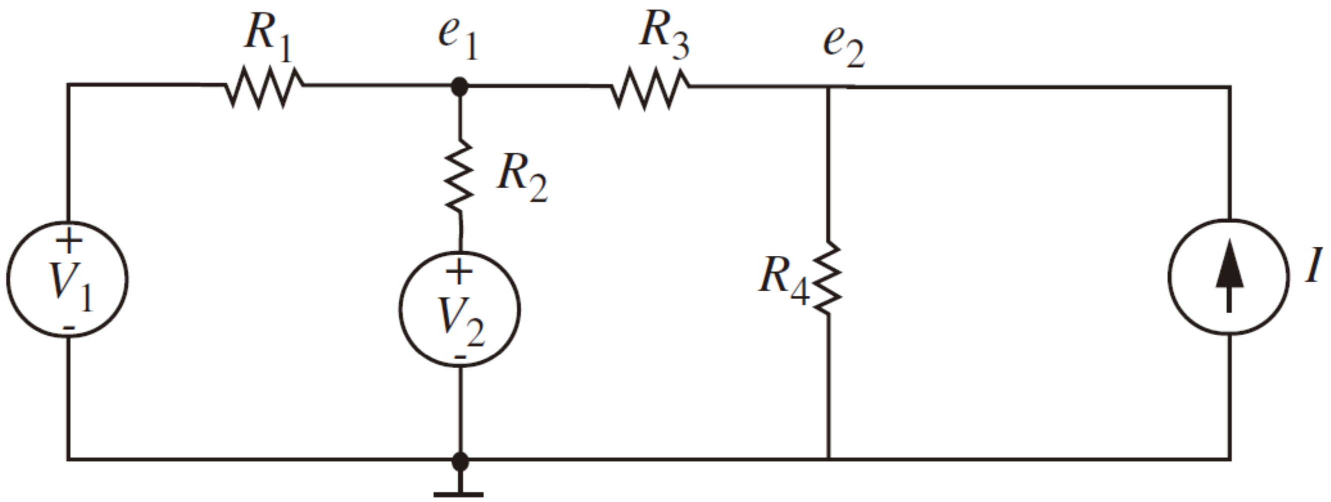
- For the circuit as shown below,



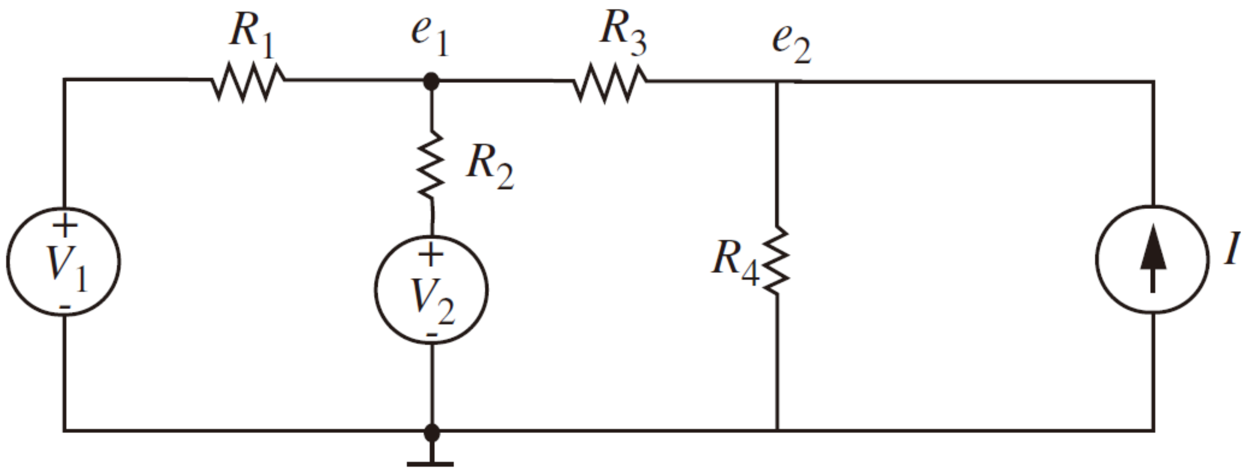
- Node voltages, e_1 and e_2, can be found from the Node analysis.

$$e_1 = \frac{(G_3 + G_4)G_1V_1 + (G_3 + G_4)G_2V_2 + G_3I}{(G_1 + G_2)(G_3 + G_4) + G_3G_4}$$

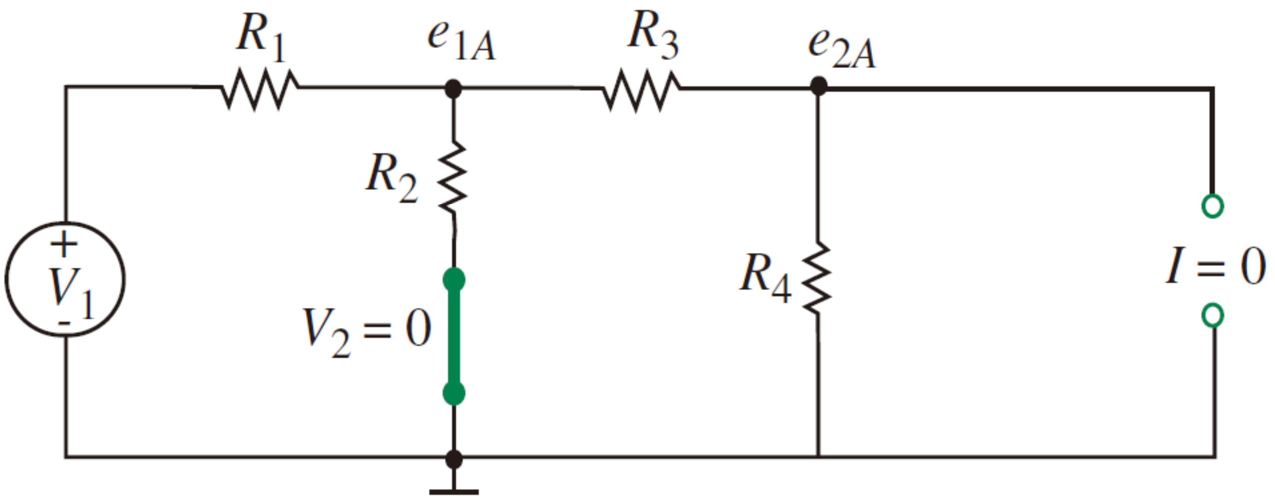
$$e_2 = \frac{G_3G_1V_1 + G_2V_2 + (G_1 + G_2 + G_3)I}{(G_1 + G_2)(G_3 + G_4) + G_3G_4}$$

Example (cont.)

- Let's us find e_1 by superposition,



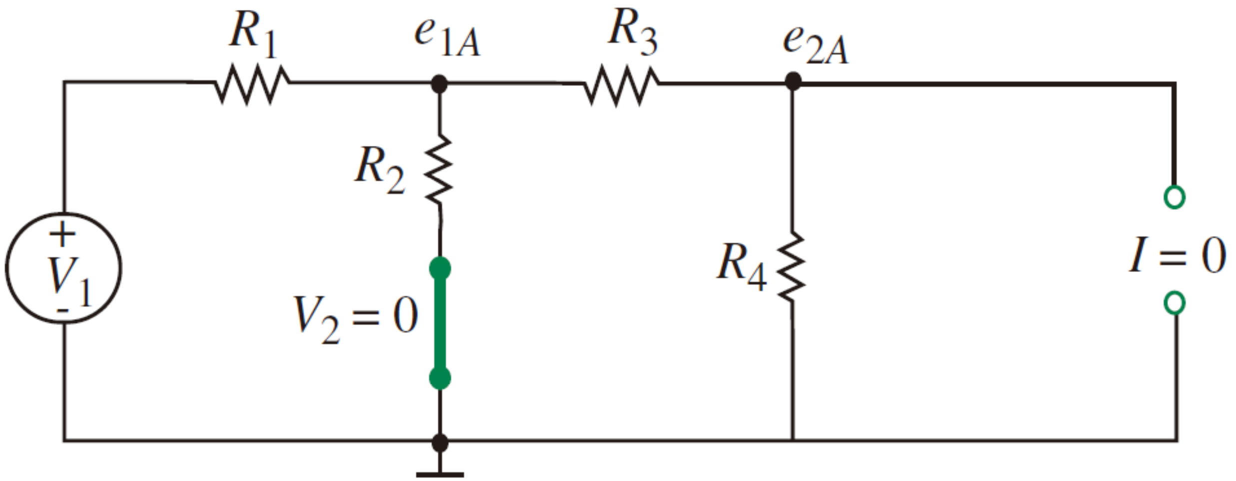
- Set V_2 and I to zero to find the voltage component e_{1A} of e_1 due to source V_1 acting alone.



Example (cont.)

- The response e_{1A} to voltage source V_1 can be found from the following circuit as:

$$e_{1A} = \frac{G_1}{G_1 + [G_2 + G_3 G_4 / (G_3 + G_4)]} V_1$$



Example (cont.)

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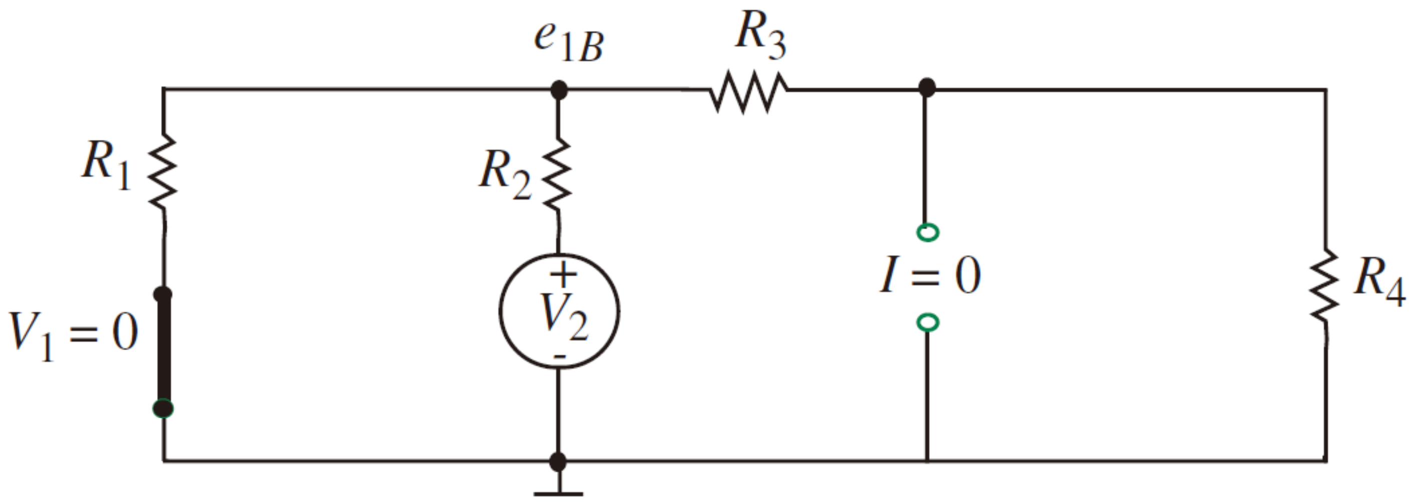
- The response e_{1B} to voltage source V_2 can be found from the following circuit as:

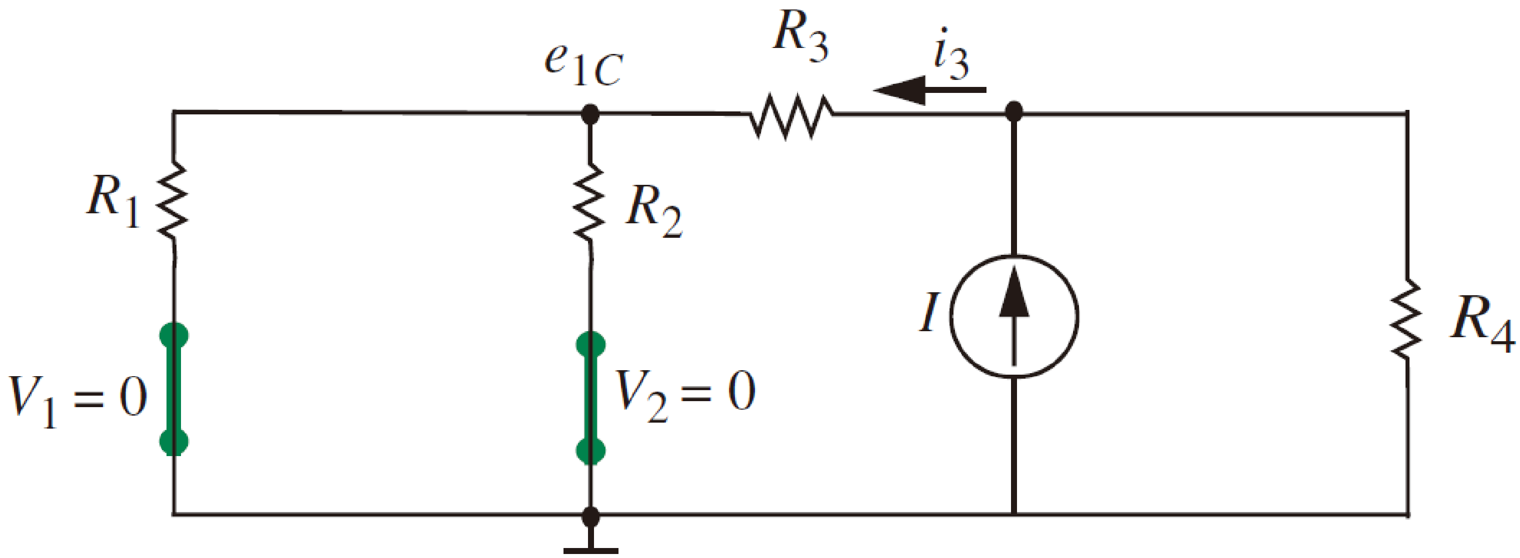
$$e_{1B} = \frac{G_2}{G_2 + [G_1 + G_3 G_4 / (G_3 + G_4)]} V_1$$



Example (cont.)

- The response e_{1C} to current source I can be found from the following circuit as:

$$e_{1C} = \frac{G_3}{(G_1 + G_2)(G_3 + G_4) + G_3 G_4} I$$

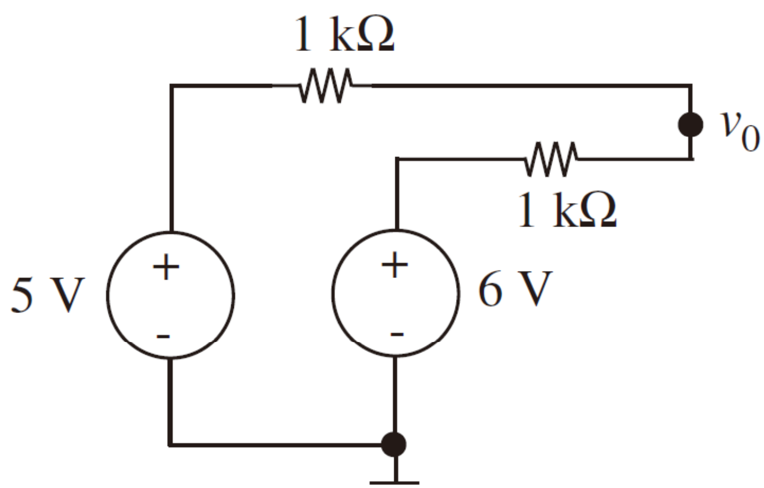


- The total response e_1 by summing together each of the individual responses as:

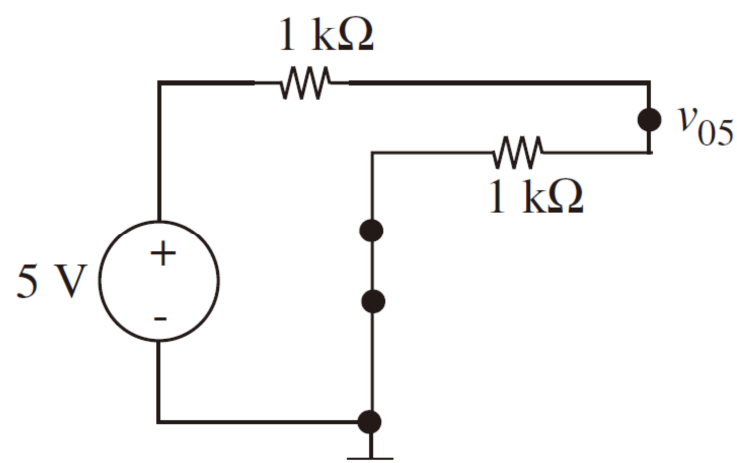
$$e_1 = e_{1A} + e_{1B} + e_{1C}$$

Averaging Circuit

- Find v_0 for the circuit as shown below:



- Circuit with 5-V source acting alone.



$$v_{05} = \frac{1\,\mathrm{k\Omega}}{1\,\mathrm{k\Omega} + 1\,\mathrm{k\Omega}} \times 5\ \mathrm{V} = \frac{5}{2}\ \mathrm{V}$$

Averaging Circuit (cont.)

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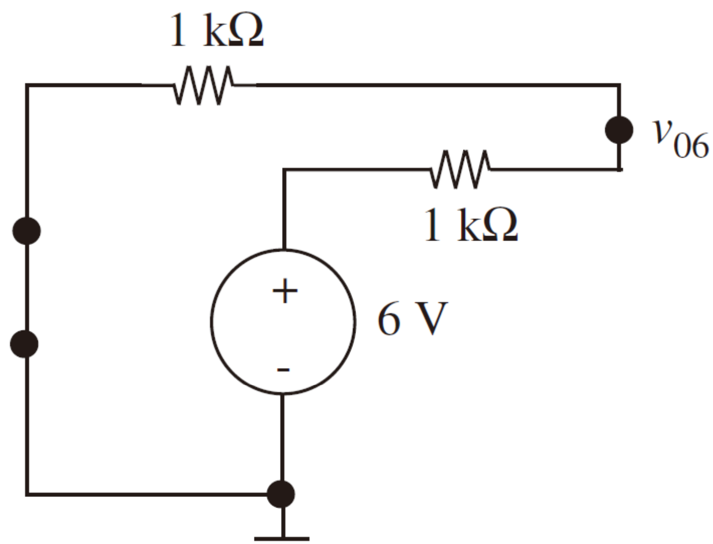
- Circuit with 6-V source acting alone .
- v_{06}, the response of the 6-V source acting alone, become.



$$v_{06} = \frac{1\,\text{k}\Omega}{1\,\text{k}\Omega + 1\,\text{k}\Omega} \times 6\text{ V} = \frac{6}{2}\text{ V}$$

- And v_0 is sum the two partial responses.

$$v_0 = v_{05} + v_{06} = \frac{5\text{ V} + 6\text{ V}}{2} = \frac{11}{2}\text{ V}$$

- Note that v_0 is the average of the two input voltages.

The Thévenin method

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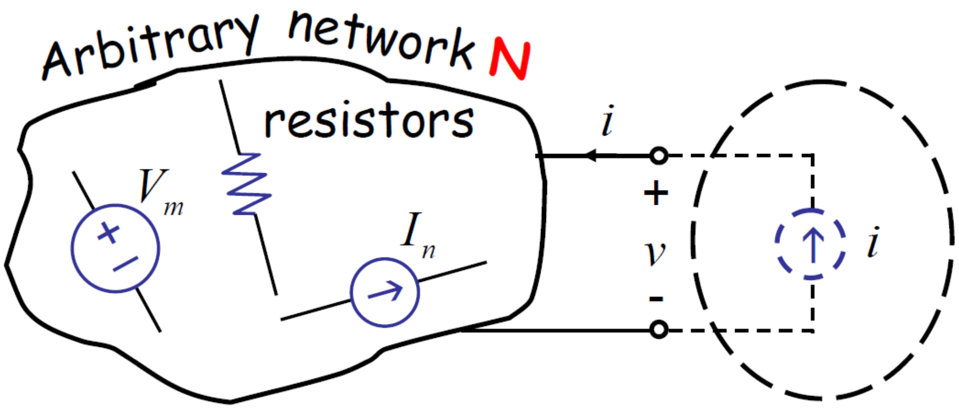
- Consider:



- Let's us choose to apply a test current source to the terminals. To find the response *v* by superposition.
- By superposition

$$v = \sum_m \alpha_m V_m + \sum_n \beta_n I_n + R_{TH} i$$

- The first two terms is independent of external excitation and behaves like a voltage source v_{TH} .
- The coefficient of the last term is independent of external excitement i and behaves like a resistor R_{TH}.

The Thévenin method

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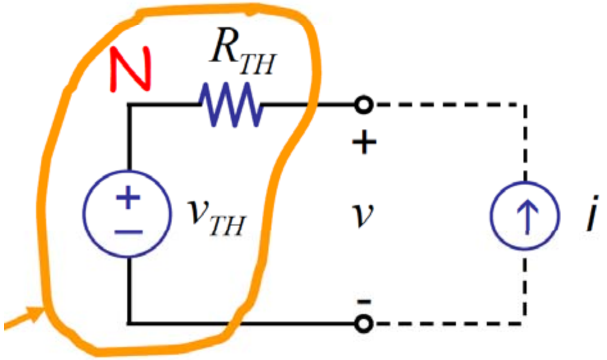
- Or $v = v_{TH} + R_{TH} i$

- As far as the external world is concerned (for the purpose of I-V relation),"Arbitrary network N"is indistinguishable from:



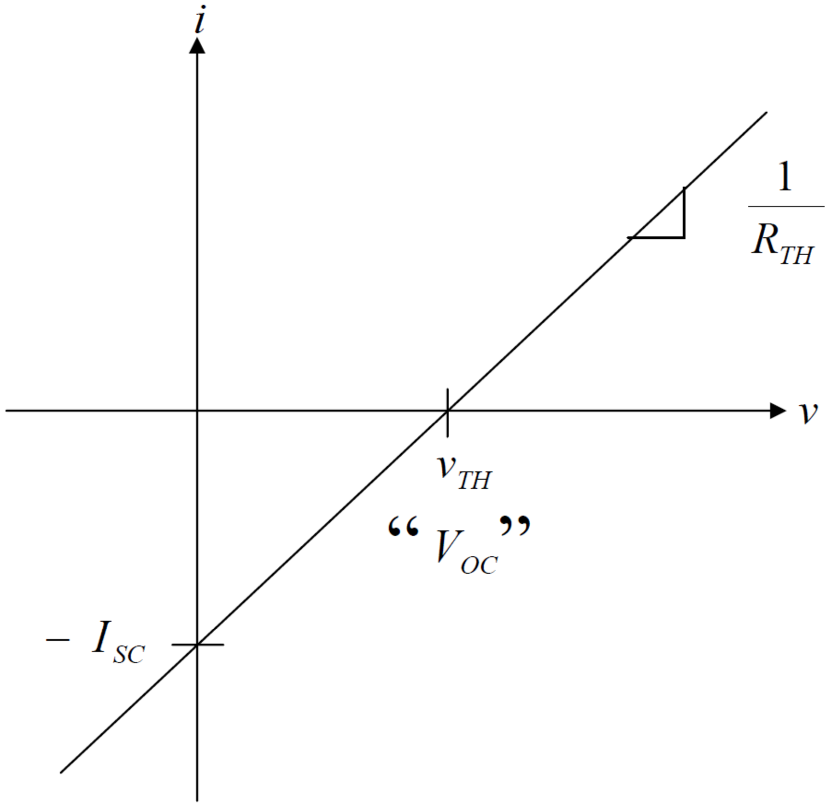
Thévenin equivalent network

- Open circuit voltage at port: v_{TH}

- Resistance of network seen from port: R_{TH}

Graphical Expression

- Graphically for $v = v_{TH} + R_{TH} i$



- Open circuit ($i \equiv 0$)

$$v = v_{TH} = V_{OC}$$

- Short circuit ($v \equiv 0$)

$$i = -\frac{v_{TH}}{R_{TH}} = -I_{SC}$$

Method 5:The Thévenin Method

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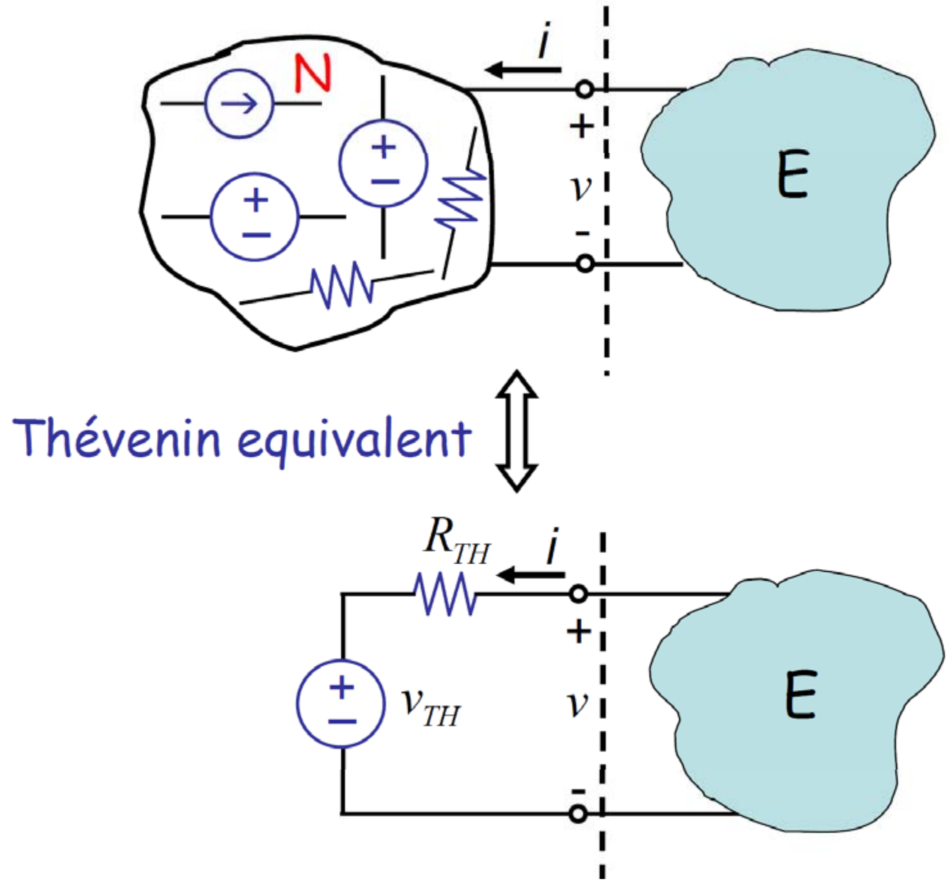
- Developed at 1883 by M. Leon Thévenin (1857–1926), a French telegraph engineer.
- Replace network N with its Thévenin equivalent, then solve external network E



A Method for Determining the Thévenin Equivalent Circuit

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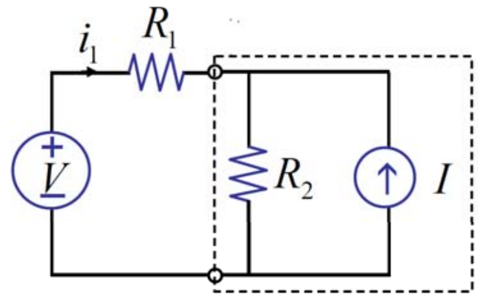
- The Thévenin equivalent circuit for any linear network at a given pair of terminals consists of a voltage source v_{TH} in series with a resistor R_{TH}. The voltage v_{TH} and resistance R_{TH} can be obtained as follows:

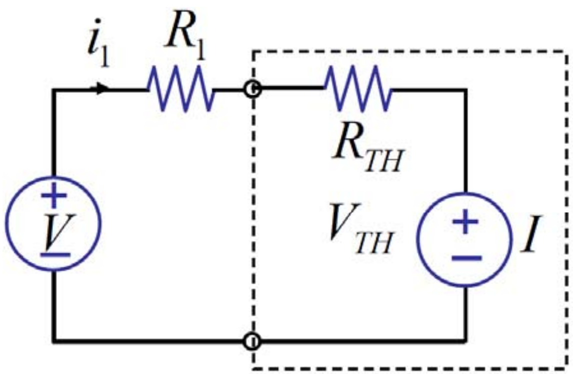
1. v_{TH} can be found by calculating or measuring the open-circuit voltage at the designated terminal pair on the original network.
2. R_{TH} can be found by calculating or measuring the resistance of the open-circuit network seen from the designated terminal pair with all independent sources internal to the network set to zero. That is, with independent voltage sources replaced with short circuits, and independent current sources replaced with open circuits. (Dependent sources must be left intact, however.)

Example

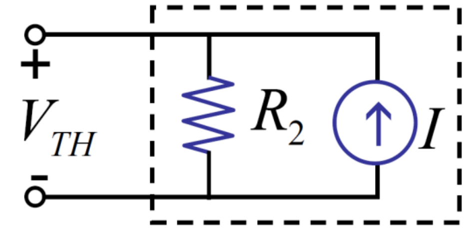
- To find current i_1 of the circuit



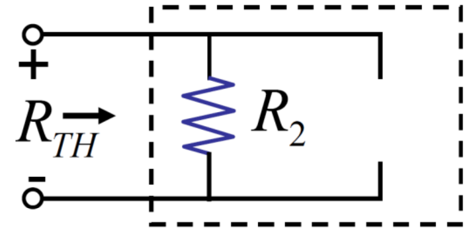
- Replacing network N with its Thévenin equivalent



$$V_{TH} = IR_2$$



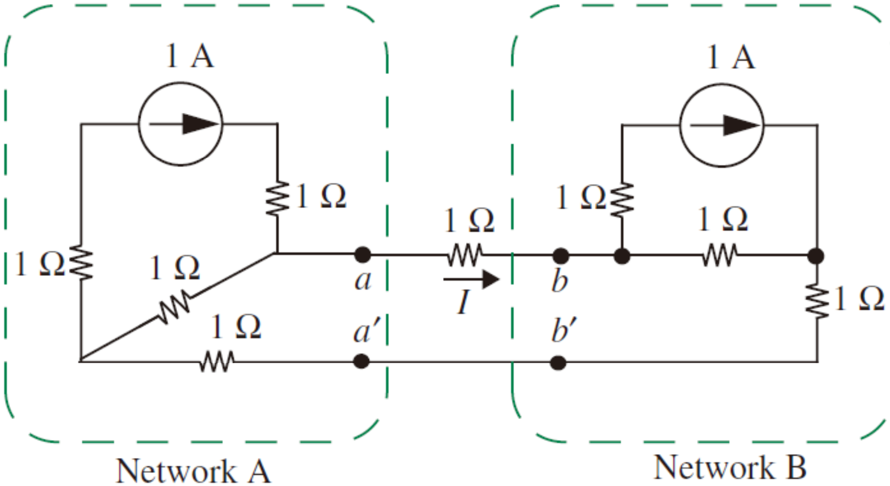
$$R_{TH} = R_2$$



- Current i_1 is $i_1 = \dfrac{V - V_{TH}}{R_1 + R_{TH}} = \dfrac{V - IR_2}{R_1 + R_2}$

Example

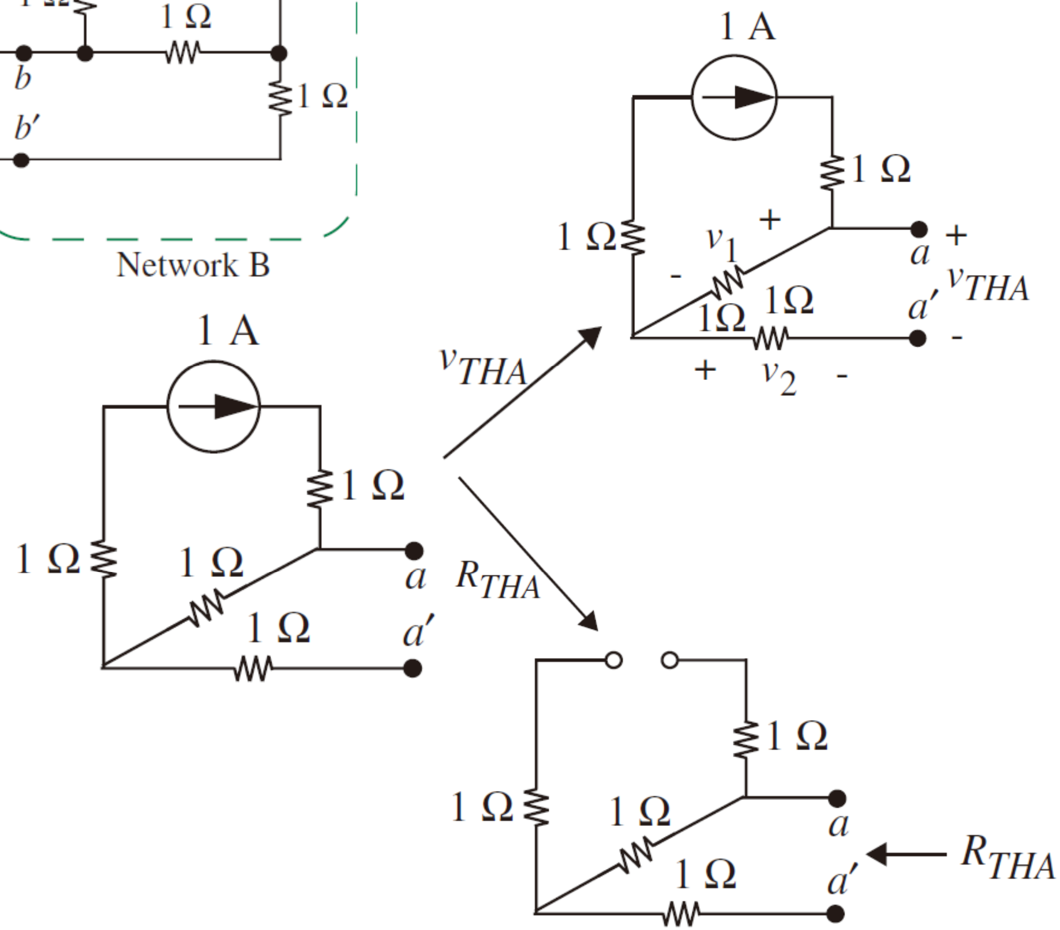
- To find current *I* of the circuit



- Replacing network A with its Thévenin equivalent

$$V_{THA} = 1\ \text{V}$$

$$R_{THA} = 2\ \text{Ohm}$$

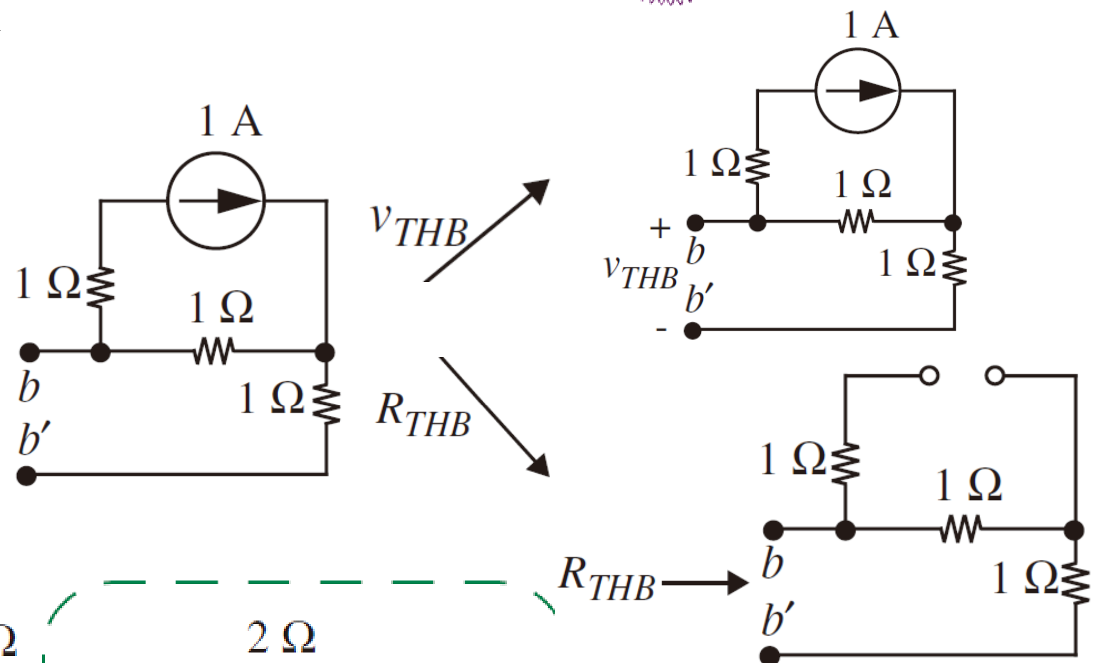


Example (cont.)

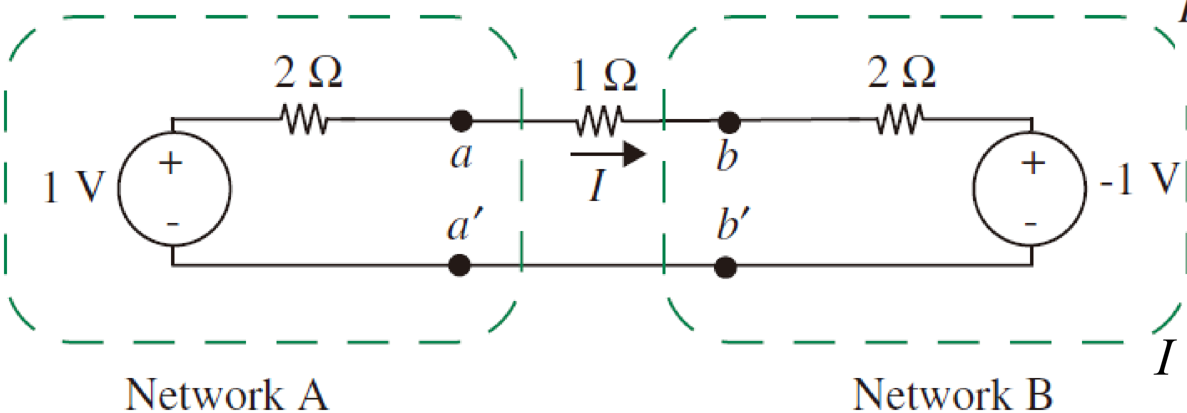
- Replacing network B with its Thévenin equivalent

$$V_{THB} = -1\text{ V}$$

$$R_{THB} = 2\text{ Ohm}$$



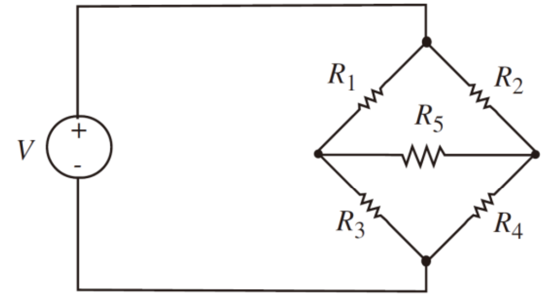
- The circuit become:



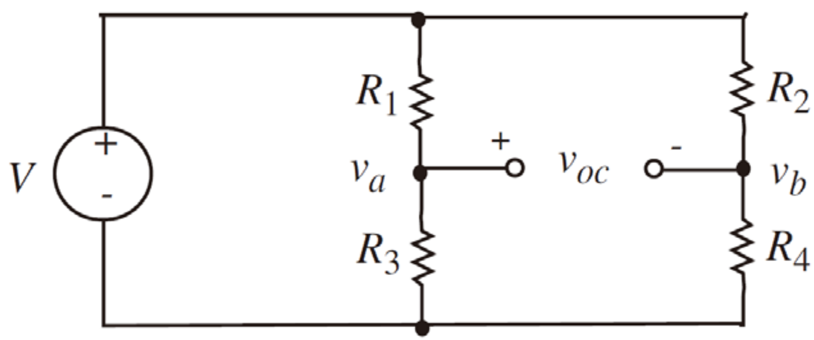
$$I = \frac{1\text{ V} - (-1\text{ V})}{2\ \Omega + 1\ \Omega + 2\Omega} = \frac{2}{5}\text{ A}$$

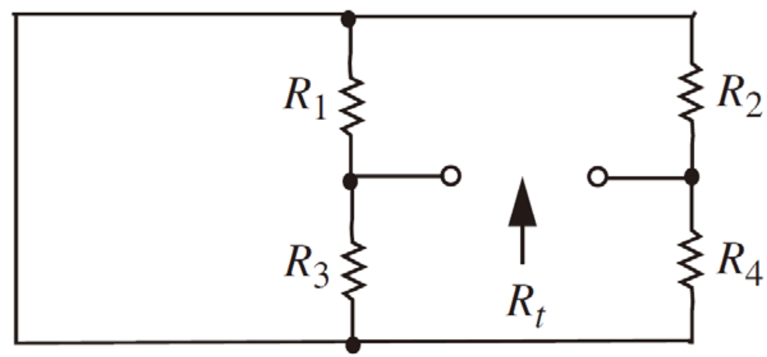
More Example

- Find condition that voltage across R_5 is zero.



- Replacing network N with its Thévenin equivalent



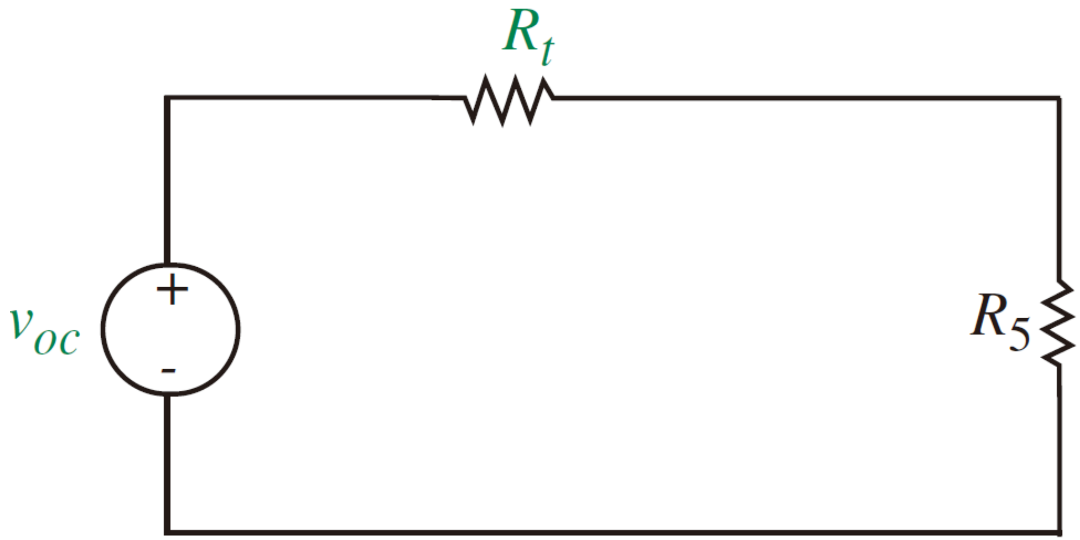


$$v_{OC} = v_a - v_b = V\left(\frac{R_3}{R_1 + R_3} - \frac{R_4}{R_2 + R_4}\right)$$

$$R_t = R_1 \| R_3 + R_2 \| R_4$$

More Example (cont.)

- The circuit is equivalent to:

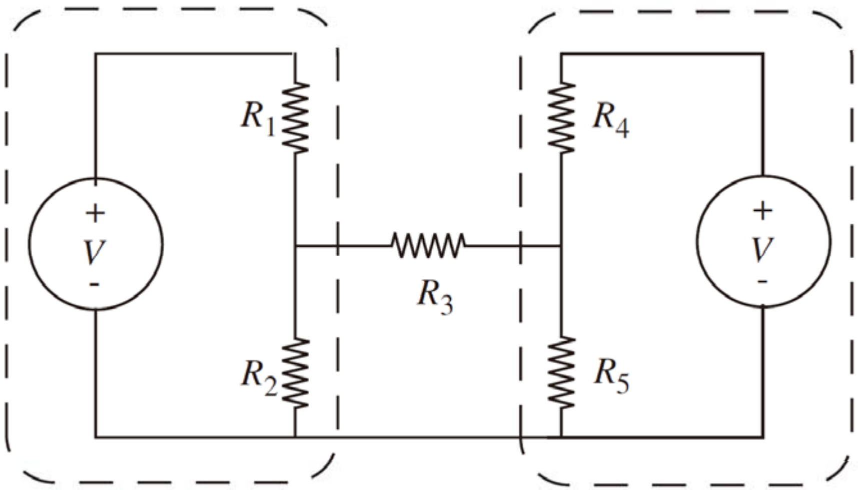


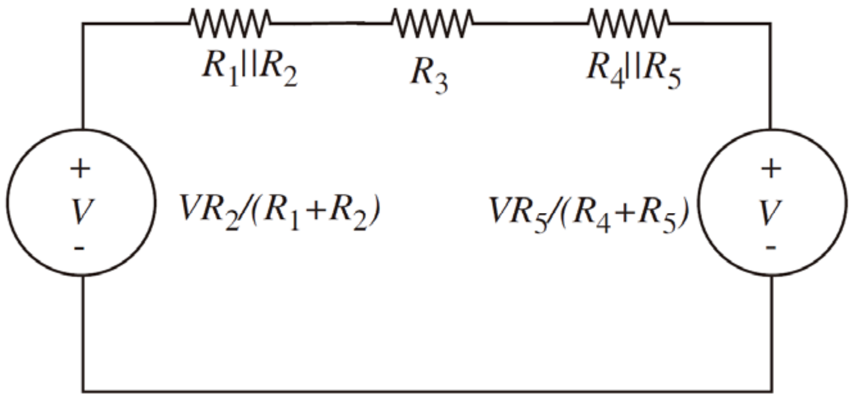
- The condition that voltage across R_5 is zero is:

$$v_{OC} = V\left(\frac{R_3}{R_1+R_3} - \frac{R_4}{R_2+R_4}\right) = 0 \quad \text{or} \quad \frac{R_3}{R_1+R_3} = \frac{R_4}{R_2+R_4} \quad \text{or} \quad \frac{R_3}{R_1} = \frac{R_4}{R_2}$$

Another way

- The circuit is equivalent to:





- The condition that voltage across R_3 is zero is:

$$v_{TH\,1} = V\frac{R_2}{R_1+R_2} = v_{TH\,2} = V\frac{R_5}{R_4+R_5} \text{ or } \frac{R_2}{R_1+R_2} = \frac{R_5}{R_4+R_5} \text{ or } \frac{R_2}{R_1} = \frac{R_5}{R_4}$$

The Norton Method

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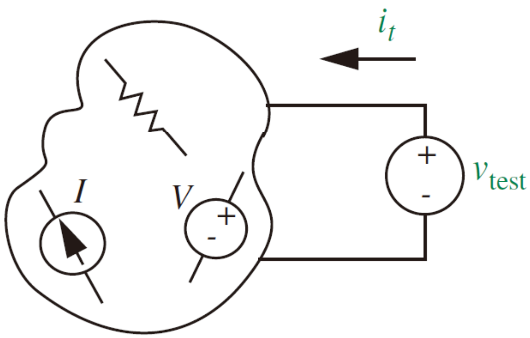
- Consider:



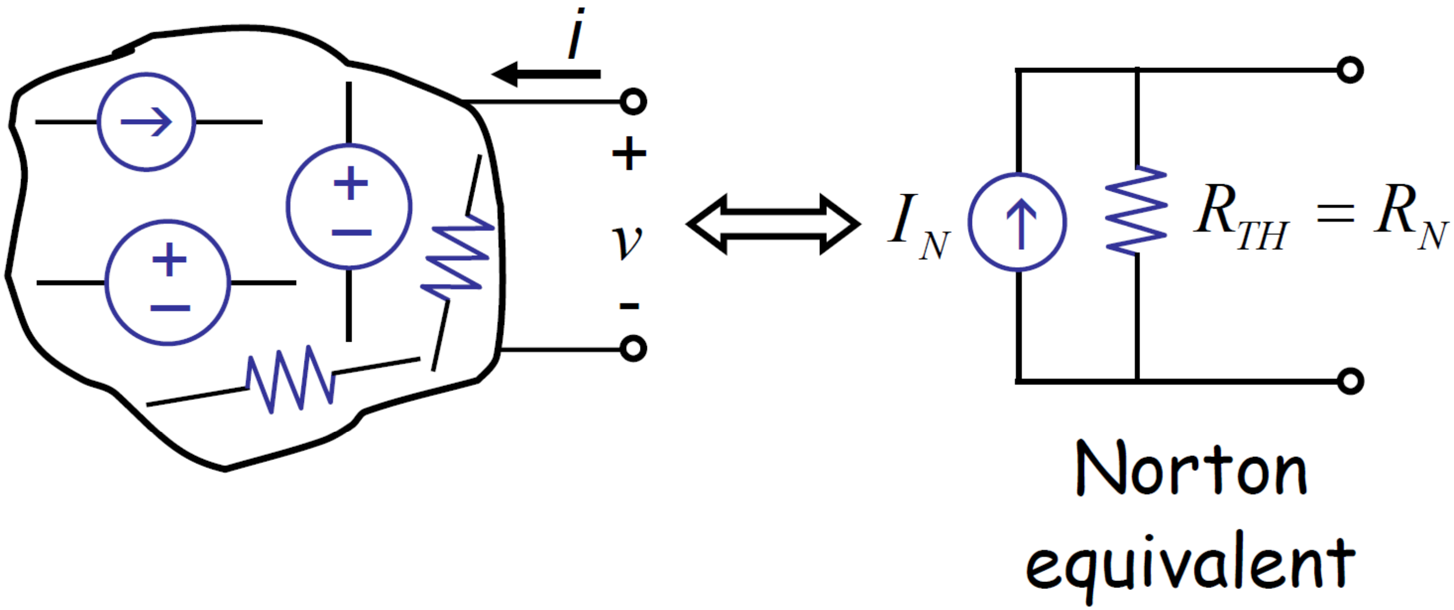
- This time let's us choose to apply a test voltage source to the terminals. To find the response *i* by superposition.
- By superposition

$$i = \sum_m \alpha_m V_m + \sum_n \beta_n I_n + \frac{v}{R_N}$$

- The first two terms is independent of external excitation and behaves like a voltage source i_N .
- The coefficient of the last term is independent of external excitement i and behaves like a resistor R_N .

The Norton Method

- Developed at 1926 by E. L. Norton, an American engineer at Bell Telephone Laboratory.
- Replace network N with its Norton equivalent.



The Norton method

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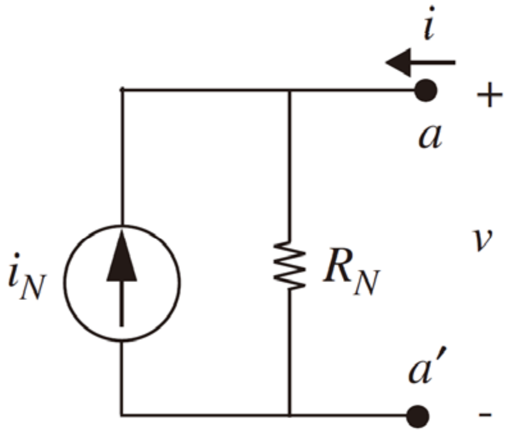
- Or $$i = i_N + \frac{v}{R_N}$$

- As far as the external world is concerned (for the purpose of *i*-*v* relation),"Arbitrary network N"is indistinguishable from:

Norton equivalent network

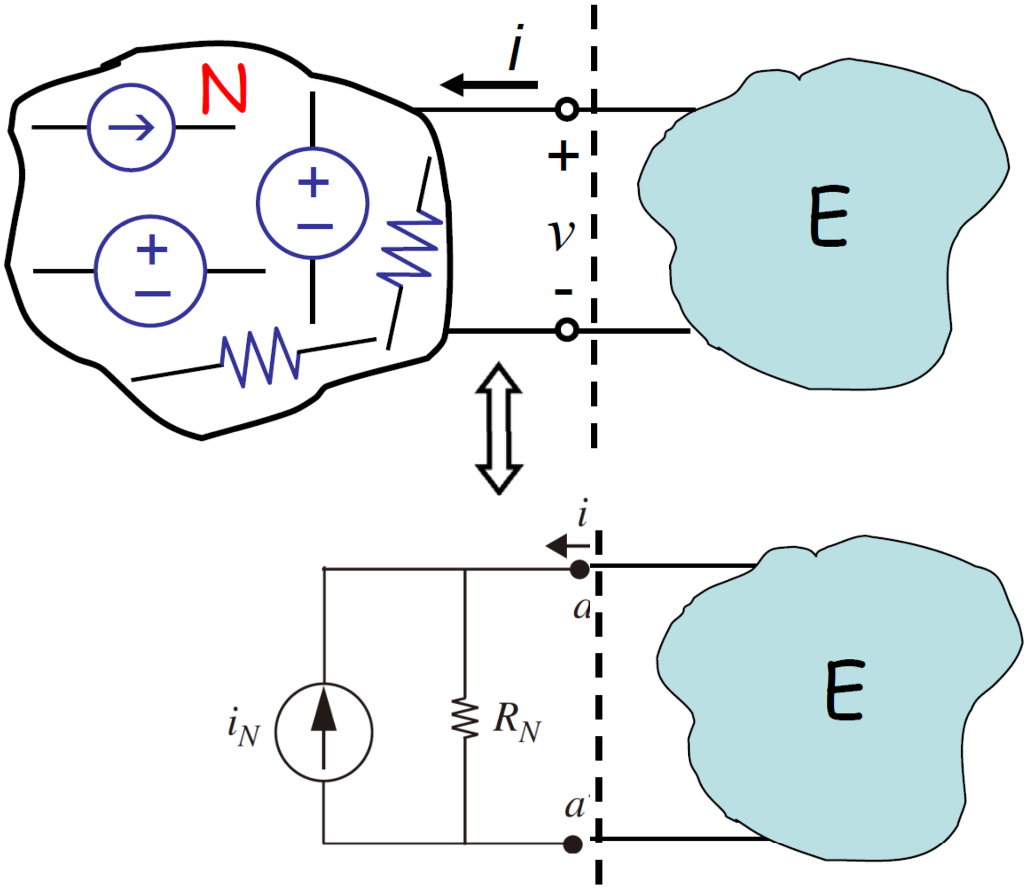


- Short circuit current at port: i_N

- Resistance of network seen from port: R_N

Method 6:The Norton Method

- Replace network N with its Norton equivalent, then solve external network E.



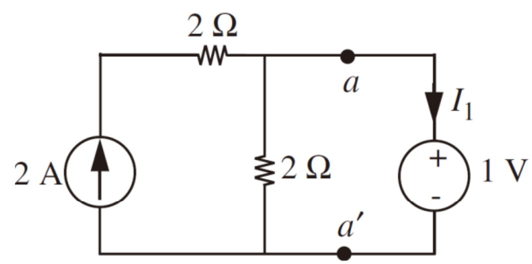
A Method for Determining the Norton Equivalent Circuit

- The Norton equivalentcircuit for any linear network at a given pair of terminals consists of a current source i_N in parallel with a resistor R_N. The current i_N and resistance R_N can be obtained as follows:

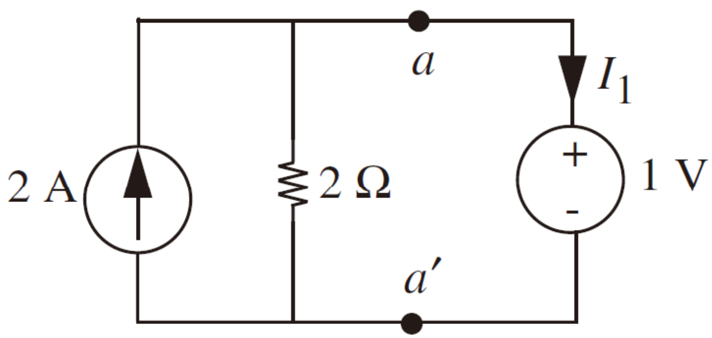
1. i_N can be found by applying a short at the designated terminal pair on the original network and calculating or measuring the current through the short circuit.
2. R_N can be found in the same manner as R_{TH}, that is, by calculating or measuring the resistance of the open-circuit network seen from the designated terminal pair with all independent sources internal to the network set to zero; that is, with voltage sources replaced with short circuits, and current sources replaced with open circuits. (Dependent sources must be left intact, however.)

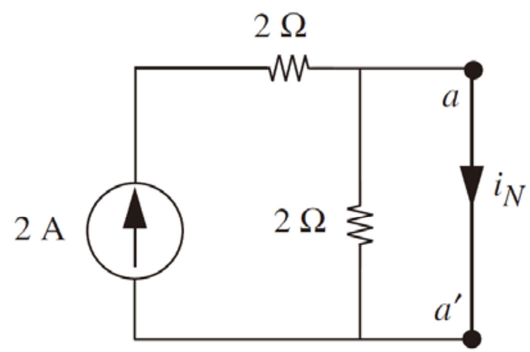
Example

- To find current I_1 of the circuit



- Replacing network left of aa′ with its Norton equivalent



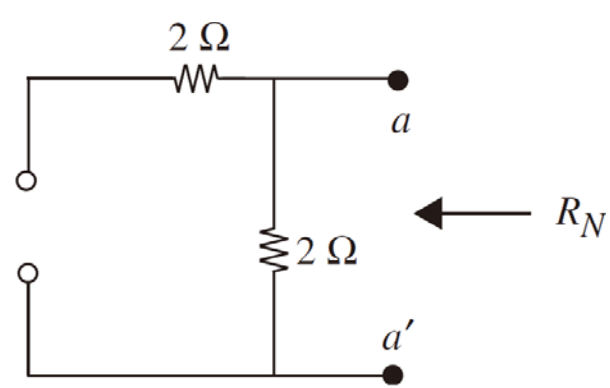


$$i_N = 2\ \text{A}$$

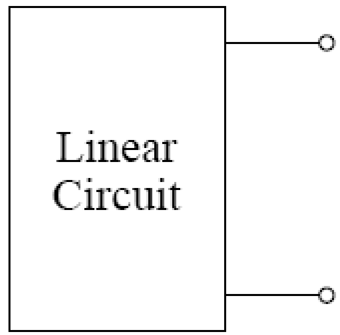
$$R_N = 2\ \Omega$$

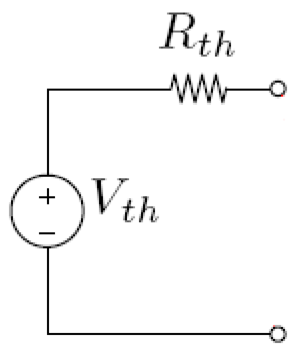
- Current I_1 is

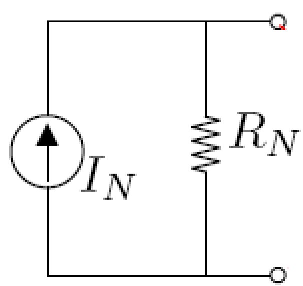
$$I_1 + \frac{1\ \text{V}}{2\ \Omega} - 2\ \text{A} = 0 \Rightarrow I_1 = 1.5\ \text{A}$$



Thévenin′s and Norton′s







- $R_{th} = R_N$.
- Open circuit voltage:
  - For the Thévenin′s:

    $v_{oc} = V_{TH}$;
  - For the Norton′s:

    $v_{oc} = I_N R_N$;
  - $V_{th} = I_N R_N$.
- Short circuit current:
  - For the Thévenin′s:

    $i_{sc} = V_{TH}/R_{TH}$;
  - For the Norton′s:

    $i_{sc} = I_N$;
  - $I_N = V_{TH}/R_N$.

Maximum Power Transfer

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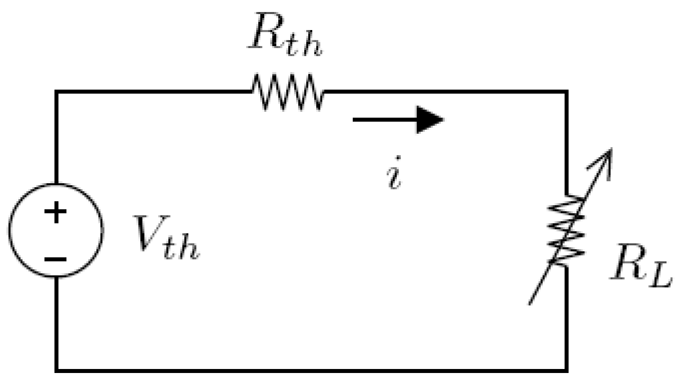

$$i = \frac{V_{TH}}{R_{TH} + R_L} \quad \rightarrow \quad P_L = \left(\frac{V_{TH}}{R_{TH} + R_L}\right)^2 R_L$$

To find the maximum of P_L:

$$\frac{dP_L}{dR_L} = V_{TH}^2 \frac{(R_{TH} - R_L)}{(R_{TH} + R_L)^3} = 0 \quad \rightarrow \quad R_{TH} = R_L$$

And the maximum of P_L is $\frac{V_{TH}^2}{4R_{TH}}$

Conclusions

- Discretize matter

  Physics $\rightarrow$ EE

  Lump Matter Discretization (LMD) → Lump Circuit Abstraction (LCA)

- *R*, *I*, *V* forms ***Linear*** networks
- Analysis methods (linear and nonlinear)
  - KVL, KCL, I-V
  - Combination rules
  - Node method (and Mesh Method)
- Analysis methods (linear)
  - Superposition
  - Thévenin
  - Norton

Mesh Analysis (Loop Analysis)

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Mesh: A loop which does not contain any other loops within it.

1. Assign mesh currents $i_1, i_2, \ldots, i_m$ to the *m* meshes.

2. Apply Kirchhoff's Voltage Law (KVL) to each of the m meshes. Use Ohm's law to express the voltage relationship within each mesh in terms of mesh currents $i_1, i_2, \ldots, i_m$.

3. Solve the *m* simultaneous equations to obtain the mesh currents $i_1, i_2, \ldots, i_m$.

Mesh Analysis

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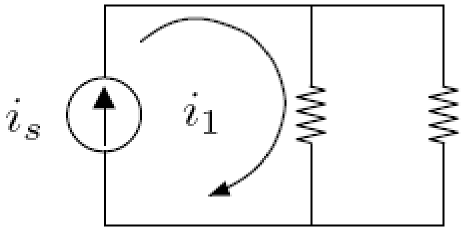
Special Case 1

When a current source exists in one mesh, then the current of this mesh is equal to this current source.

$$i_1 = i_s$$



Mesh Analysis

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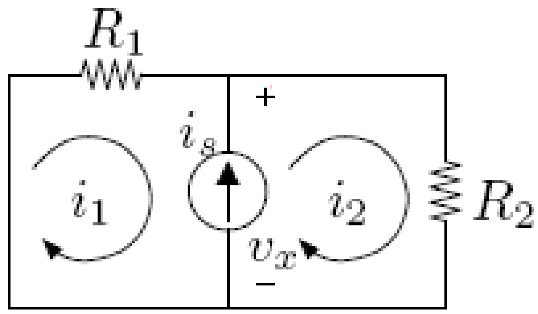
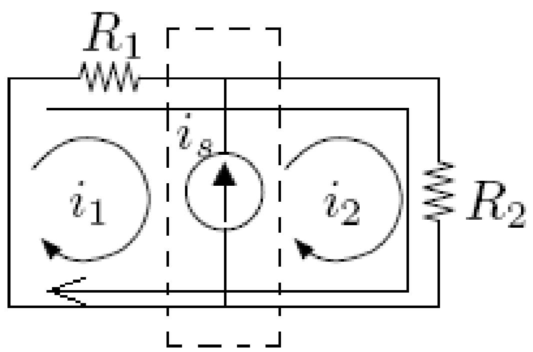
Special Case 2

When a current source exists between two meshes, a supermesh is defined by *excluding the current source and any components in series connection with it.*



$$i_1R_1 + v_x = 0;\ -v_x + i_2R_2 = 0$$
$$\rightarrow i_1R_1 + i_2R_2 = 0$$



$$(-i_1) + i_2 = i_s$$
$$i_1R_1 + i_2R_2 = 0$$

Summary for Node and Mesh Analysis

- For a circuit with b branches and n nodes
  - Node Analysis: $(n - 1)$ equations by applying KCL to each non-reference nodes.
  - Mesh Analysis: $m = b - (n - 1)$ equations by applying KVL to each meshes.