Second Order Circuits

Chenhsin Lien and Po-Tai Cheng
CENTER FOR ADVANCED POWER TECHNOLOGIES
Dept. of Electrical Engineering
National Tsing Hua University
Hsinchu, TAIWAN

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

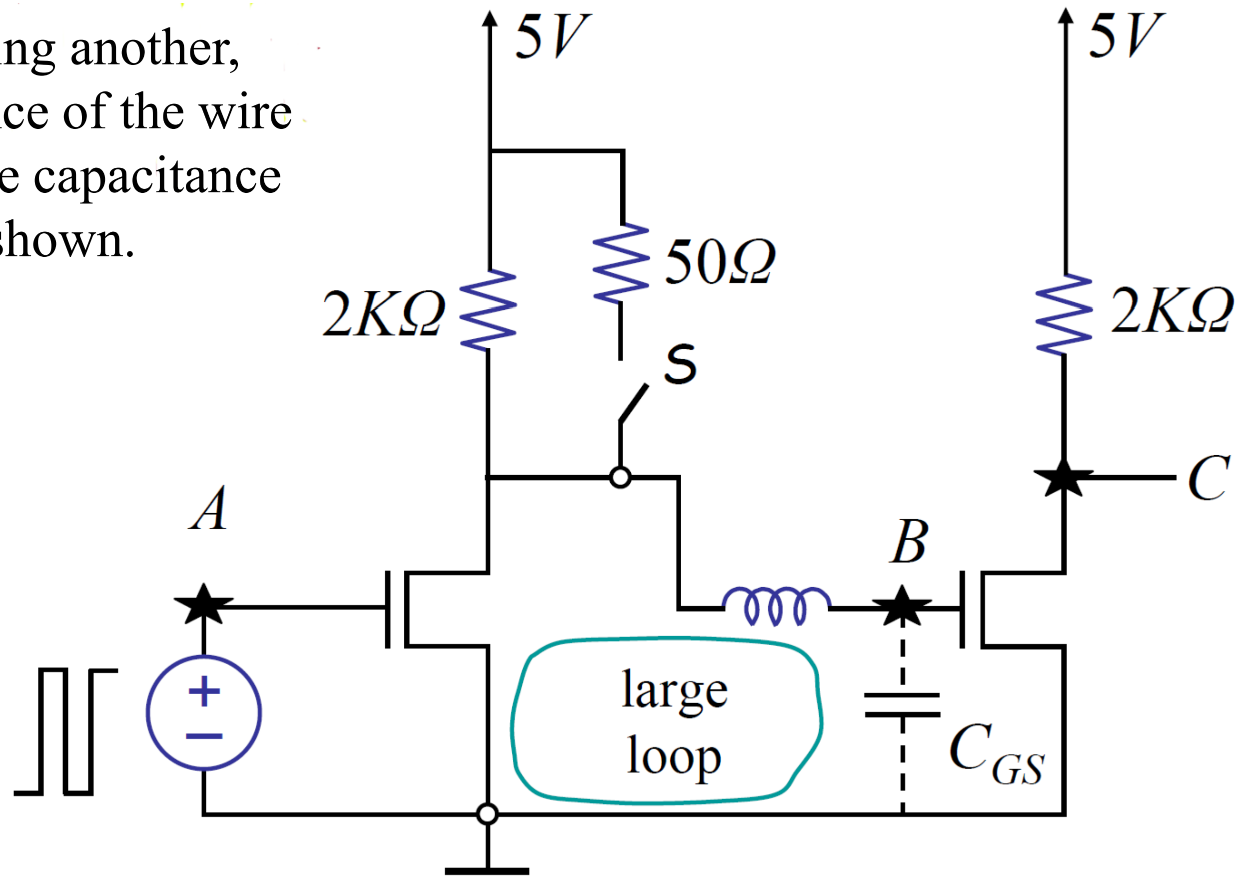
Overview

- Second order circuits can be characterized by circuit contain two independent energy storage elements.
- Second order circuits can be characterized second order differential equations .
- The LC circuit.
- The series $R - L - C$ circuit.
  - Over damped.
  - Critically damped.
  - Under damped.
- The Intuitive Analysis
- Parallel $R - L - C$ circuit.
- The State-variable analysis

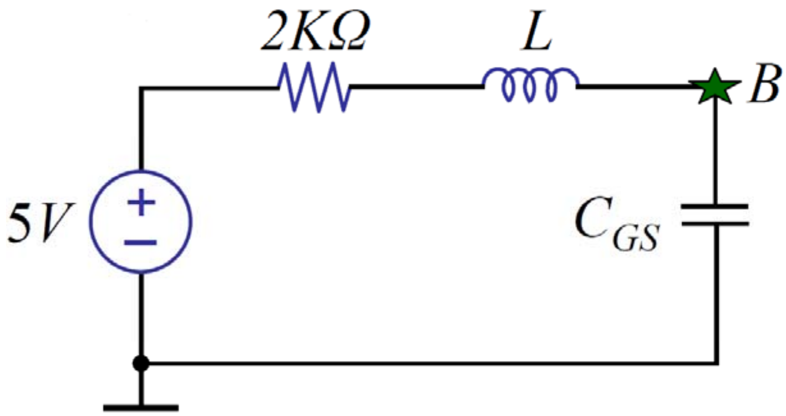
The Inverter Chain

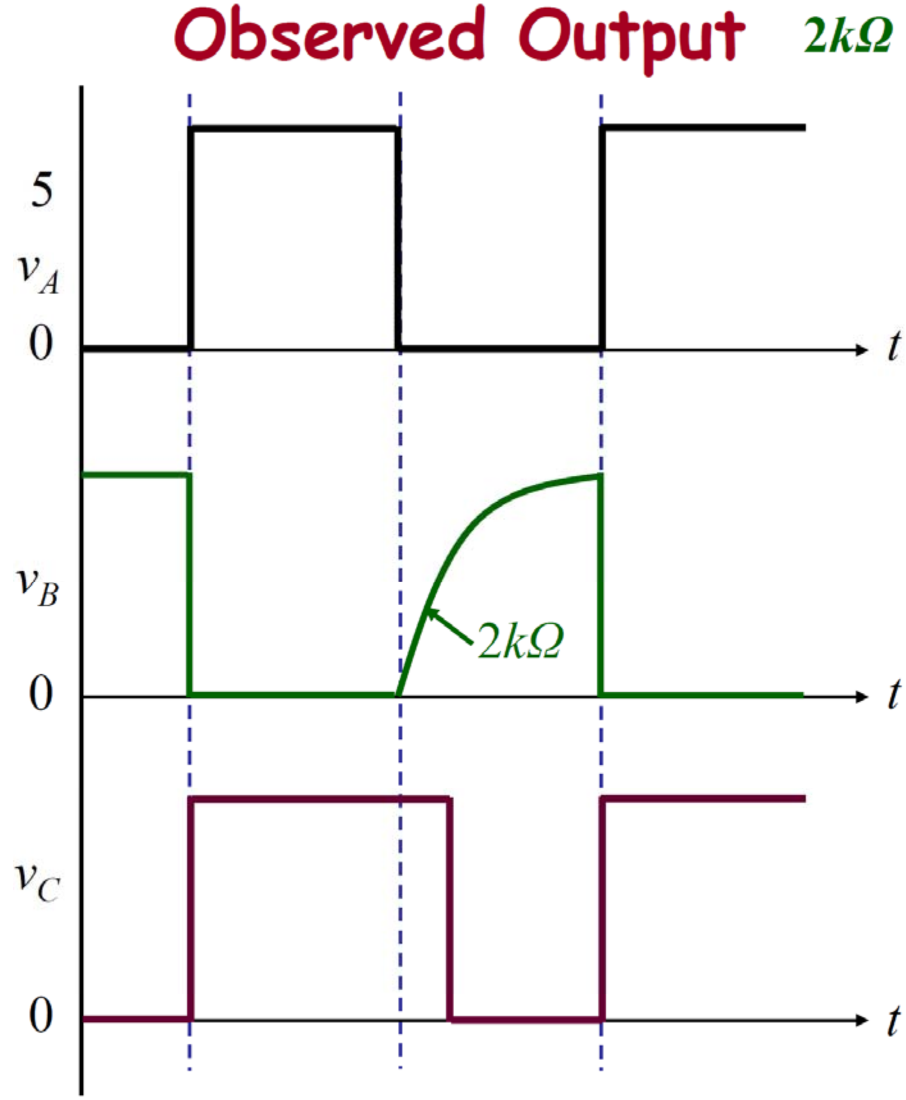
CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

For this inverter driving another, the parasitic inductance of the wire and the gate-to-source capacitance of the MOSFET are shown.



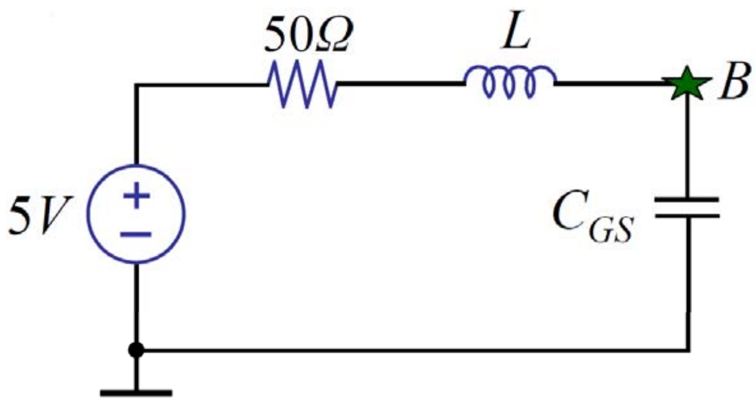
With 2 kΩ Load Resistance



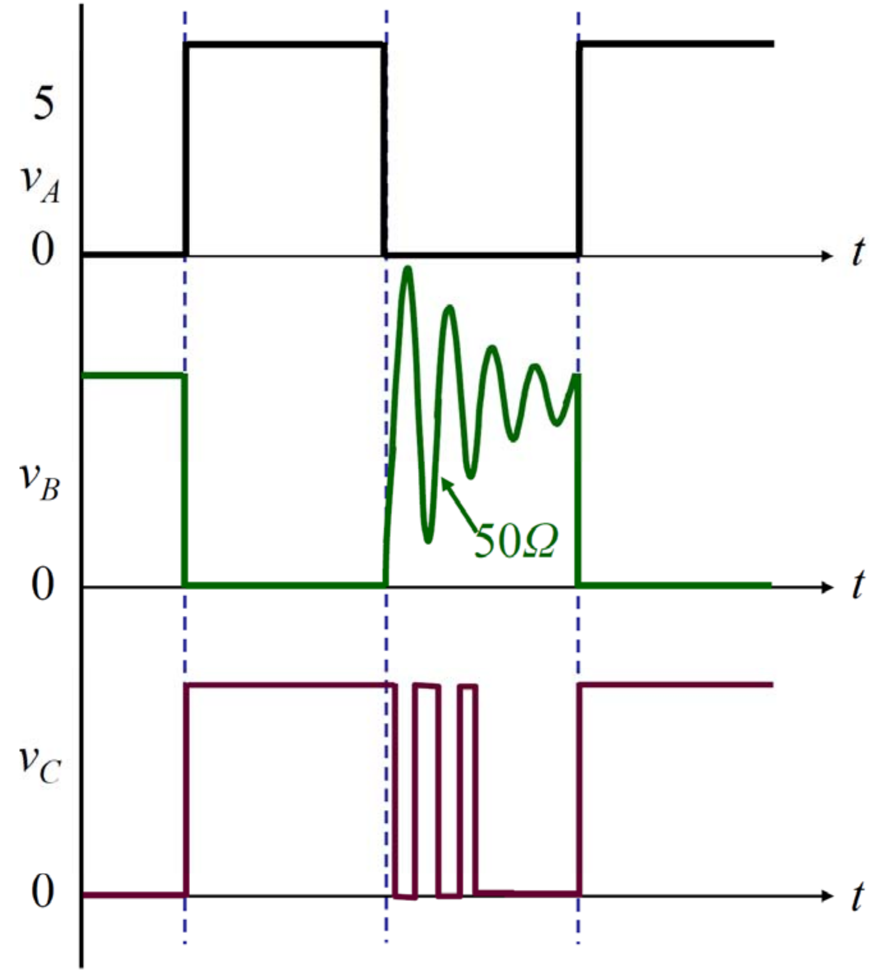


Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance.

With 50 Ω Load Resistance



Observed Output ~50Ω

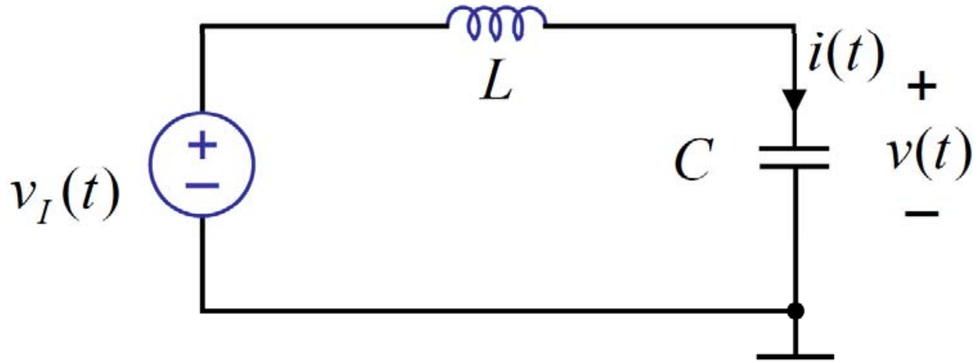


In addition to the speedy rising time. There are additional unexpected ringing.

LC Network

Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- To understand this, let's analyze the LC network first (instead of RLC).



- Node method:

$$i(t) = C\frac{dv}{dt} \qquad v_I - v = L\frac{di}{dt} \Rightarrow i = \frac{1}{L}\int_{-\infty}^{t}(v_I - v)dt$$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)dt = C\frac{dv}{dt} \Rightarrow \frac{1}{L}(v_I - v) = C\frac{d^2v}{dt^2}$$

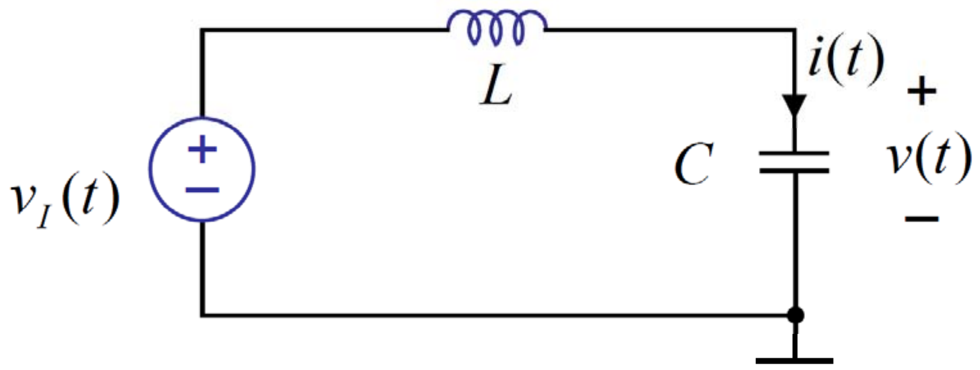
$$LC\frac{d^2v}{dt^2} + v = v_I$$

- v, i are state variables.

Unit of LC is Time^2

Method of homogeneous and particular solutions

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

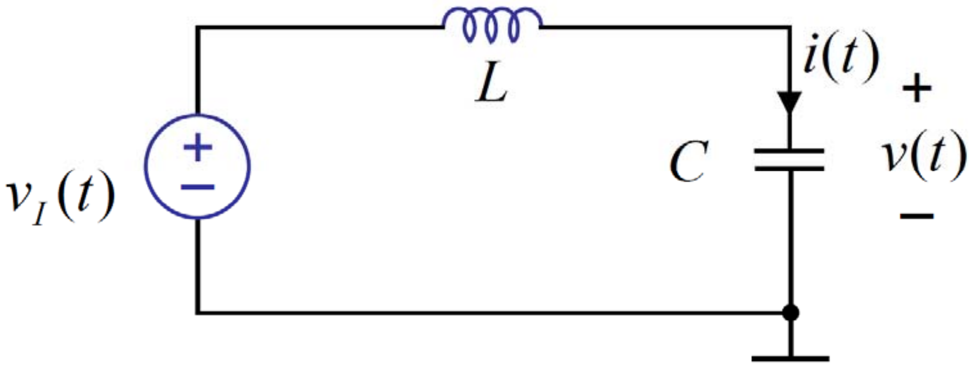


- Find the particular solution, v_P.
- Find the homogeneous solution , v_H.
- The total solution is the sum of the particular and homogeneous solutions , $v = v_P + v_H$.
- Use the initial conditions to solve for the remaining constants.

LC Circuit

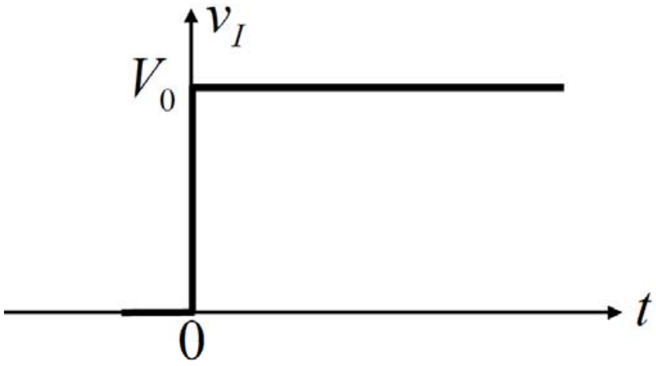
- Solve:

$$LC\frac{d^2v}{dt^2}+v=v_I$$



- With input:

$$v_I(t)=V_0u(t)$$



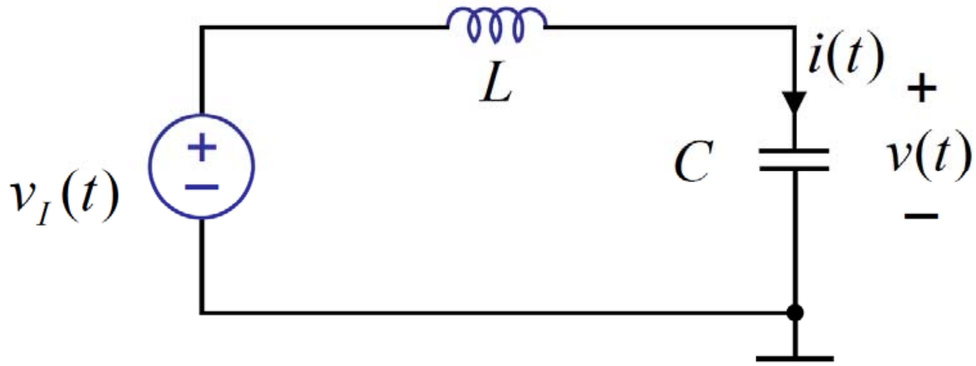
- And zero initial state:

$$v(0)=0$$
$$i(0)=0$$

- Zero State Response (ZSR).

The Particular solution

Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN



- Find the particular solution for $LC\dfrac{d^2v_P}{dt^2}+v_P=v_I$

- Use trial and error : Try $v_P = K$, .

$$LC\frac{d^2K}{dt^2}+K=V_0 \Rightarrow 0+K=V_0 \Rightarrow K=V_0 \Rightarrow v_P=V_0$$

The Homogeneous Solution

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- Find the homogeneous solution , v_H, for $LC\dfrac{d^2 v_H}{dt^2} + v_H = 0$
- Assume solution is of this form : $v_H = Ae^{st}$

$$LC\frac{d^2 Ae^{st}}{dt^2} + Ae^{st} = 0 \Rightarrow LCAs^2 e^{st} + Ae^{st} = 0 \Rightarrow LCs^2 + 1 = 0$$

- Characteristic equation: $LCs^2 + 1 = 0$
- Root: $\Rightarrow s = \pm j\sqrt{\dfrac{1}{LC}} = \pm j\omega_0$ where $\omega_0 = \sqrt{\dfrac{1}{LC}}$
- ω_0 is the natural frequency.
- The homogeneous solution , v_H: $v_H = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$

The Total solution

- The total solution is the sum of the particular and homogeneous solutions:

$$v = v_P + v_H = V_0 + A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

- Use the initial conditions: $v(0) = 0\ \mathrm{V}$ and $i(0) = 0\ \mathrm{A}$

$$v(0) = 0 = V_0 + A_1 e^{j\omega_0 0} + A_2 e^{-j\omega_0 0} = V_0 + A_1 + A_2$$

$$i(0) = 0 = C\frac{dv}{dt} = CA_1 j\omega_0 e^{j\omega_0 0} - CA_2 j\omega_0 e^{-j\omega_0 0} = CA_1 j\omega_0 - CA_2 j\omega_0$$

$$\Rightarrow A_1 = A_2 = -\frac{V_0}{2}$$

- The total solution v : $$v = V_0 - \frac{V_0}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

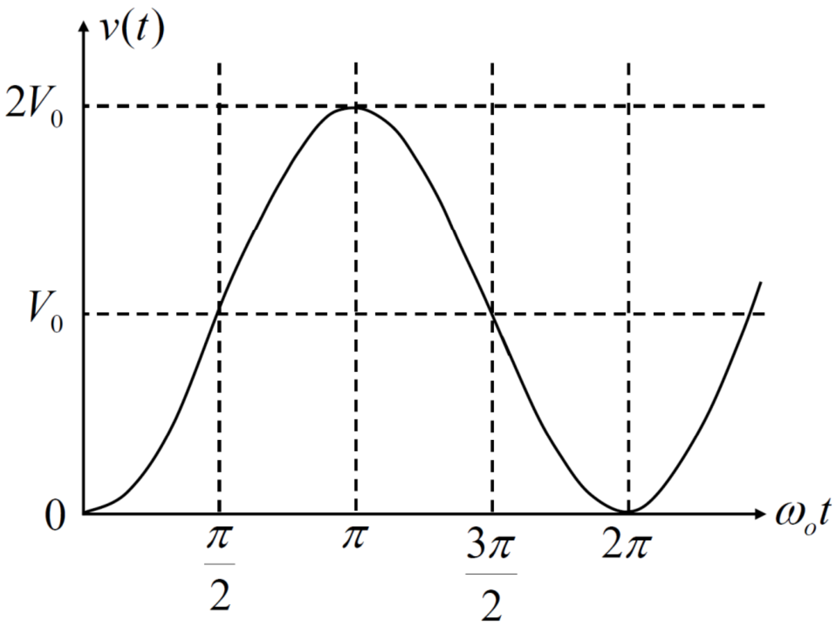
The Total solution

- The total solution *v* and *i*:

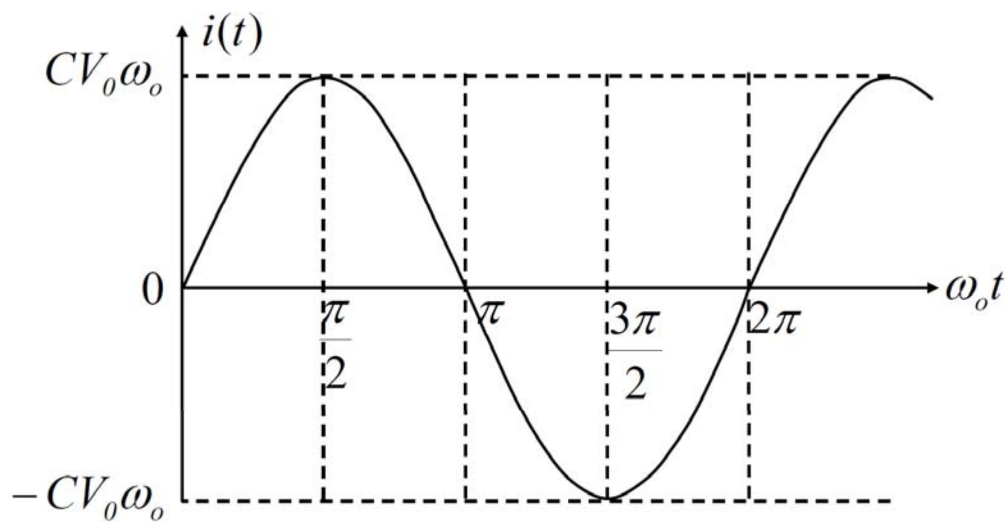
$$v = V_0 - V_0 \cos \omega_0 t$$

$$i = CV_0 \omega_0 \sin \omega_0 t$$

- The output looks sinusoidal.



$$v = V_0 - V_0 \cos \omega_0 t$$



$$i = CV_0 \omega_0 \sin \omega_0 t$$

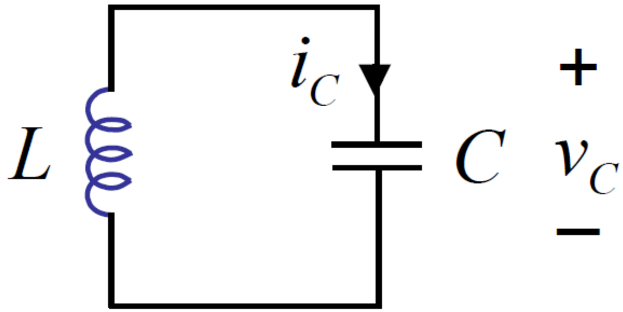
Summary of the Method

- ① Write the DE for the circuit by applying the node method.
- ②Find particular solution v_P by guessing and trial & error.
- ③Find homogeneous solution v_H by.
  - Assume the solution of the form $v_H = Ae^{st}$
  - Obtain the characteristic equation.
  - Solve the characteristic equation for roots s_j.
  - Form v_H by summing the $A_i e^{s_i t}$ terms.
- ④Total solution is $v = v_P + v_H$, and solving for the remaining constants by using the initial conditions.

Undriven LC Network

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The undriven response is also the Zero Input Response (ZIR) of the circuit.



$$LC\frac{d^2v}{dt^2}+v=0$$

- With zero input $v_I = 0$.
- And nonzero initial state

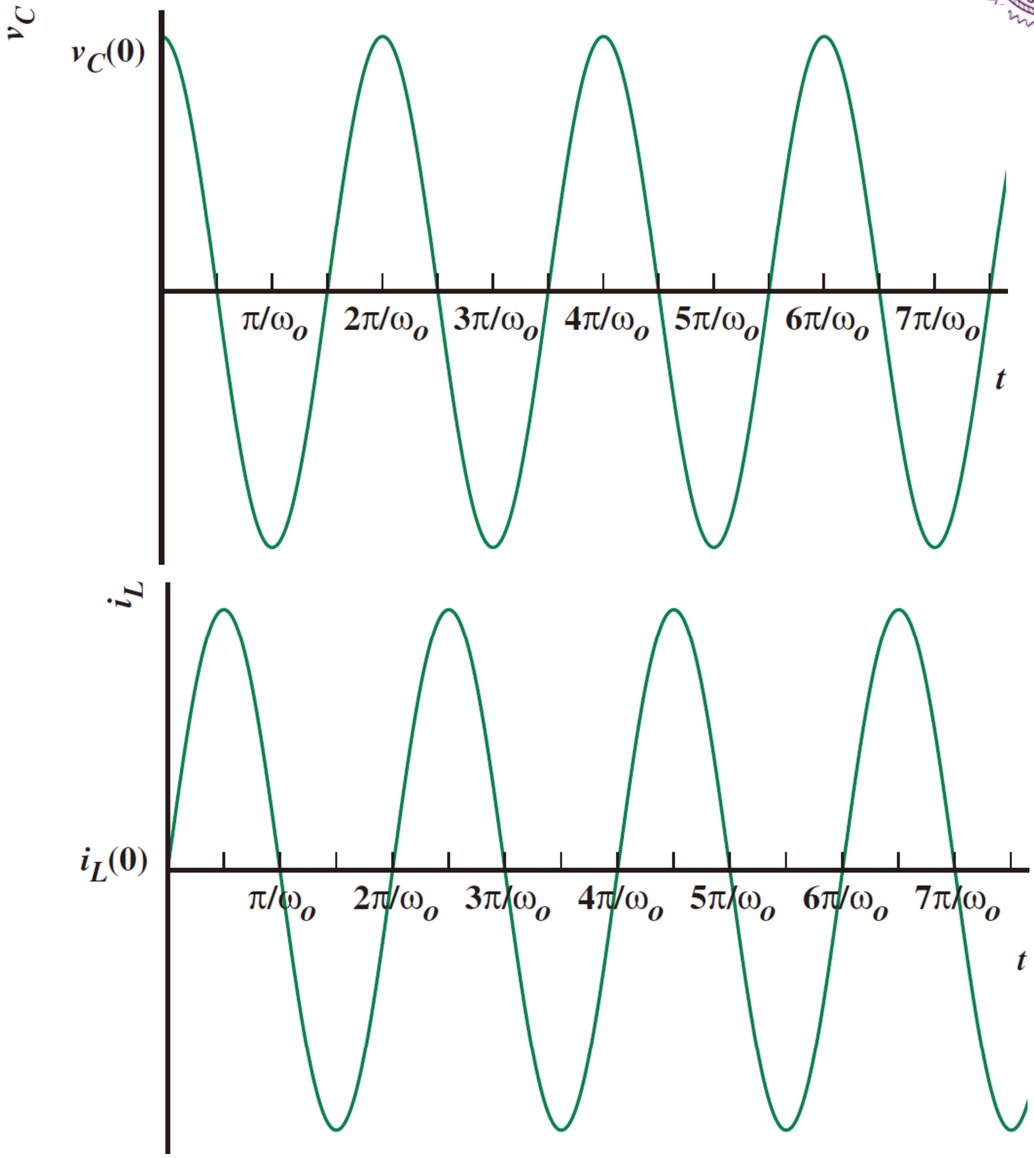
$$v_C(0)=V$$
$$i_C(0)=0$$

$$v_C(0)=V=A_1e^{j\omega_0 0}+A_2e^{-j\omega_0 0} \Rightarrow V=A_1+A_2$$
$$i(0)=0=CA_1 j\omega_0 - CA_2 j\omega_0 \Rightarrow A_1=A_2$$
$$A_1=A_2=\frac{V}{2}$$

- The Solutions

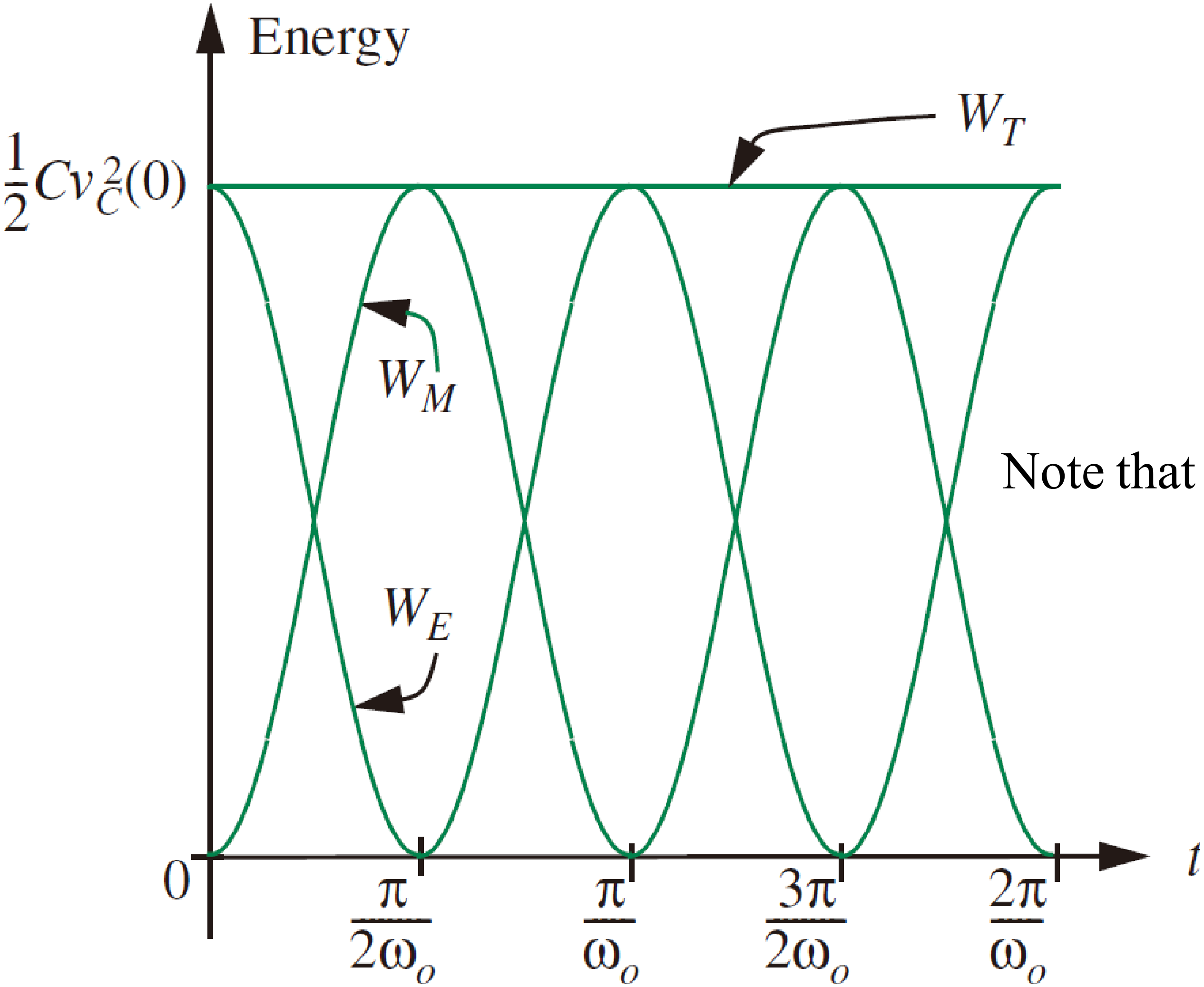
$$v=V\cos\omega_0 t$$
$$i=-CV\omega_0\sin\omega_0 t$$

Undriven LC Network



The Energy

- Total energy in the system is a constant, but it sloshes back and forth between the Capacitor and the inductor.

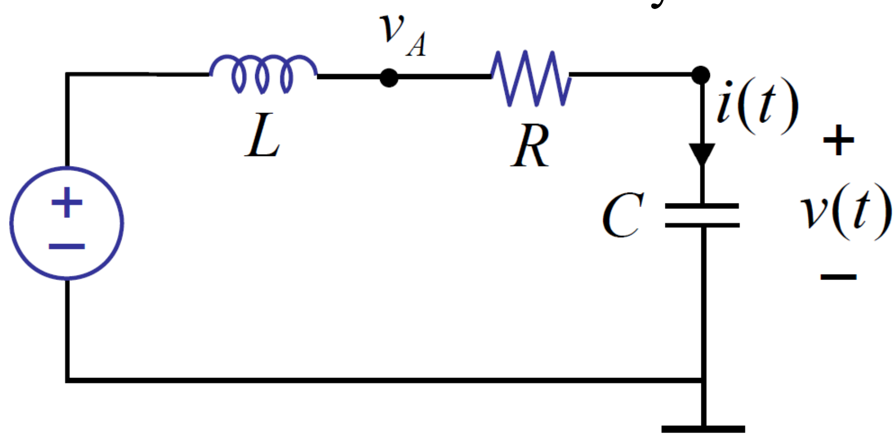


Note that $\frac{1}{2}Cv_C^2 + \frac{1}{2}Li_C^2. = \frac{1}{2}CV^2$

RLC Network (Damped Oscillator)

- Now, let's add a resistor to the LC network and analyze the RLC network.

$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v=v_I$$



- Node method:

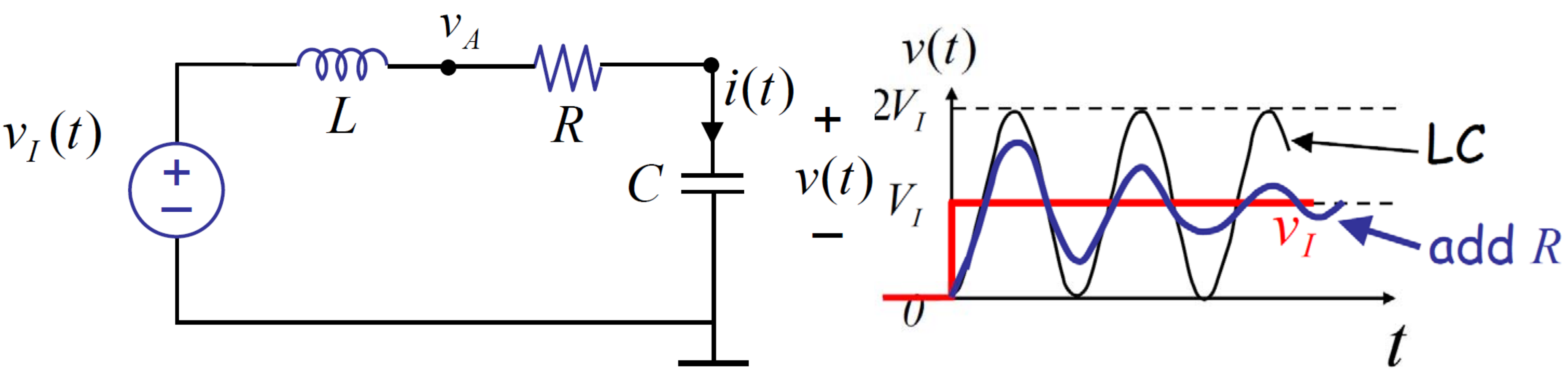
Node v_A: $\frac{1}{L}\int_{-\infty}^{t}(v_I-v_A)dt=i=\frac{v_A-v}{R}$

Node v: $C\frac{dv}{dt}=i=\frac{v_A-v}{R}\Rightarrow RC\frac{dv}{dt}=v_A-v$

$$C\frac{d^2v}{dt^2}=\frac{1}{R}\frac{d(v_A-v)}{dt}=\frac{v_I-v_A}{L}\Rightarrow LC\frac{d^2v}{dt^2}=v_I-v_A$$

$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}=v_I-v_A+v_A-v\Rightarrow LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v=v_I$$

Method of homogeneous and particular solutions

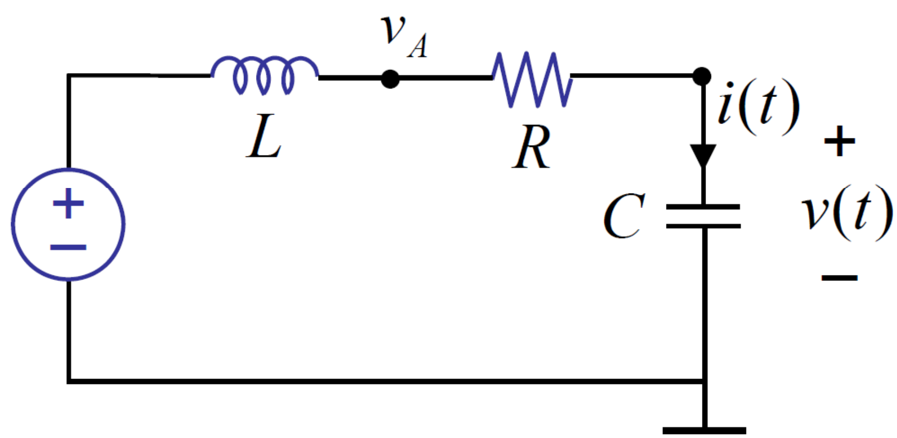


- Find the particular solution, v_P.
- Find the homogeneous solution , v_H.
- The total solution is the sum of the particular and homogeneous solutions , $v = v_P + v_H$.
- Use the initial conditions to solve for the remaining constants.

RLC Circuit

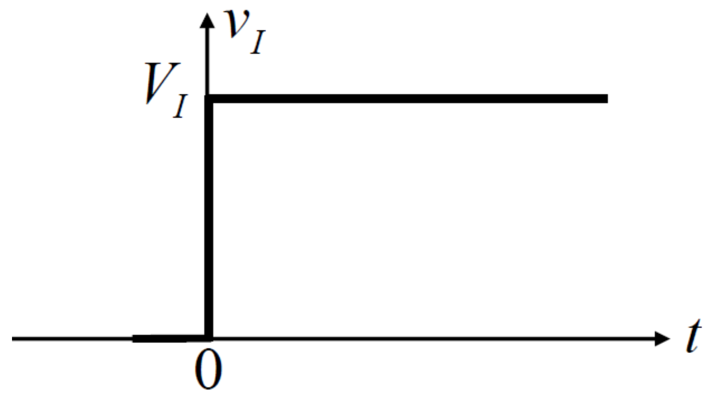
- Solve:

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}v_I$$



- With input:

$$v_I(t)=V_I u(t)$$



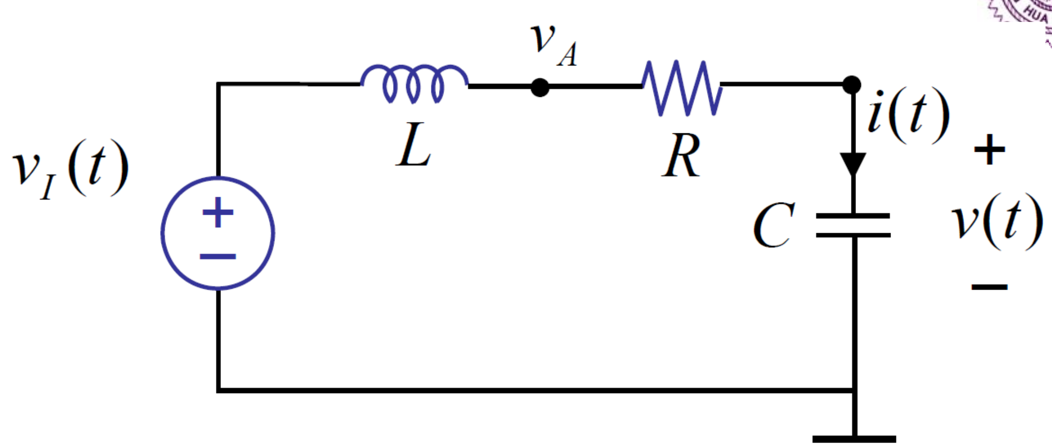
- And zero initial state:

$$v(0)=0$$
$$i(0)=0$$

- Zero State Response (ZSR).

The Particular solution

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN



- Find the particular solution for $$\frac{d^2 v_P}{dt^2} + \frac{R}{L}\frac{dv_P}{dt} + \frac{1}{LC}v_P = \frac{1}{LC}V_I$$
- Use trial and error : Try $v_P = K$, .

$$\frac{d^2 K}{dt^2} + \frac{R}{L}\frac{dK}{dt} + \frac{1}{LC}K = \frac{1}{LC}V_I \Rightarrow 0 + 0 + \frac{1}{LC}K = \frac{1}{LC}V_I \Rightarrow K = V_I$$

$$\Rightarrow v_P = V_I$$

The Homogeneous Solution

- Find the homogeneous solution , v_H, for $\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC}v_H = 0$

- Assume solution is of this form : $v_H = Ae^{st}$

$$\frac{d^2 Ae^{st}}{dt^2} + \frac{R}{L}\frac{dAe^{st}}{dt} + \frac{1}{LC}Ae^{st} = 0 \Rightarrow s^2 Ae^{st} + \frac{R}{L}sAe^{st} + \frac{1}{LC}Ae^{st} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- Characteristic equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

where $\alpha = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$

The Homogeneous Solution

- Characteristic equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0 \quad \text{where} \quad \alpha = \frac{R}{2L} \text{ and } \omega_0^2 = \frac{1}{LC}$$

- Root: $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- The homogeneous solution , v_H: $v_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$v_H = A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)t}$$

The Total solution

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The total solution is the sum of the particular and homogeneous solutions:

$$v(t) = v_P + v_H = V_I + A_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)t}$$

- Use the initial conditions: $v(0) = 0\text{ V}$ and $i(0) = 0\text{ A}$

$$v(0) = V_I + A_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)0} + A_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)0} = V_I + A_1 + A_2 = 0$$

$$i(0) = C\frac{dv}{dt} = \left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)A_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)0} + \left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)A_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)0} = 0$$

$$\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)A_1 + \left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)A_2 = 0$$

If $\alpha \neq \omega_0 \Rightarrow A_1 = -\dfrac{\alpha+\sqrt{\alpha^2-\omega_0^2}}{2\sqrt{\alpha^2-\omega_0^2}}V_I$ and $A_1 = -\dfrac{-\alpha+\sqrt{\alpha^2-\omega_0^2}}{2\sqrt{\alpha^2-\omega_0^2}}V_I$.

Solutions for Damped 2nd Order Circuit

- Let's stare at the total solution for a little bit longer…

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2 - \omega_0^2}\right)t}$$

- There are 3 possible cases: $\alpha > \omega_0$, $\alpha = \omega_0$, and $\alpha < \omega_0$.
  - The case for $\alpha > \omega_0$ is called ***overdamped***.

    $$v(t) = V_I + A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$

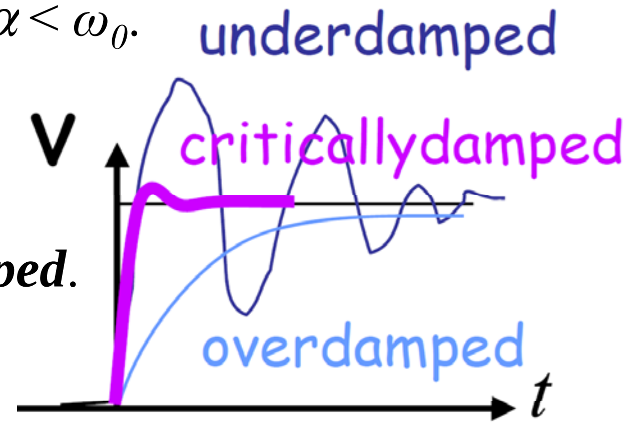
  - The case for $\alpha = \omega_0$ is called ***critically damped***.

    $$v(t) = V_I - V_I t e^{-\alpha t}$$

  - The case for $\alpha < \omega_0$ is called ***underdamped***.

    $$v(t) = V_I + A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

    $$v(t) = V_I + K_1 e^{-\alpha t} \cos \omega_d t + K_2 \sin \omega_d t$$



where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

Damp

Damp:

Noun

1. moisture in the air; humidity.
2. Lowness of spirits; depression.
3. A restraint or check; a discouragement.

Transitive verb

1. To make damp or moist; moisten.
2. To restrain or check; discourage.
3. (Music). To slow or stop the vibrations of (the strings of a keyboard instrument) with a damper.
4. (Physics) To decrease the amplitude of (an oscillating system).

Underdamped

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- Let's look at the underdamped case more closely.
- $\alpha < \omega_0$

$$v(t) = V_I + K_1 e^{-\alpha t} \cos \omega_d t + K_2 e^{-\alpha t} \sin \omega_d t$$

- Use the initial conditions: $v(0) = 0$ V and $i(0) = 0$ A

$$v(0) = V_I + K_1 = 0 \Rightarrow K_1 = -V_I$$

$$i(0) = C\left(-\alpha K_1 e^{-\alpha 0} \cos \omega_d 0 - \omega_d K_1 e^{-\alpha t} \sin \omega_d t - \alpha K_2 e^{-\alpha 0} \sin \omega_d 0 + \omega_d K_2 e^{-\alpha 0} \cos \omega_d 0\right)$$

$$i(0) = -C\alpha K_1 + C\omega_d K_2 = 0 \Rightarrow K_2 = \frac{\alpha}{\omega_d} K_1 = \frac{\alpha}{\omega_d} V_I$$

- The total solution for underdamped case, $\alpha < \omega_0$, is:

$$v(t) = V_I - V_I e^{-\alpha t} \cos \omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t} \sin \omega_d t$$

Underdamped

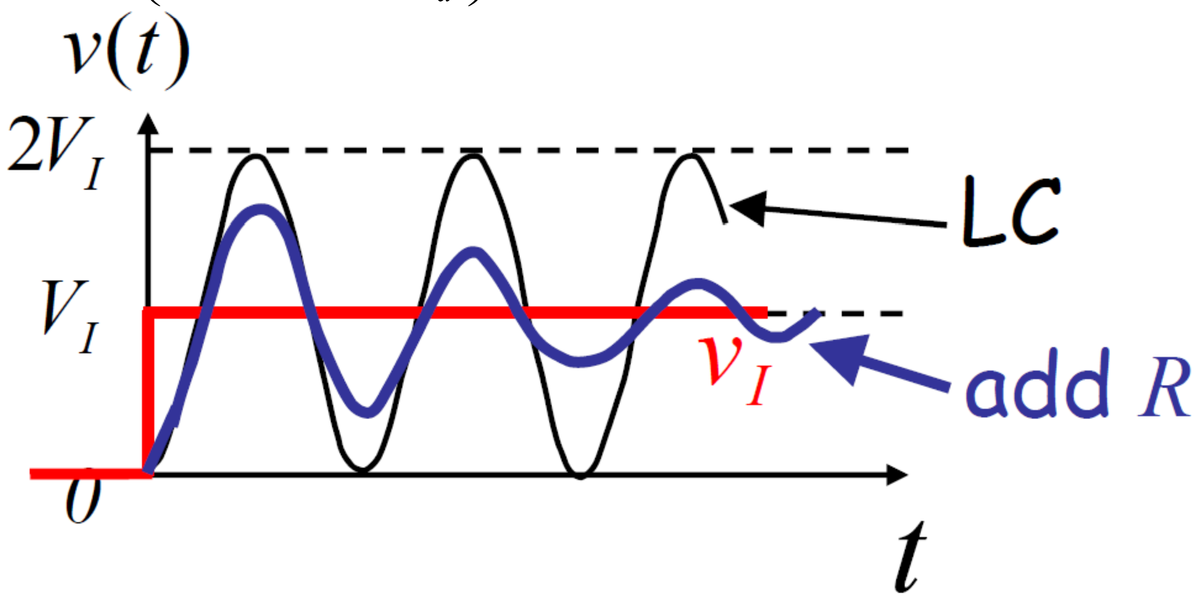
CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The total solution for underdamped case, $\alpha < \omega_0$, is:

$$v(t) = V_I - V_I e^{-\alpha t} \cos \omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t} \sin \omega_d t$$

- Since the scaled sum of sines (of the same frequency) are also sines, let's rewrite the total solution as:

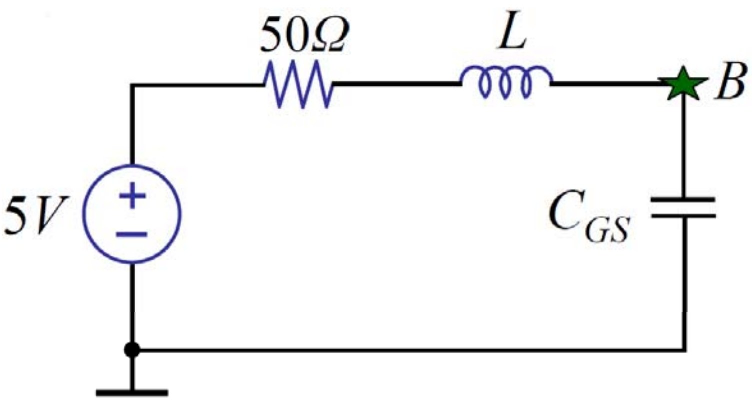
$$v(t) = V_I - V_I \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1} \frac{\alpha}{\omega_d} \right)$$



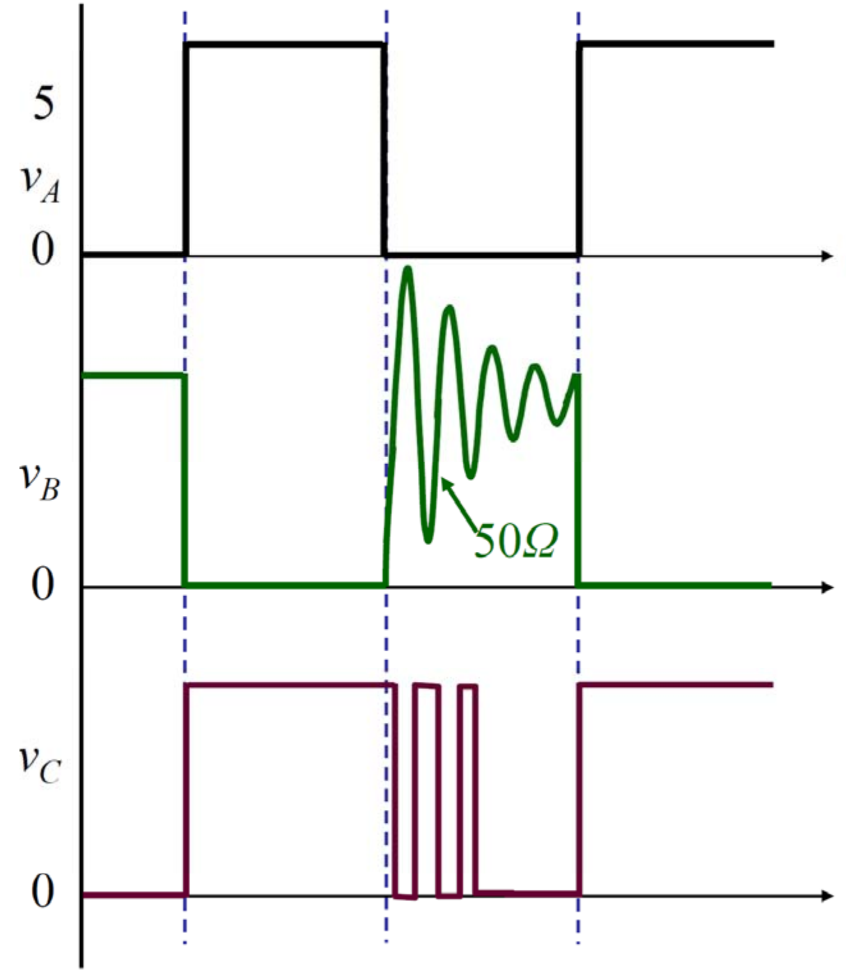
With 50 Ω Load Resistance

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

Observed Output ~50Ω



Under smaller *R* of 50 Ω, the series RLC circuit become ***underdamped*** and the *ringing* occurs.



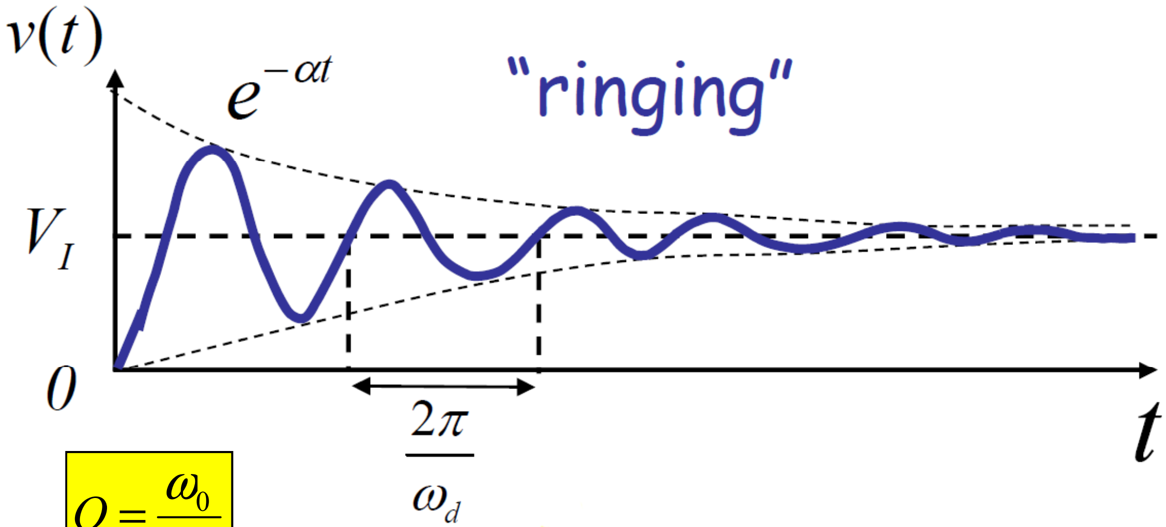
Intuitive Analysis

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The total solution for underdamped case, $\alpha < \omega_0$, is:

$$v(t) = V_I - V_I \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$

- Characteristic equation: $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$ $\quad s^2 + 2\alpha s + \omega_0^2 = 0$
- $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ is the oscillation frequency.
- α governs the decay rate.
- V_I is the final steady state value.
- $v(0)$ and $i(0)$ gives the initial value and slope.
- Q is the quality factor (approximately the number of cycles of ringing)



$$Q = \frac{\omega_0}{2\alpha}$$

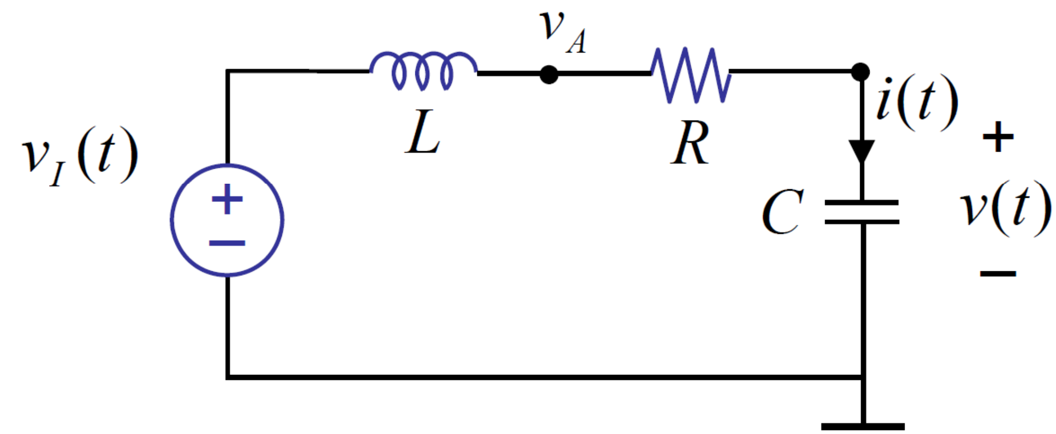
Intuitive Analysis RLC Circuit

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The circuit:

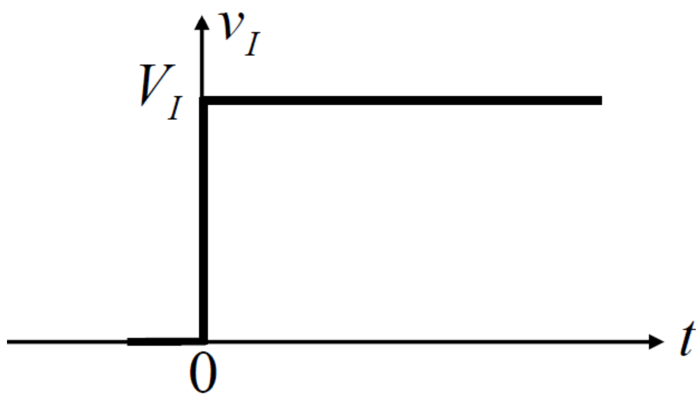
  $R = 0.2\ \Omega$, $L = 100\ \mu\text{H}$,

  $C = 100\ \mu\text{F}$



- With input:

$$v_I(t) = V_I u(t) \text{ where } V_I = 1\text{ V}$$



- With initial states:

$$v(0) = 0.5\ \text{V}$$

$$i(0) = -0.5\ \text{A}$$

Intuitive Analysis RLC Circuit

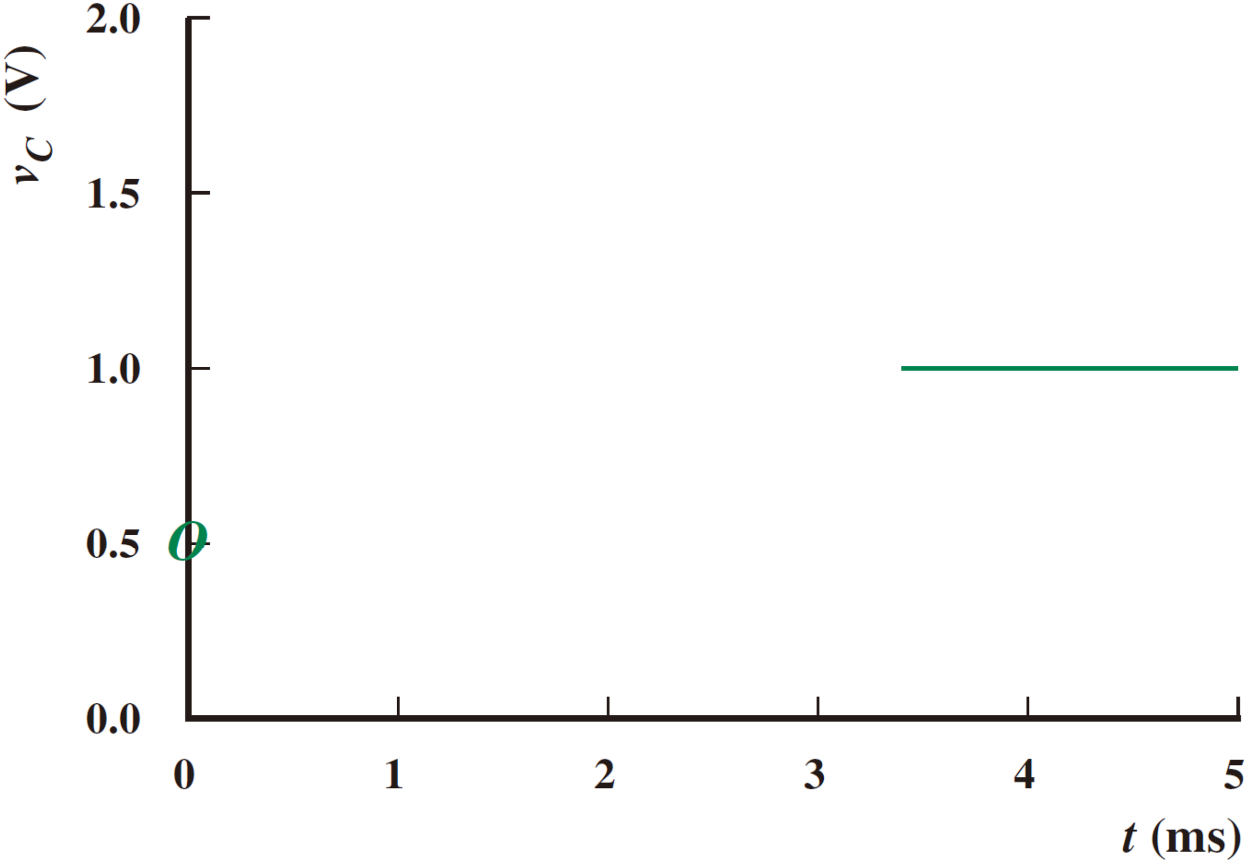
CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- In the steady state, the capacitor behaves like an open circuit. Therefore, the inductor current vanishes and the input drive appears across the capacitor.

$$v(\infty) = 1 \text{ V}$$

$$i(\infty) = 0 \text{ A}$$

$$v(0) = 0.5 \text{ V}$$

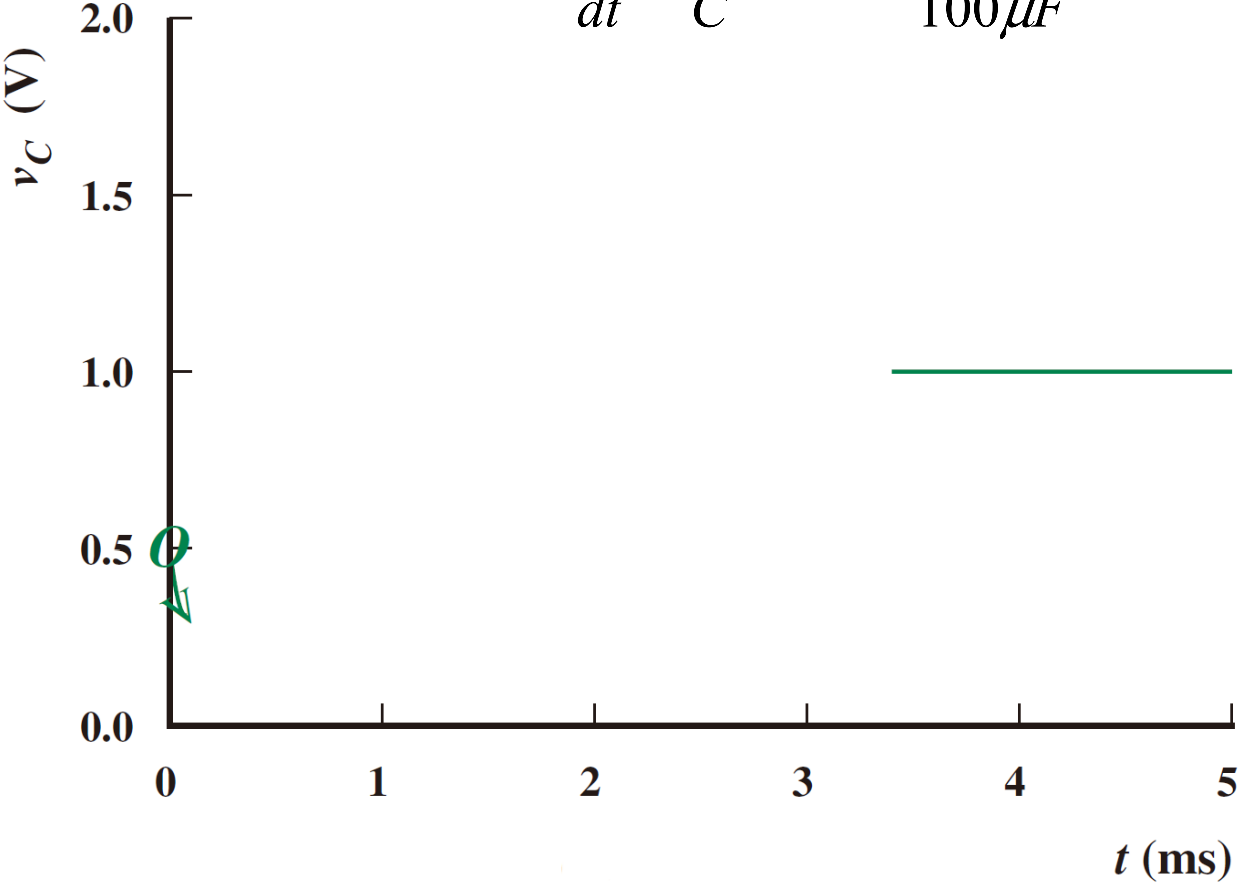


Intuitive Analysis RLC Circuit

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The initial trajectory of the capacitor voltage (increasing or decreasing) starting from its initial value of 0.5 V is:

$$\frac{dv}{dt} = \frac{1}{C} i(0) = -\frac{0.5A}{100\mu F} = 5000 \ \frac{\text{V}}{\text{sec}}$$



Intuitive Analysis RLC Circuit

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- Characteristic equation:

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0 \quad \text{i.e.}\ \ s^2+2\alpha s+\omega_0^2=0$$

$$\alpha=\frac{R}{2L}=10^3\ \text{rad/sec} \qquad \omega_0=\sqrt{\frac{1}{LC}}=10^4\ \text{rad/sec}$$

- *Since* $\alpha < \omega_0$, We conclude that the system is under-damped. The oscillation frequency is given by

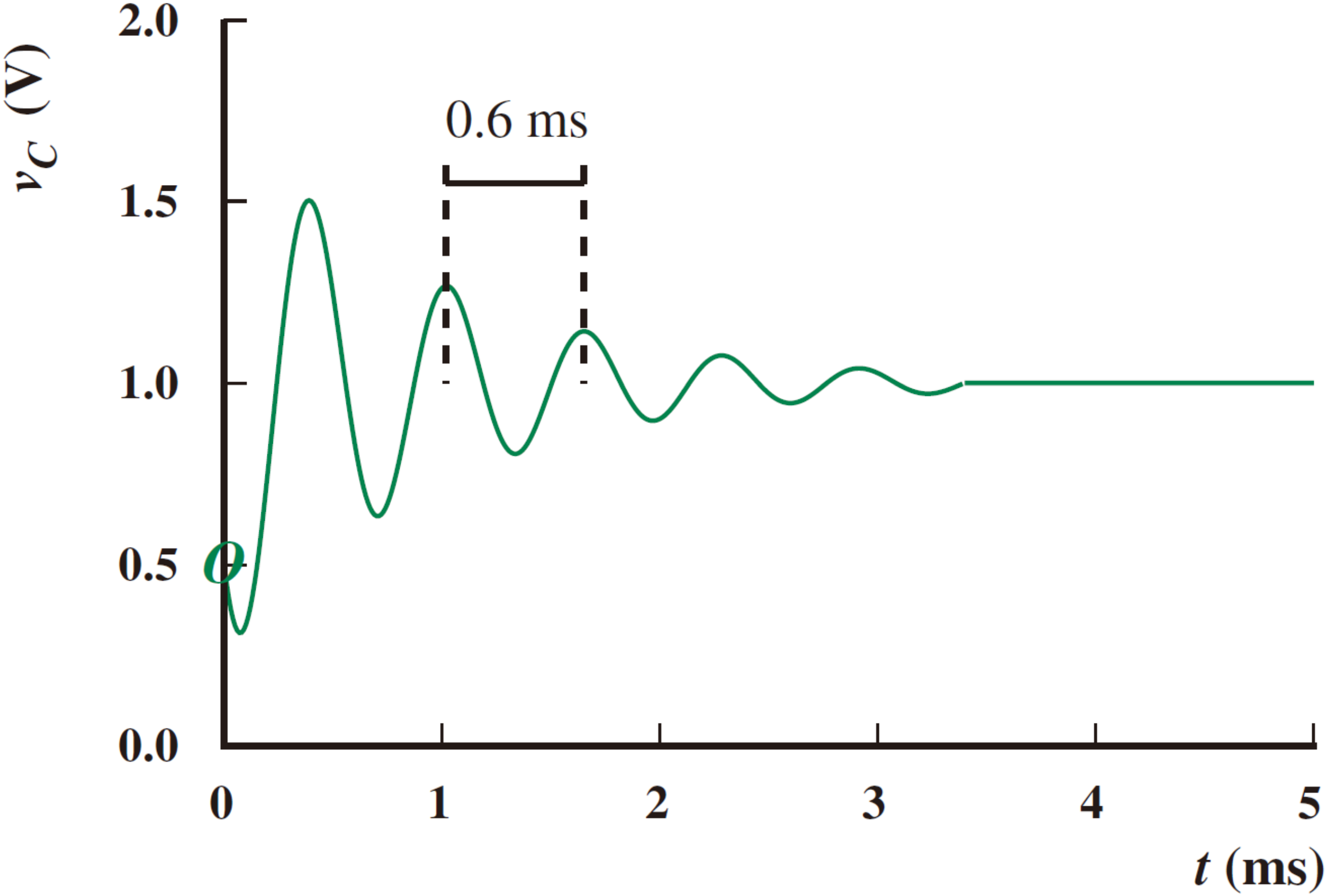
$$\omega_d=\sqrt{\omega_0^2-\alpha^2}\approx 9950\ \ \text{rad/sec}$$

- Quality factor, Q: $Q=\dfrac{\omega_0}{2\alpha}\approx 5$, i.e. the system will ring for approximately 5 cycles.

Intuitive Analysis RLC Circuit

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

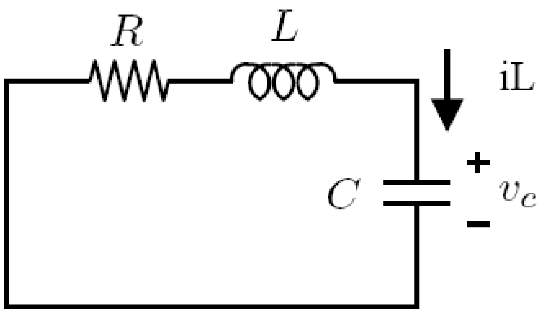
- Knowing the initial trajectory, we can stitch in a sinusoid that decays over about 5 cycles with the correct initial trajectory.



Undriven RLC Network

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

- The undriven response is also the Zero Input Response (ZIR) of the circuit. For the following circuit, If $L = 0.04$ H and $C = 0.01$ F, find $v_C(t)$ and $i_L(t)$ for the following $R = 5\ \Omega$, $4\ \Omega$, and $1\ \Omega$.



$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC} v_C = 0$$

- Nonzero initial state:

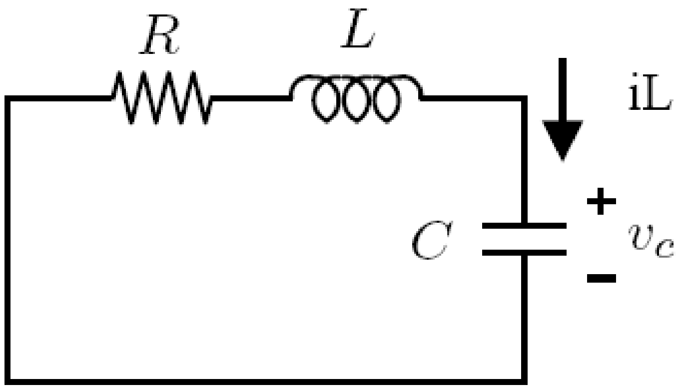
$$v_C(0) = 3\ \text{V}$$

$$i_L(0) = 0\ \ \text{A}$$

- The Equations: $$\frac{d^2 v_C}{dt^2} + 25R\frac{dv_C}{dt} + 2500 v_C = 0$$

Undriven RLC Network

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN



- Characteristic equation:

$$s^2 + 25Rs + 2500 = 0$$

$$b^2 - 4ac = 625R^2 - 10000$$

- **R = 5 Ω** , $625R^2 - 10000 = 5625 > 0,$ ***overdamped***

$$s_1 = -25 \text{ and } s_2 = -100 \qquad v_C(t) = A_1 e^{-25t} + A_2 e^{-100t}$$

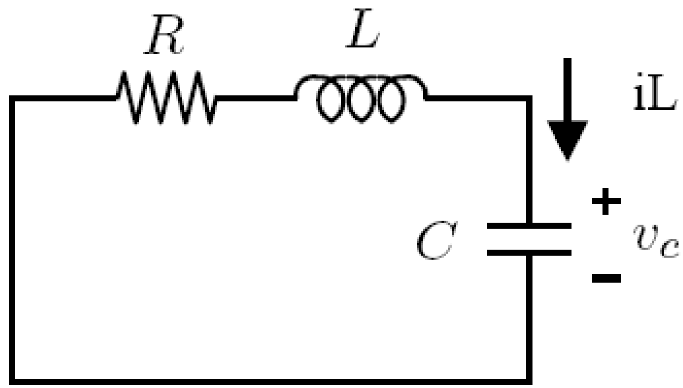
- **R = 4 Ω** , $625R^2 - 10000 = 0$, ***critically damped***:

$$s_1 = s_2 = -50 \qquad v_C(t) = A_1 e^{-50t} + A_2 t e^{-50t}$$

- **R = 1 Ω** , $625R^2 - 10000 = -9375 < 0$, ***underdamped*** :

$$s_{1,2} = -12.5 \pm j48.4 \quad v_C(t) = Ke^{-12.5t}\sin(48.4\text{t} + \varphi)$$

Undriven RLC Network



- Determine the coefficient from

  Initial conditions:

  $$v_C(0) = 3\text{ V}$$

  $$i_L(0) = 0 \text{ A} = i_L(0) = C\frac{dv_C}{dt}$$

- ***R* = 5 Ω , *overdamped***

  $$A_1 + A_2 = 3 \text{ and } -25A_1 - 100A_2 = 0 \Rightarrow A_1 = 4 \quad A_2 = -1$$

  $$v_C(t) = 4e^{-25t} - e^{-100t}$$

- ***R* = 4 Ω , *critically damped*:**

  $$A_1 = 3 \text{ and } -50A_1 + A_2 = 0 \Rightarrow A_1 = 3 \quad A_2 = 150$$

  $$v_C(t) = 3e^{-50t}(1 + 50t)$$

- ***R* = 1 Ω , *underdamped* :**

  $$K\sin\varphi = 3 \text{ and } -12.5\,K\sin\varphi + 48.4K\sin\varphi = 0 \Rightarrow K = 3.1 \quad \varphi = 75.5°$$

  $$v_C(t) = 3.1e^{-12.5t}\sin(48.4t + 75.5°)$$

CAPT Center for Advanced Power Technologies

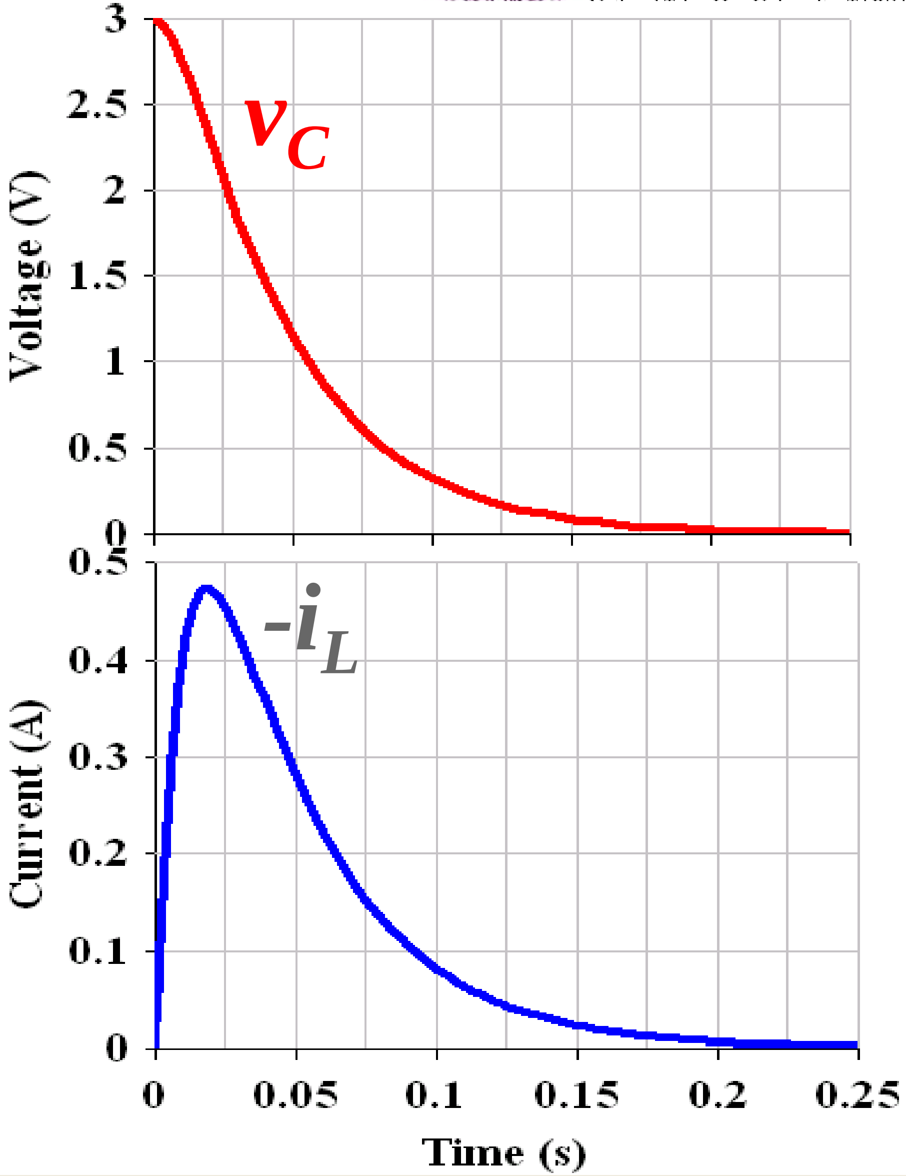
Overdamped

- $R = 5\ \Omega$, *overdamped*

$$v_C(t) = 4e^{-25t} - e^{-100t}\ \text{V}$$

$$i_L(t) = i_C(t) = C\frac{dv_C}{dt}$$

$$i_L(t) = -(e^{-25t} - e^{-100t})\ \text{A}$$

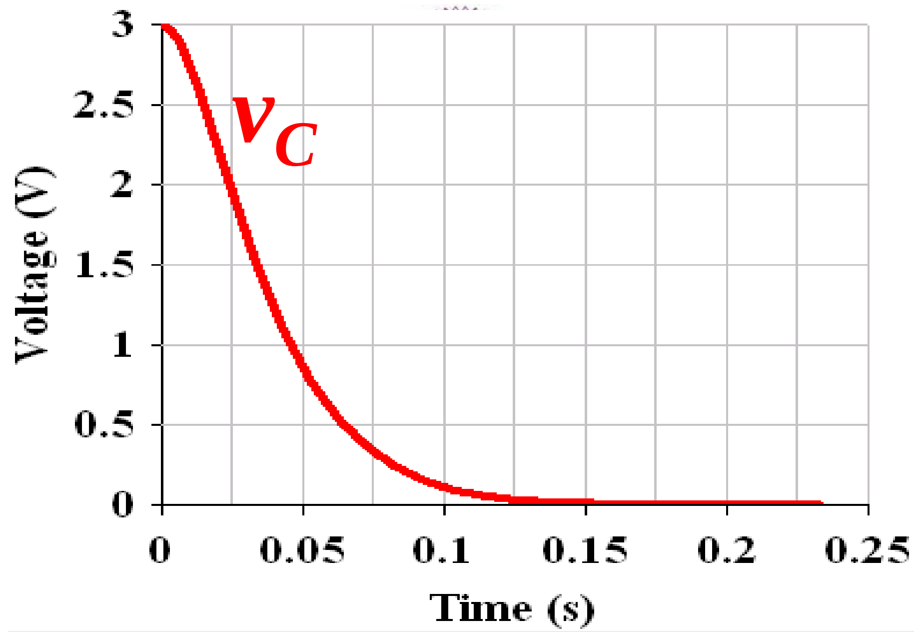


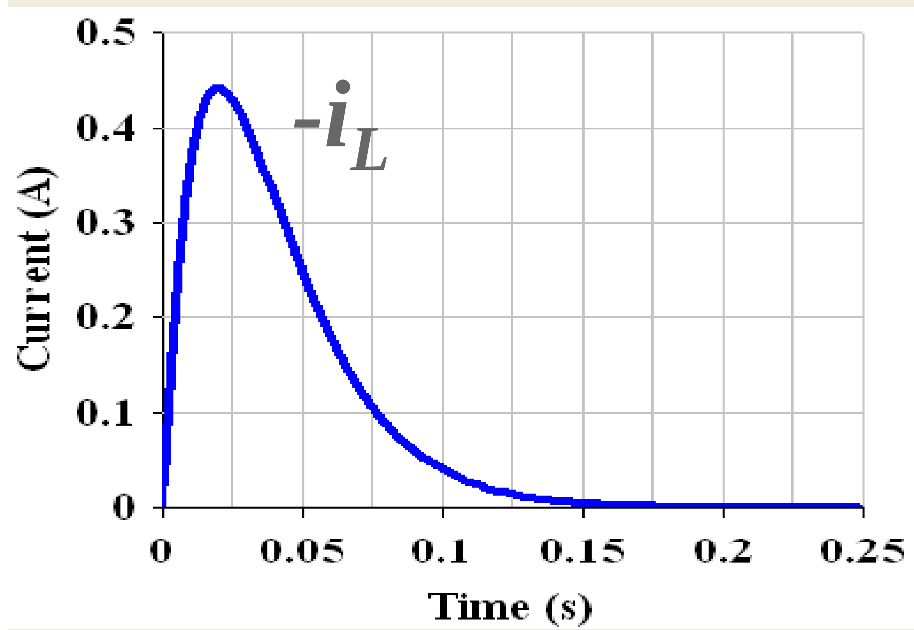
Critically Damped

- ***R* = 4 Ω , *Critically damped***

$$v_C(t) = 3e^{-50t}(1 + 50t)\ \text{V}$$

$$i_L(t) = -75te^{-50t}\ \ \text{A}$$



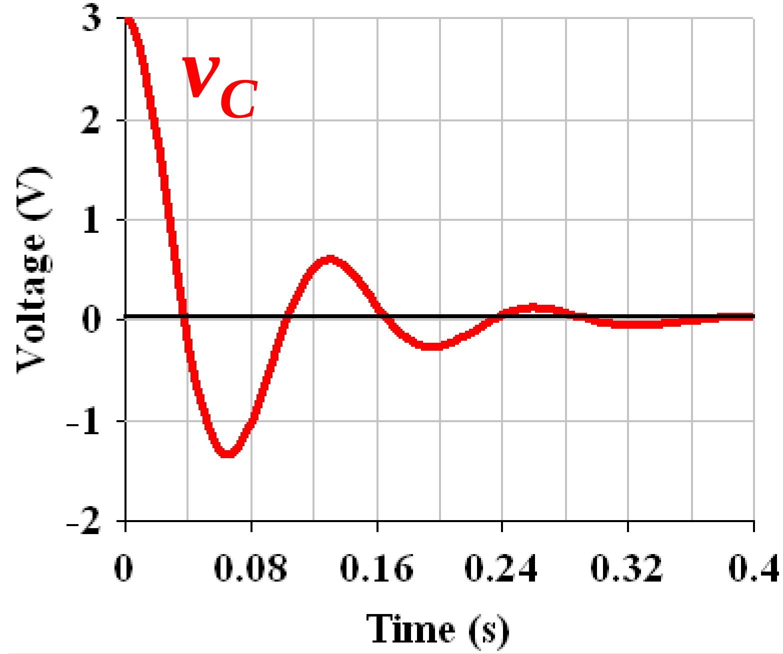


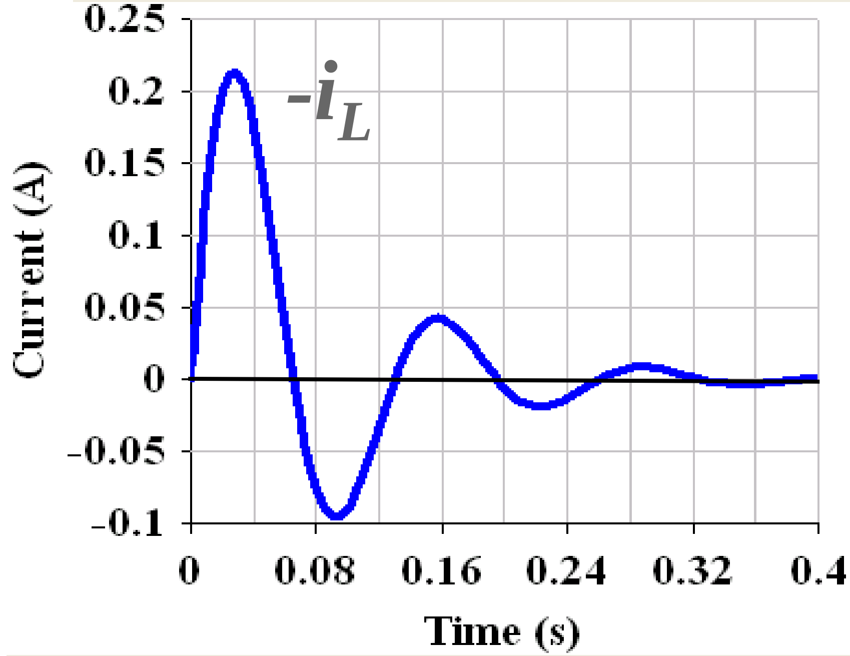
Under Damped

- ***R* = 1 Ω , *under damped***

$$v_C(t) = 3.1e^{-12.5t}\sin(48.4\text{t} + 75.5°)\ \text{V}$$

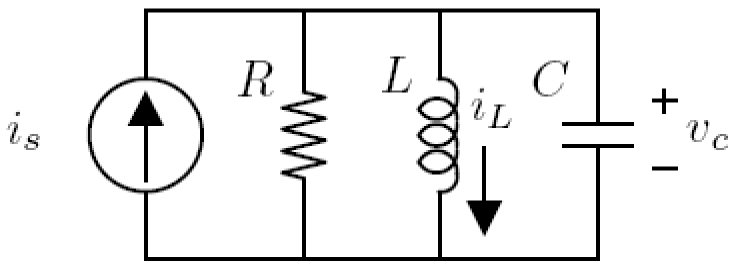
$$i_L(t) = -1.55e^{-12.5t}\sin(48.4\text{t})\ \ \text{A}$$





Natural Response; RLC in parallel

- The *RLC* circuit can be analyzed by KCL:

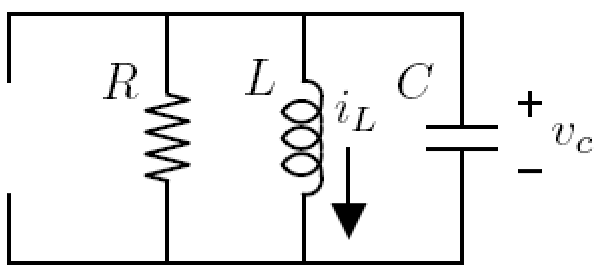


$$i_s = \frac{v_c}{R} + i_L + C\frac{dv_c}{dt}$$

$$= \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt^2}$$

$$\rightarrow \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{i_s}{LC}$$

- For $i_s = 0$, the natural response of the circuit can be derived:



$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0$$

Natural Response; RLC in parallel

Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

Assume $i_L = Ae^{st}$,

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0$$

$$\rightarrow \underline{s^2 Ae^{st} + \frac{1}{RC} s\, Ae^{st} + \frac{1}{LC} Ae^{st}} = 0$$

$$\rightarrow \underline{Ae^{st}(s^2 + \frac{1}{RC} s + \frac{1}{LC}) = 0}$$

$$\rightarrow s_1, s_2 = \underline{-\frac{1}{2RC} + \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}}}\, ; \underline{-\frac{1}{2RC} - \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}}}$$

$$= \underline{-\alpha + \sqrt{\alpha^2 - \omega_0^2}}\, ; \quad \underline{-\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

where $\alpha = \frac{1}{2RC}$, and $\omega_0 = \sqrt{\frac{1}{LC}}$

The solution can be expressed as follows:

$$i_L = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Natural Response; RLC in parallel

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

Characteristic equation:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0, \quad s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

- Two distinct real roots: $\alpha^2 > \omega_0^2 \rightarrow \frac{1}{4R^2C^2} > \frac{1}{LC} \rightarrow L > 4R^2C$
  - s_1 and s_2 are negative real numbers.
  - $i_L = \underline{A_1 e^{s_1 t} + A_2 e^{s_2 t}}$.
  - i_L decays exponentially without any oscillations; <u>over-damped</u> .
- Double roots: $\alpha^2 = \omega_0^2 \rightarrow \frac{1}{4R^2C^2} = \frac{1}{LC} \rightarrow L = 4R^2C$
  - $s_1 = s_2 = -\alpha = -\frac{R}{2L}$
  - $i_L = \underline{(A_1 t + A_2)e^{-\alpha t}}$.
  - i_L decays at a moderate pace, this is referred to as <u>critically-damped</u>

Natural Response; RLC in parallel

Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

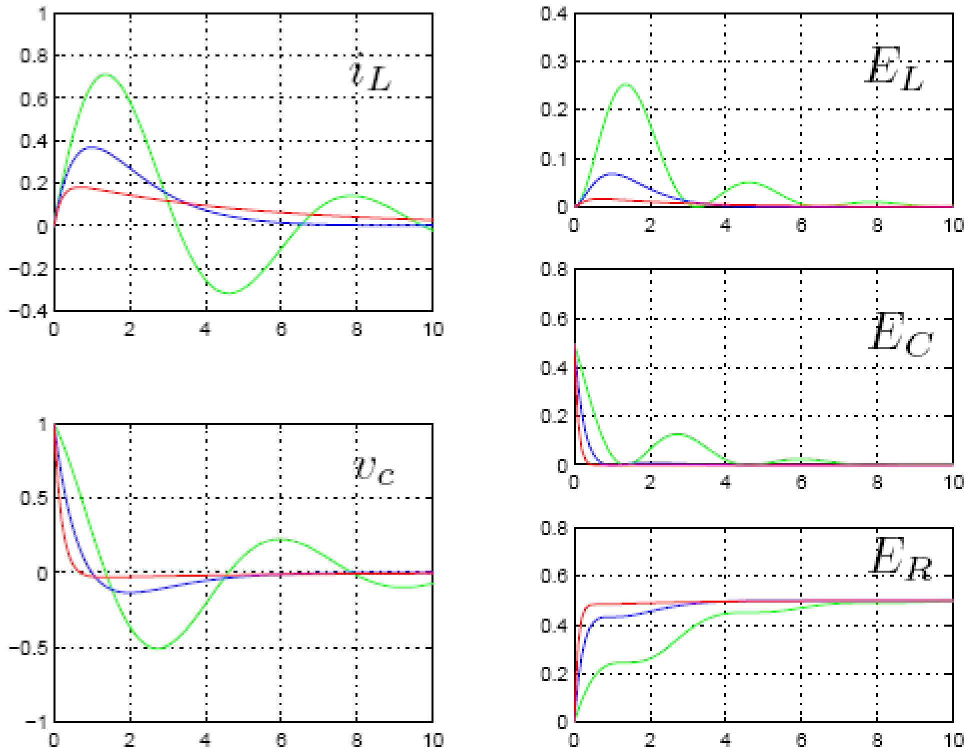
Characteristic equation:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0, \quad s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

- Two complex roots: $\alpha^2 - \omega_0^2 < 0 \quad \rightarrow \quad L < 4R^2C;$
  - $s_1 = s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
  - $i_L = \underline{e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))}.$
  - i_L oscillates within a exponentially-decaying envelope; under-damped.

Natural Response; RLC in parallel

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN



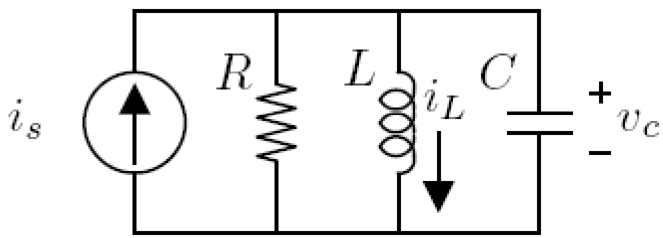
$$L = 1.0\,\mathrm{H}; \quad i_L\,|_{t=0} = 0$$

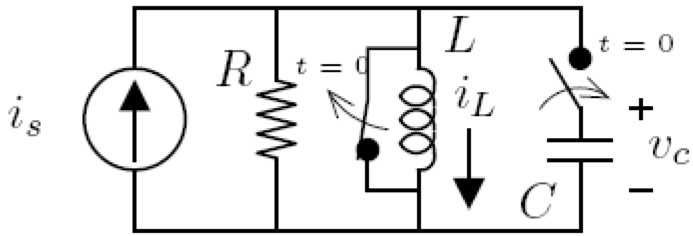
$$C = 1.0\,\mathrm{F}; \quad v_C\,|_{t=0} = 1\,\mathrm{V}$$

$$R = \begin{cases} 2.0\,\Omega; & \text{under-damped} \\ 0.5\,\Omega; & \text{critically-damped} \\ 0.2\,\Omega; & \text{over-damped} \end{cases}$$

Step Response; RLC in parallel

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN





$$i_s = I_s u(t)$$

$$v_c \,|_{t=0} = V_{c0}$$

$$i_L \,|_{t=0} = I_{L0}$$

By KCL,

$$i_s = \frac{v_c}{R} + i_L + C\frac{dv_c}{dt} = \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt^2} \rightarrow \quad \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{i_s}{LC}$$

Characteristic equation:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0, \quad s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

Step Response; RLC in parallel

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

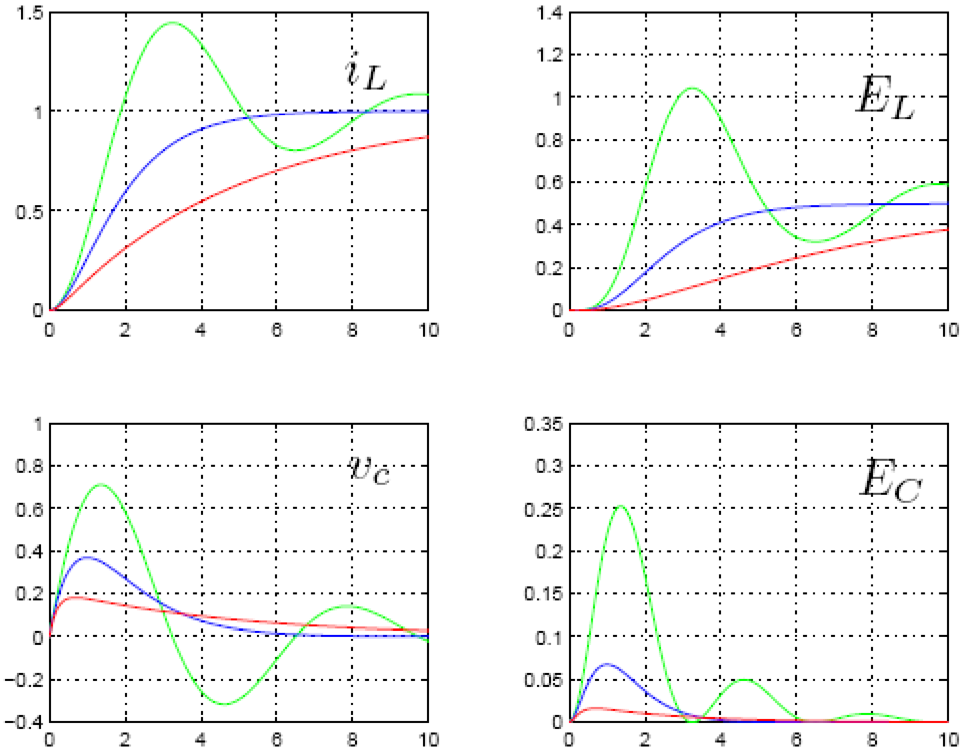
$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{i_s}{LC} = \frac{I_s u(t)}{LC}$$

Step response at steady state: $i_{Lf} = \underline{I_s}$

- Two distinct real roots: $\alpha^2 > \omega_0^2 \rightarrow \frac{1}{4R^2C^2} > \frac{1}{LC} \rightarrow L > 4R^2C$
  - s_1 and s_2 are <u>negative real numbers</u>.
  - $i_L = \underline{A_1 e^{s_1 t} + A_2 e^{s_2 t}} + \underline{I_s}$
- Double roots: $\alpha^2 = \omega_0^2 \rightarrow \frac{1}{4R^2C^2} = \frac{1}{LC} \rightarrow L = 4R^2C$
  - $s_1 = s_2 = -\alpha = -\frac{R}{2L}$
  - $i_L = \underline{(A_1 t + A_2)e^{-\alpha t}} + \underline{I_s}$
- Complex roots: $\alpha^2 - \omega_0^2 < 0 \rightarrow \frac{1}{4R^2C^2} < \frac{1}{LC} \rightarrow L < 4R^2C$;
  - $s_1 = s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.
  - $i_L = \underline{e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))} + \underline{I_s}$

Step Response; RLC in parallel

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN



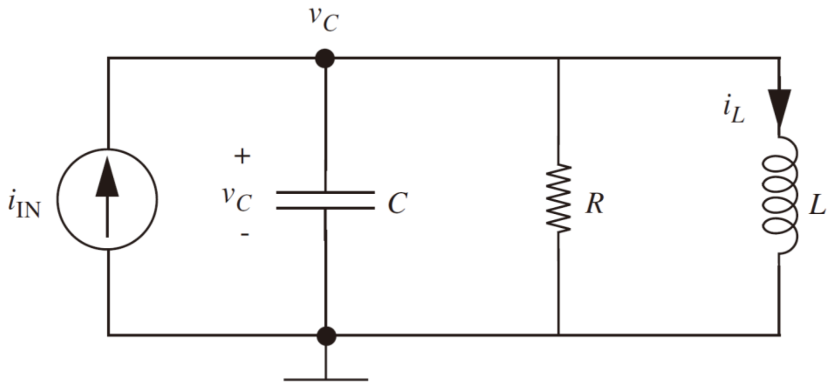
$$L = 1.0\,\mathrm{H};\ i_L\ |_{t=0} = 0$$

$$C = 1.0\,\mathrm{F};\ v_C\ |_{t=0} = 0$$

$$R = \begin{cases} 2.0\,\Omega; & \text{under-damped} \\ 0.5\,\Omega; & \text{critically-damped} \\ 0.2\,\Omega; & \text{over-damped} \end{cases}$$

State-variable Method

- When the circuit states are of primary interest, we can obtain the equations which govern the state evolution, and hence a more direct way to determine the states themselves .

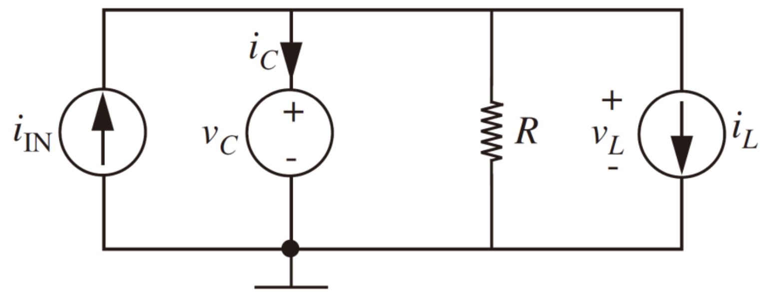


- To find the state equations for the parallel RLC circuit shown above, first, let's chose v_C and i_L as State variables.
- To analyze this circuit we replace the capacitor by a voltage source and the inductor by a current source.

State Equations

- To analyze this circuit, first we replace the capacitor by a voltage source and the inductor by a current source.
- How to find State Equations for $v_C(t)$ and $i_L(t)$?
- Find the corresponding i_C and v_L for state $v_C(t)$ and $i_L(t)$ and excitations i_{IN}.

	$v_C(t)$	$i_L(t)$	i_{IN}
$i_C=dv_C/dt$	$-1/R$	-1	1
$v_L=di_L/dt$	1	0	0



- State Equations for finding $v_C(t)$ and $i_L(t)$.

$$C\frac{dv_C}{dt}=i_C=-\frac{v_C}{R}-i_L+i_{IN}$$

$$L\frac{di_L}{dt}=v_L=v_C-0i_L+0i_{IN}$$

$$\begin{bmatrix}\frac{dv_C}{dt}\\ \frac{di_L}{dt}\end{bmatrix}=\begin{bmatrix}-\frac{1}{RC} & -\frac{1}{C}\\ \frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}v_C\\ i_L\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}i_{IN}\end{bmatrix}$$

- To analyze this circuit we replace the capacitor by a voltage source and the inductor by a current source

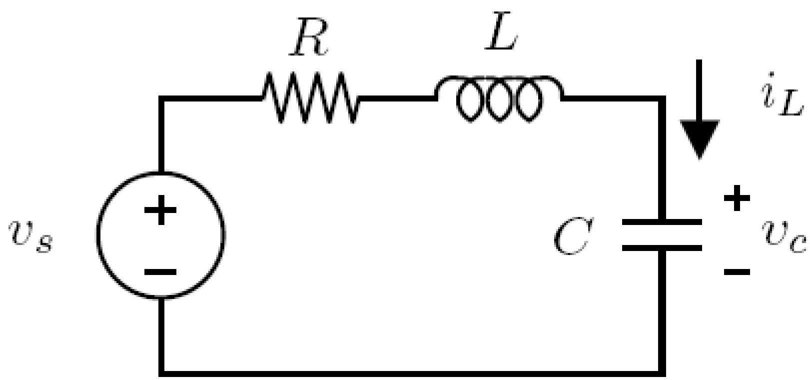
CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

Summary

- Second Order Circuits have two energy storage elements.
  - Natural frequency ω_0. $\omega_0 = \sqrt{\frac{1}{LC}}$
  - Damping factor α. Series RLC: $\alpha = \frac{R}{2L}$ and Parallel RLC: $\alpha = \frac{1}{2RC}$
  - Damped natural frequency. $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
  - Quality factor. $Q = \frac{\omega_0}{2\alpha}$
- Natural Response depends on circuit parameters and the initial conditions of energy storage elements; There are two energy storage elements.
  - Over damped, $\alpha > \omega_0$.
  - Critically damped, $\alpha = \omega_0$.
  - Under damped, $\alpha < \omega_0$.
- The Intuitive method.
- State-variable method.

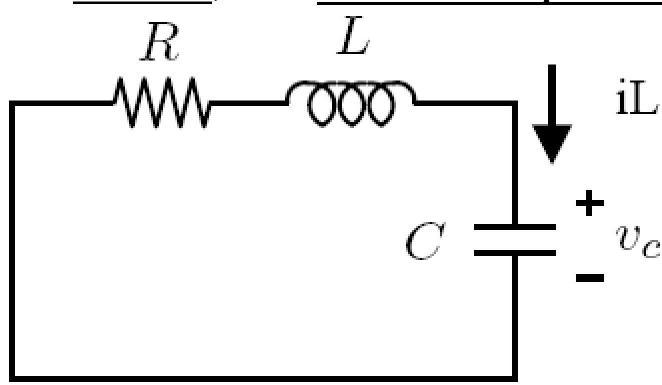
Natural Response; RLC in series

- The *RLC* circuit can be analyzed by KVL:



$$v_s = Ri_L + v_L + v_c$$
$$= RC\frac{dv_c}{dt} + LC\frac{d^2v_c}{dt^2} + v_c$$
$$\rightarrow \frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{v_s}{LC}$$

- For $v_s = 0$, the natural response of the circuit can be derived:



$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = 0$$

Natural Response; RLC in series

Assume $v_c = Ae^{st}$,

$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC} v_c = 0$$

$$\rightarrow \underline{s^2 Ae^{st} + \frac{R}{L} s\, Ae^{st} + \frac{1}{LC} Ae^{st}} = 0$$

$$\rightarrow \underline{Ae^{st}(s^2 + \frac{R}{L} s + \frac{1}{LC})} = 0$$

$$\rightarrow s_1, s_2 = \underline{-\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}}; \underline{-\frac{R}{2L} - \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}}$$

$$= \underline{-\alpha + \sqrt{\alpha^2 - \omega_0^2}}; \quad \underline{-\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

$$\text{where } \alpha = \frac{\text{R}}{2\text{L}}, \text{ and } \omega_0 = \sqrt{\frac{1}{LC}}$$

The solution can be expressed as follows:

$$v_c = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Natural Response; RLC in series

Characteristic equation:

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0,\quad s_1,s_2=-\frac{R}{2L}\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2};$$

- Two distinct real roots: $\alpha^2>\omega_0^2 \rightarrow \frac{R^2}{4L^2}>\frac{1}{LC} \rightarrow R^2>\frac{4L}{C}$
  - s_1 and s_2 are negative real numbers.
  - $v_c = \underline{A_1e^{s_1t} + A_2e^{s_2t}}$.
  - v_c decays exponentially without any oscillations; this is referred to as <u>over-damped</u> .
- Double roots: $\alpha^2=\omega_0^2 \rightarrow \frac{R^2}{4L^2}=\frac{1}{LC} \rightarrow R^2=\frac{4L}{C}$
  - $s_1 = s_2 = -\alpha = -\frac{R}{2L}$
  - $v_c = \underline{(A_1t + A_2)e^{-\alpha t}}$.
  - v_c decays at a moderate pace, this is referred to as <u>critically-damped</u>

Natural Response; RLC **in series**

Characteristic equation:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0, \quad s_1, s_2 = -\frac{R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

- Two complex roots: $\alpha^2 - \omega_0^2 < 0 \rightarrow R^2 < \frac{4L}{C}$

  $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d$,where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

  Let

$$y_1 = e^{(-\alpha + j\omega_d)t} = e^{-\alpha t}(\cos(\omega_d t) + j\sin(\omega_d t))$$
$$y_2 = e^{(-\alpha - j\omega_d)t} = e^{-\alpha t}(\cos(\omega_d t) - j\sin(\omega_d t))$$

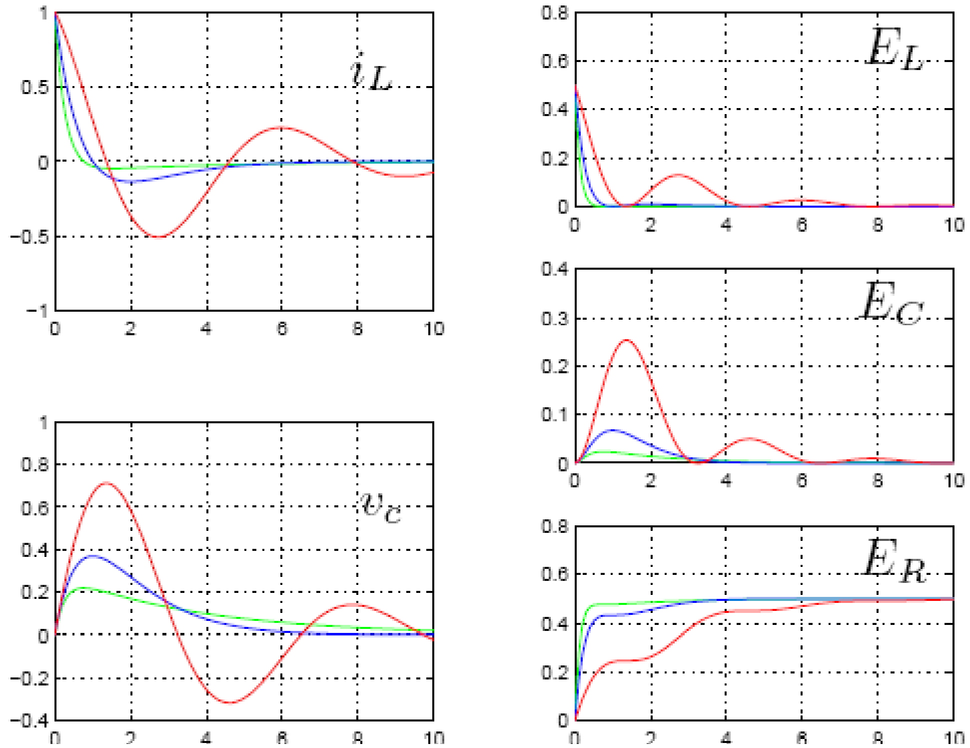
Any linear combination of y_1 and y_2 will be a valid solution to the differential equation.

$$v_c = A_1(\frac{1}{2}(y_1 + y_2)) + A_2(\frac{1}{j2}(y_1 - y_2)) = e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))$$

v_c oscillates within an exponentially-decaying envelope, this is referred to as under-damped

Natural Response; RLC in series

CAPT
Center for Advanced Power Technologies
National Tsing Hua University, TAIWAN

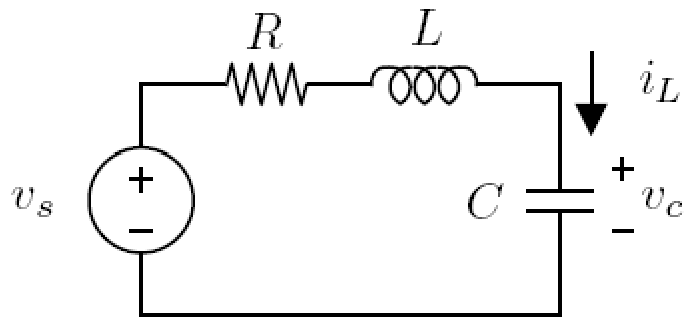


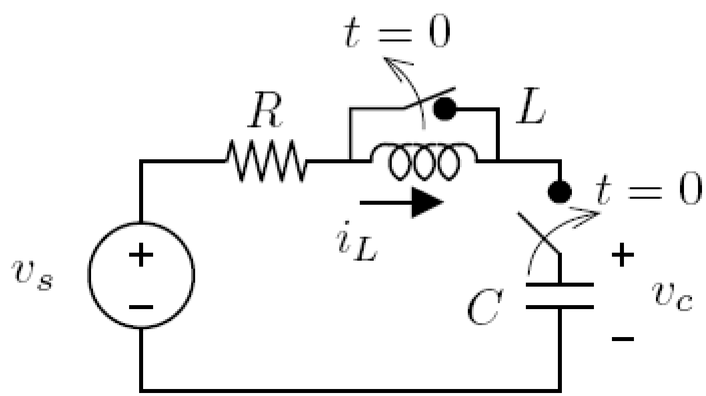
$$L = 1.0\,\mathrm{H}; \quad i_L\,|_{t=0} = 1\,\mathrm{A}$$

$$C = 1.0\,\mathrm{F}; \quad v_C\,|_{t=0} = 0$$

$$R = \begin{cases} 0.5\,\Omega; & \text{under-damped} \\ 2.0\,\Omega; & \text{critically-damped} \\ 4.0\,\Omega; & \text{over-damped} \end{cases}$$

Step Response; RLC in series





$$v_s = V_s u(t)$$
$$v_c \,\big|_{t=0} = V_{c0}$$
$$i_L \,\big|_{t=0} = I_{L0}$$

By KVL,

$$v_s = Ri_L + v_L + v_C = RC\frac{dv_c}{dt} + LC\frac{d^2 v_c}{dt^2} + v_c \rightarrow \quad \frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{v_s}{LC}$$

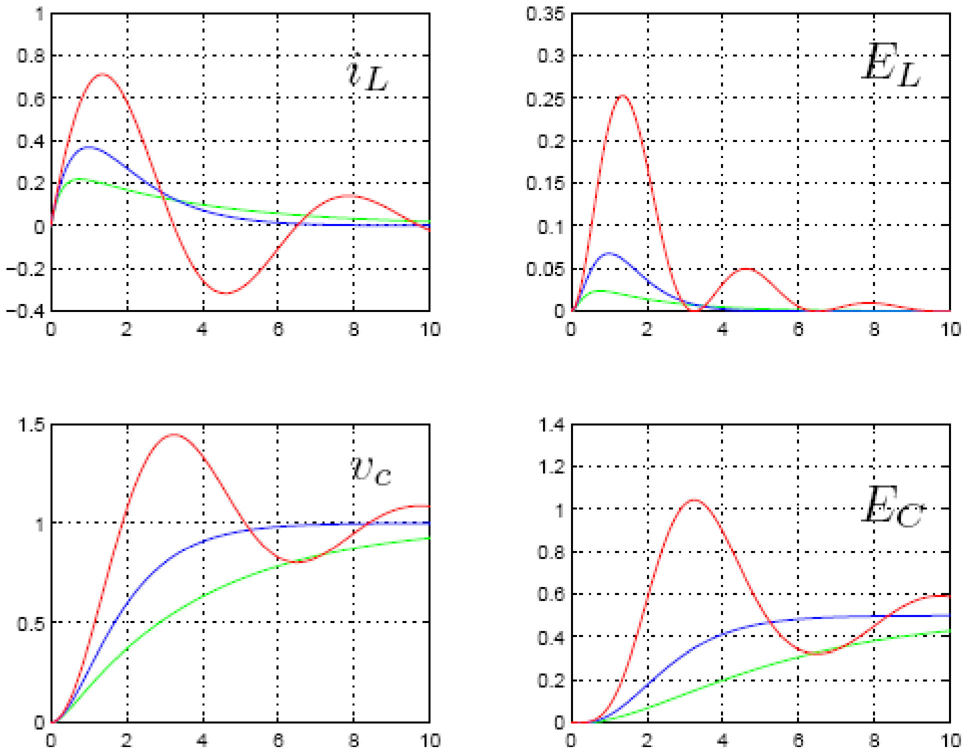
Step Response; RLC in series

$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{v_s}{LC} = \frac{V_s u(t)}{LC}$$

Step response at steady state: $v_{cf} = \underline{V_s}$

- Two distinct real roots: $\alpha^2 > \omega_0^2 \rightarrow \frac{R^2}{4L^2} > \frac{1}{LC} \rightarrow R^2 > \frac{4L}{C}$;
  - s_1 and s_2 are negative real numbers.
  - $v_c = \underline{A_1 e^{s_1 t} + A_2 e^{s_2 t}} + \underline{V_s}$
- Double roots: $\alpha^2 = \omega_0^2 \rightarrow \frac{R^2}{4L^2} = \frac{1}{LC} \rightarrow R^2 = \frac{4L}{C}$;
  - $s_1 = s_2 = -\alpha = -\frac{R}{2L}$
  - $v_c = \underline{(A_1 t + A_2)e^{-\alpha t}}. + \underline{V_s}$
- Complex roots: $\alpha^2 - \omega_0^2 < 0 \rightarrow R^2 < \frac{4L}{C}$;
  - $s_1 = s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.
  - $v_c = \underline{e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))} + \underline{V_s}$

Step Response; RLC in series



$$L = 1.0\,\mathrm{H};\ i_L\ |_{t=0} = 0;$$
$$C = 1.0\,\mathrm{F};\ v_C\ |_{t=0} = 0;$$

$$R = \begin{cases} 0.5\,\Omega; & \text{under-damped} \\ 2.0\,\Omega; & \text{critically-damped} \\ 4.0\,\Omega; & \text{over-damped} \end{cases}$$