First Order Circuits

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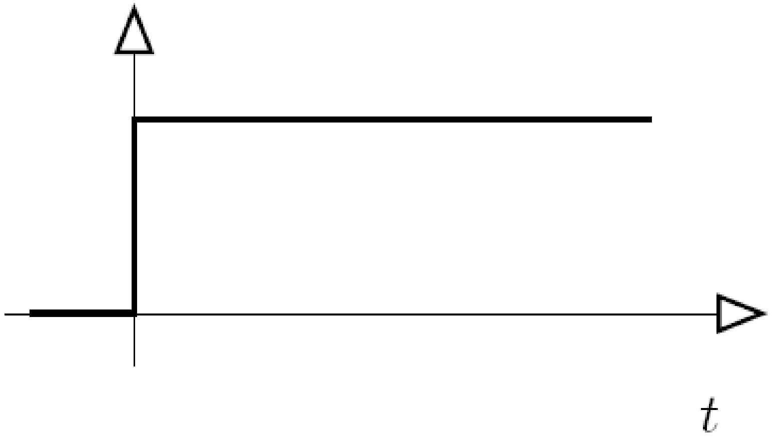
Overview

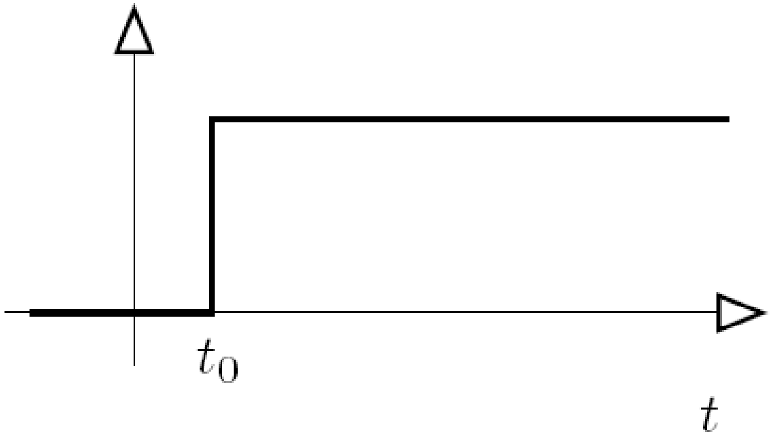
- Excitations
- First order circuits are characterized by first order differential equations.
- Natural response of first order circuits:
  - $R - C$ circuit;
  - $R - L$ circuit.
- Forced response of first order circuits:
  - $R - C$ circuit;
  - $R - L$ circuit.
- Intuitive Method
- State and Memory.

Excitations

- Step function;



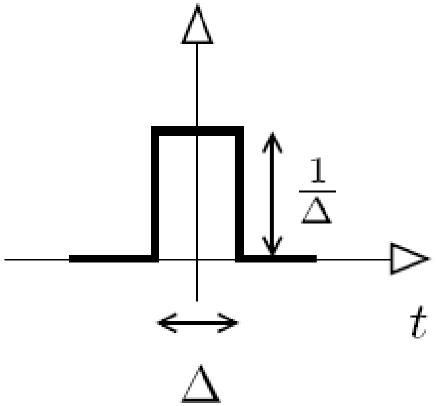
$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$$u(t - t_0) = \begin{cases} 0 & \text{if } t < t_0 \\ 1 & \text{if } t \geq t_0 \end{cases}$$

Excitations

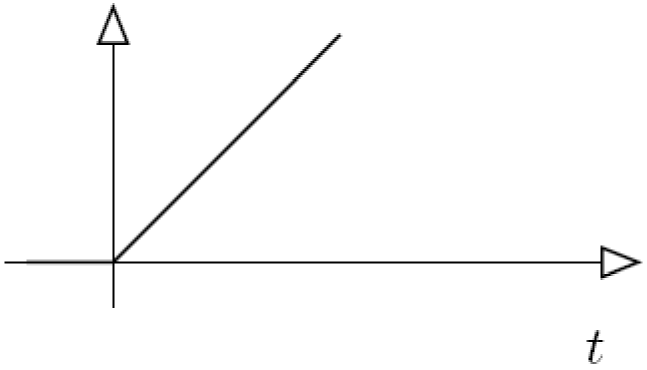
- Impulse function



$$\delta(t) = \frac{du(t)}{dt} = \begin{cases} 0 & t > \frac{\Delta}{2},\ t < -\frac{\Delta}{2},\ \Delta \to 0. \\ \frac{1}{\Delta} & -\frac{\Delta}{2} < t < \frac{\Delta}{2}, \Delta \to 0. \end{cases}$$

$$\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \delta(t)dt = 1$$

- Ramp function



$$r(t) = \int u(t)dt = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$

RC Circuit

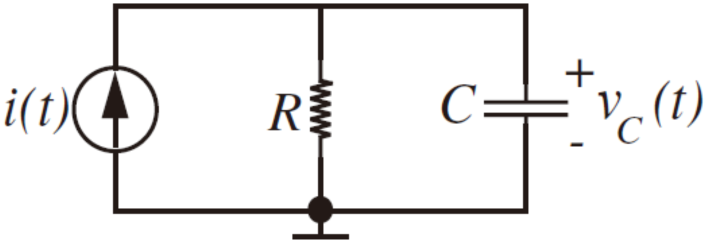
- Apply node method:

$$i(t) = \frac{v_C}{R} + C\frac{dv_C}{dt}$$

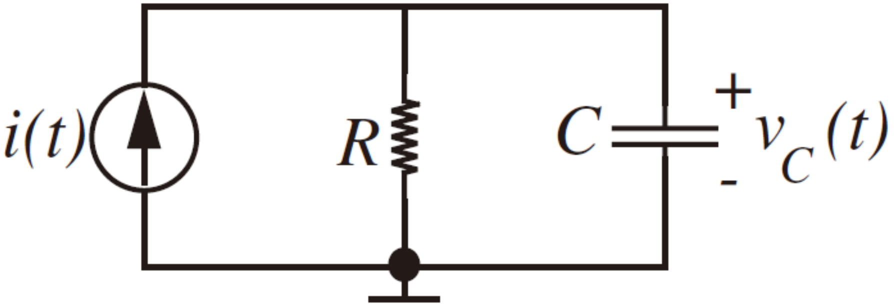
$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{i(t)}{C}$$

$$i(t) = I_0 u(t)$$

$$\boxed{\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_0}{C} \quad \text{for } t \geq 0}$$



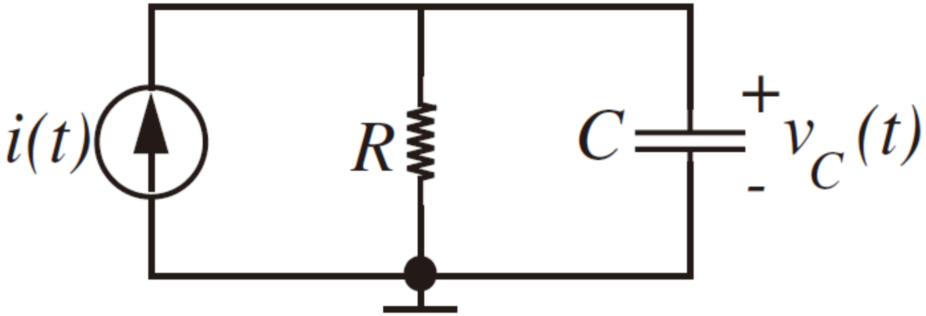
Method of homogeneous and particular solutions



$$\boxed{\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_0}{C} \text{ for } t \geq 0}$$

- Find the particular solution, v_{CP}.
- Find the homogeneous solution , v_{CH}.
- The total solution is the sum of the particular and homogeneous solutions , $v_C = v_{CP} + v_{CH}$.
- Use the initial conditions to solve for the remaining constants.

The Particular solution



$$\boxed{\frac{dv_C}{dt}+\frac{v_C}{RC}=\frac{I_0}{C} \quad \text{for } t\geq 0}$$

- The particular v_{CP} solution is also called the ***forced response*** or the forced solution because it depends on the external inputs to the circuit.
- Find the particular solution, $\frac{dv_{CP}}{dt}+\frac{v_{CP}}{RC}=\frac{I_0}{C}$
- v_{CP} :any solution that satisfies the above equation.
- Use trial and error : Try $v_{CP}=K$, .

$$\frac{dK}{dt}+\frac{K}{RC}=\frac{I_0}{C} \Rightarrow 0+\frac{K}{RC}=\frac{I_0}{C} \Rightarrow K=I_0R \Rightarrow v_{\mathrm{CP}}=I_0R$$

Forced Response

- The forced response of a circuit is its behavior (in terms of voltages and currents) under <u>external sources of excitation</u> .
- The forced response of a circuit depends on:
  - Parameters of <u>circuit components</u> ;
  - <u>Initial conditions</u> of <u>energy storage components</u> within the circuit;
  - <u>Forms</u> of external excitations.
- The forced response of a circuit can be described by a <u>non-homogeneous</u> differential equation.
- A general solution $y(x)$ of the linear non-homogeneous differential equation is the sum of a general solution of the corresponding homogeneous solution and an arbitrary particular solution.

The Homogeneous Solution

- The homogeneous solution, v_{CH}, is also called the ***natural response*** of the circuit because it depends only on the internal energy storage properties of the circuit and not on external inputs.
- Find the homogeneous solution , v_{CH}.

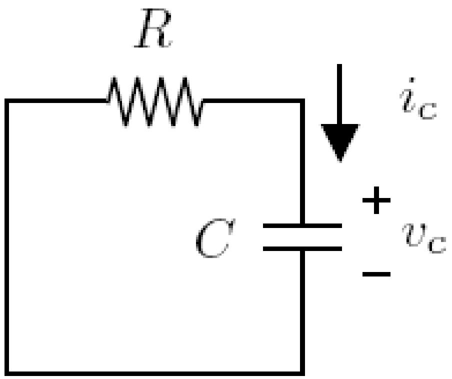
$$\frac{dv_{CH}}{dt} + \frac{v_{CH}}{RC} = 0$$

- Assume solution is of this form : $v_{CH} = Ae^{st}$

$$\frac{dAe^{st}}{dt} + \frac{Ae^{st}}{RC} = 0 \Rightarrow sAe^{st} + \frac{Ae^{st}}{RC} = 0 \Rightarrow s + \frac{1}{RC} = 0 \Rightarrow \quad RCs + 1 = 0$$

- Characteristic equation: $\boxed{RCs + 1 = 0}$ $\Rightarrow s = -\frac{1}{RC} = -\frac{1}{\tau}$
- The homogeneous solution , v_{CH}:

$$v_{CH} = Ae^{-\frac{t}{RC}}$$

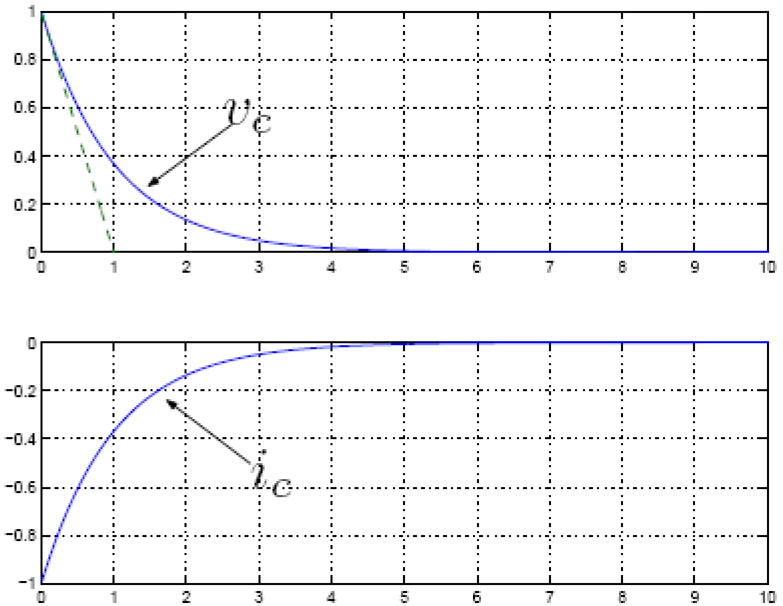


- RC is called time constant τ.

Natural response

- The natural response of a circuit is the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation ;
- The natural response depends on:
  - Component parameters of the circuit;
  - Initial conditions of the energy storage components within the circuit.
- The natural response of a first order circuit can be described by a homogeneous first order differential equation.

Natural response of RC **circuit**



$$v_C = V_0 e^{-\frac{1}{RC}t}$$

$$i_C = C\frac{dv_C}{dt} = -\frac{V_0}{R} e^{-\frac{1}{RC}t}$$

$$P_R = v_C \cdot (-i_C) = \frac{V_0^2}{R} e^{-\frac{1}{RC}2t}$$

$$W_R = \int P_R\, dt = \frac{1}{2} C V_0^2 (1 - e^{-\frac{2t}{RC}})$$

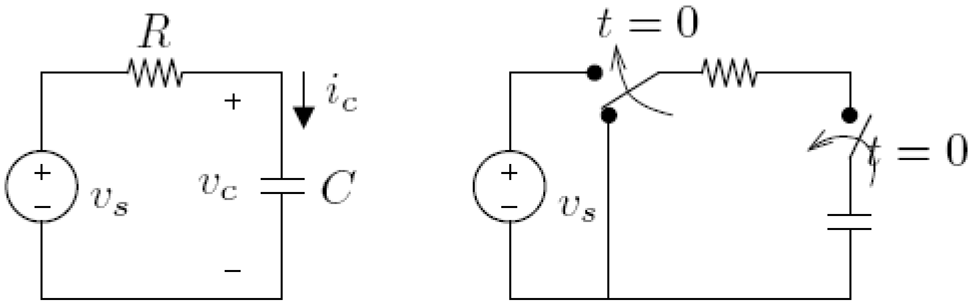
$$R = 1.0\Omega; C = 1.0F; V_0 = 1.0V$$

- Time constant $\underline{\tau = RC}$ indicates the speed of the decay.
- v_c decays to $\underline{36.8\%}$ of its initial value V_{c0} at $\underline{t = \tau}$.
- Tangent of v_c at $t = 0$ intersects the time axis at $t = \tau$.

$$\frac{dv_C}{dt}\bigg|_{t=0} = -\frac{V_0}{\tau}$$

- Because $e^{-5} = 0.0067$, it is common to assume that the system reaches steady state after $\underline{t = 5\ \tau}$.

Step Response; RC circuit



$$v_S = V_S\,u(t)$$
$$v_C\,|_{t=0} = V_{C0}$$

By KVL, $v_s = \underline{Ri + v_c} = \underline{RC\frac{dv_c}{dt} + v_c} \rightarrow \underline{\frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{v_s}{RC}}$

Homogeneous solution :

$$v_{CH} = \underline{Ae^{-\frac{t}{RC}}}$$

Particular solution :

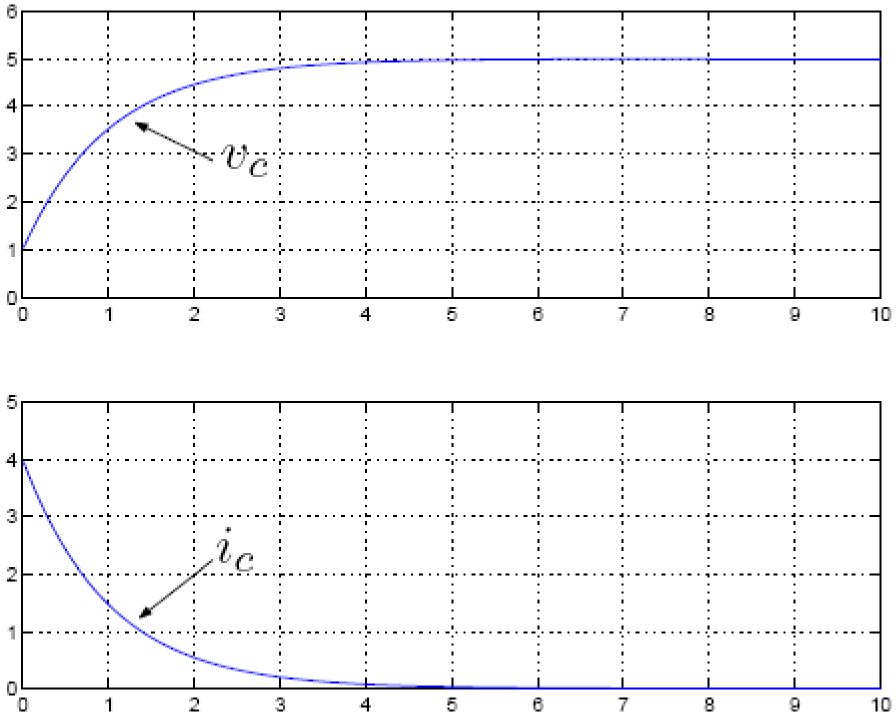
$$v_{CP} = \underline{V_S}$$

Total solution :

$$v_C = v_{CH} + v_{CP} = V_S + Ae^{-\frac{t}{RC}}; \quad v_C\,|_{t=0} = \underline{V_S + A = V_{C0}} \rightarrow A = \underline{V_{C0} - V_S}$$

$$\rightarrow v_C = \underline{V_S + (V_{C0} - V_S)}\,e^{-\frac{t}{RC}}$$

Step Response; RC circuit



$$v_c = V_s + (V_{c0} - V_s) e^{-\frac{t}{RC}}$$

$$i_c = C\frac{dv_c}{dt} = -\frac{(V_{c0} - V_s)}{R} e^{-\frac{t}{RC}}$$

$$R = 1.0\,\Omega; \quad C = 1.0\,\mathrm{F};$$

$$v_c|_{t=0} = V_{c0} = 1.0\,\mathrm{V}; \quad V_s = 5\,\mathrm{V}$$

The Total solution

- The total solution is the sum of the particular and homogeneous solutions:

$$v_C = v_{CP} + v_{CH} = I_0 R + Ae^{-\frac{t}{RC}}$$

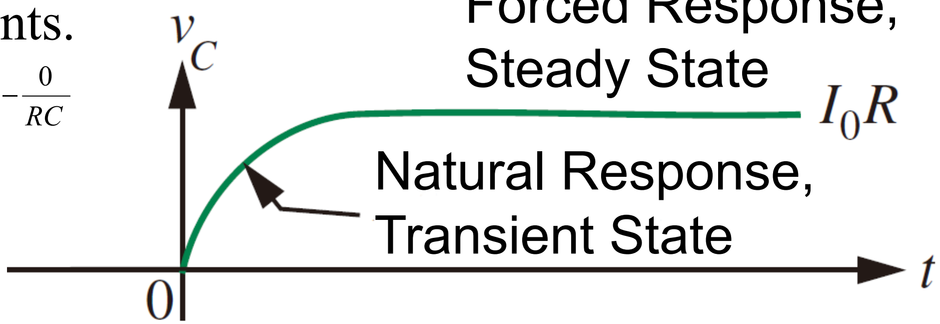
- Use the initial conditions:

$$v_C(0^+) = 0\ \text{V}$$

to solve for the remaining constants.

$$V_0 = 0 = I_0 R + Ae^{-\frac{0}{RC}} = I_0 R + Ae^{-\frac{0}{RC}}$$

$$\Rightarrow \quad A = 0 - I_0 R = -I_0 R$$



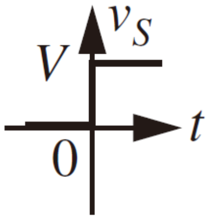
- The total solution v_C :

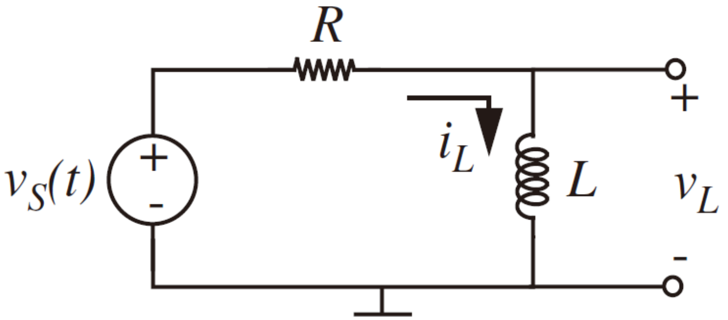
$$\boxed{v_C = I_0 R - I_0 \mathrm{R}e^{-\frac{t}{RC}}} = I_0 R\left(1 - e^{-\frac{t}{RC}}\right)$$

RL Circuit

- Apply KVL:

$$-v_S(t)+Ri_L+L\frac{di_L}{dt}=0$$



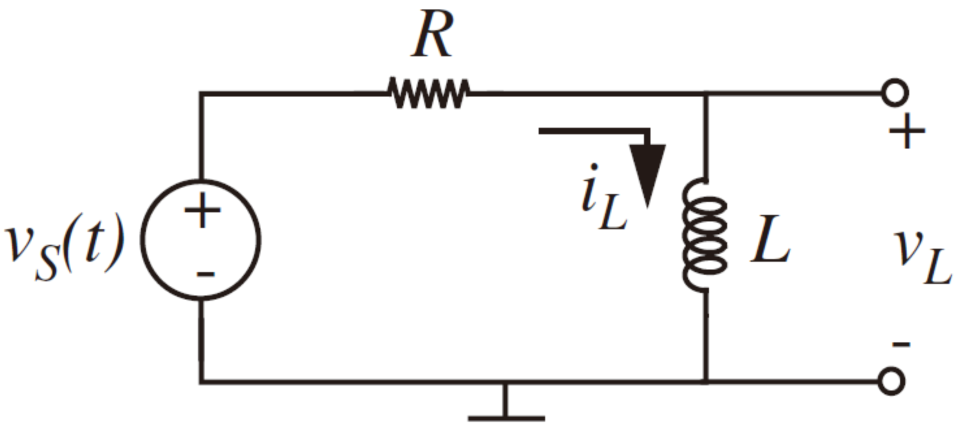


$$\frac{di_L}{dt}+\frac{R}{L}i_L=\frac{v_S(t)}{L}$$

$$v_S(t)=Vu(t)$$

$$\frac{di_L}{dt}+\frac{R}{L}i_L=\frac{V}{L}\quad \text{for } t\geq 0$$

The Particular solution



$$\frac{di_L}{dt}+\frac{R}{L}i_L=\frac{V}{L}\quad \text{for } t\geq 0$$

- The particular i_{LP} solution is also called the ***forced response*** or the forced solution because it depends on the external inputs to the circuit.

- To find the particular solution, $\frac{di_L}{dt}+\frac{R}{L}i_L=\frac{V}{L}$
- i_{LP} :any solution that satisfies the above equation.
- Use trial and error : Try $i_{LP}=K$, .

$$\frac{dK}{dt}+\frac{R}{L}K=\frac{V}{L}\Rightarrow 0+\frac{R}{L}K=\frac{V}{L}\Rightarrow K=\frac{V}{R}\Rightarrow i_{\text{LP}}=\frac{V}{R}$$

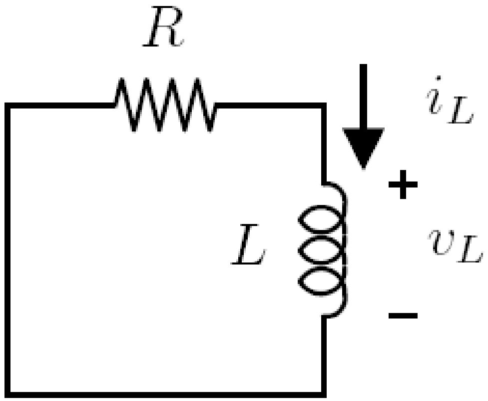
The Homogeneous Solution

- The homogeneous solution, i_{LH}, is also called the ***natural response*** of the circuit because it depends only on the internal energy storage properties of the circuit and not on external inputs.
- Find the homogeneous solution , v_{CH}. $\frac{di_{LH}}{dt}+\frac{R}{L}i_{LH}=0$
- Assume solution is of this form : $i_{LH}=Ae^{st}$

$$\frac{dAe^{st}}{dt}+\frac{R}{L}Ae^{st}=0 \Rightarrow sAe^{st}+\frac{R}{L}Ae^{st}=0 \Rightarrow s+\frac{R}{L}=0 \Rightarrow \frac{L}{R}s+1=0$$

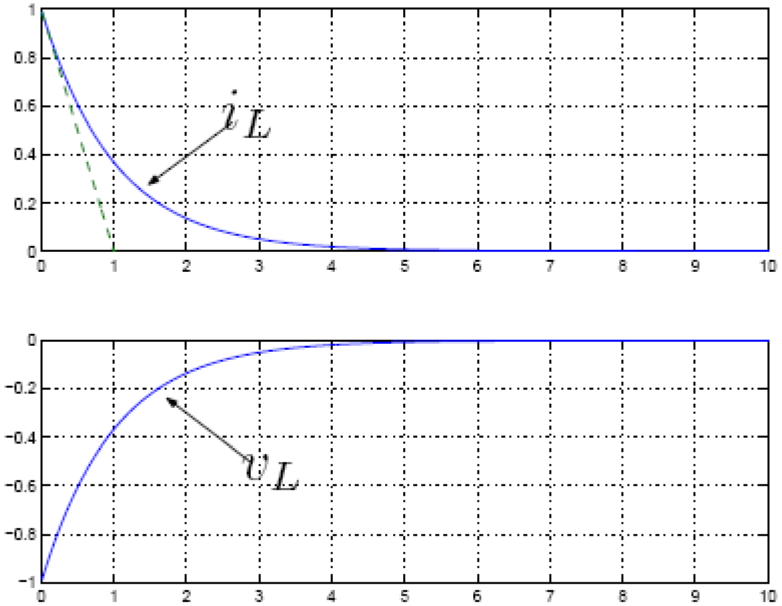
- Characteristic equation: $\boxed{\frac{L}{R}s+1=0}$ $\Rightarrow s=-\frac{R}{L}=-\frac{1}{\tau}$
- The homogeneous solution , i_{LH}:

$$i_{LH}=Ae^{-\frac{R}{L}t}$$



- L/R is called time constant τ.

Natural response; RL circuit



$$i_L = I_{L0} e^{-\frac{1}{(L/R)}t}$$

$$v_L = L\frac{di_L}{dt} = -I_{L0} R e^{-\frac{1}{(L/R)}t}$$

$$P_R = I_{L0}^2 R e^{-\frac{2t}{(L/R)}}$$

$$W_R = \frac{1}{2} L I_{L0}^2 (1 - e^{-\frac{2t}{(L/R)}})$$

$$R = 1.0\,\Omega\,;\, L = 1.0\,\text{H}\,;\, I_{L0} = 1.0\text{A}$$

- Time constant $\underline{\tau = L/R}$ indicates the speed of the decay.
- i_L decays to $\underline{36.8\%}$ of its initial value I_{L0} at $\underline{t = \tau}$.
- Tangent of i_L at $t = 0$ intersects the time axis at $t = \tau$.
- i_L reaches steady state after $\underline{t = 5\,\tau}$.

The Total solution

- The total solution is the sum of the particular and homogeneous solutions:

$$i_L = i_{LP} + i_{LH} = \frac{V}{R} + Ae^{-\frac{R}{L}t}$$

- Use the initial conditions:

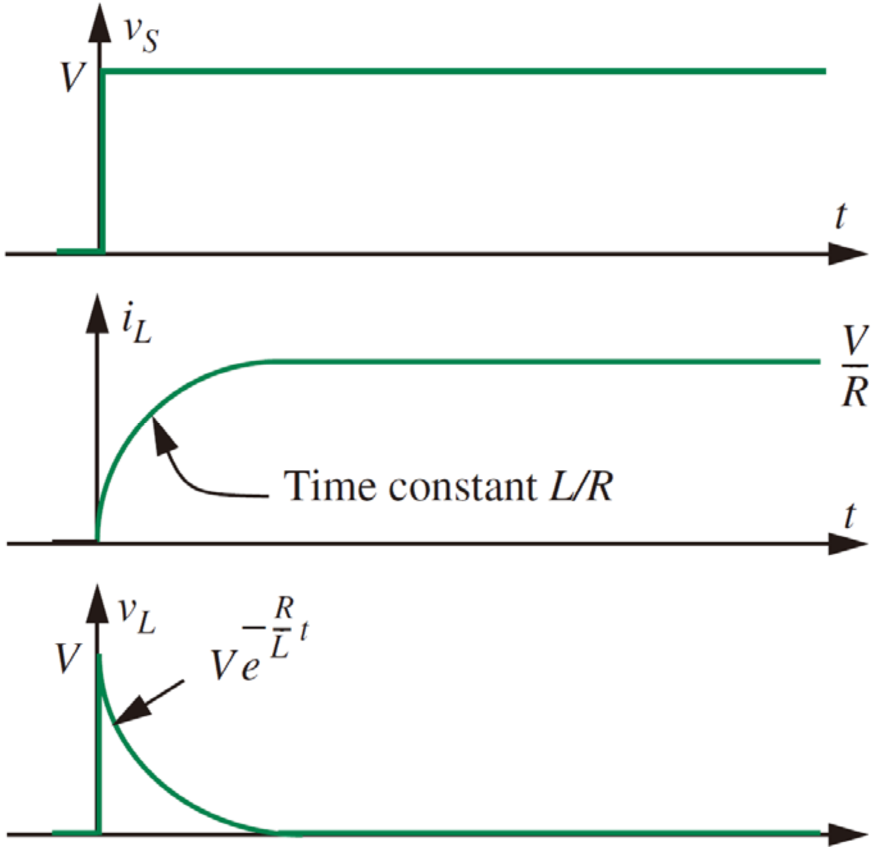
$$i_L(0^+) = 0 \text{ A}$$

to solve for the remaining constants.

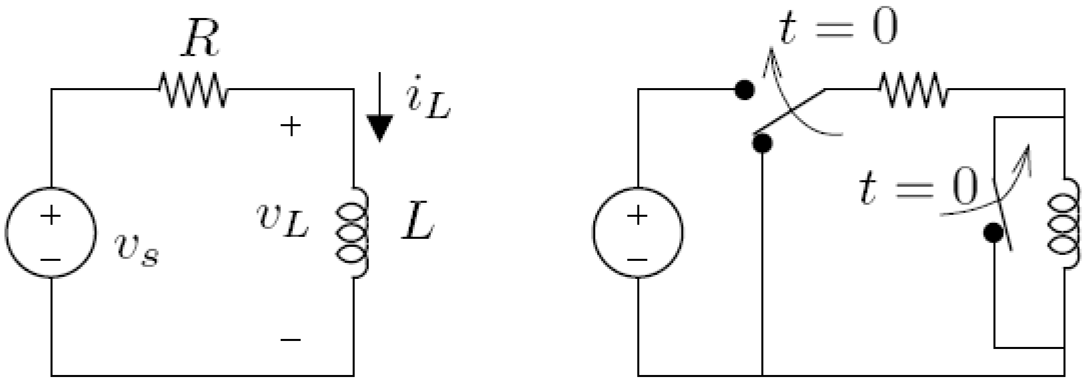
$$0 = \frac{V}{R} + Ae^{-\frac{R}{L}0} \Rightarrow A = -\frac{V}{R}$$

- The total solution i_L (and v_L):

$$\boxed{i_L = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}} = I_0R\left(1 - e^{-\frac{t}{RC}}\right) \quad \boxed{v_L = L\frac{di_L}{dt} = Ve^{-\frac{R}{L}t}}$$



Step Response; RL circuit



$v_s = V_s u(t)$

$i_L|_{t=0} = I_{L0}$

By KVL, $v_s = \underline{Ri + v_L} = \underline{Ri + L\frac{di_L}{dt}} \rightarrow \underline{\frac{di_L}{dt} + \frac{R}{L}i = \frac{v_s}{L}}$

Homogeneous solution :

$i_{LH} = \underline{Ae^{-\frac{t}{L/R}}}$

Particular solution :

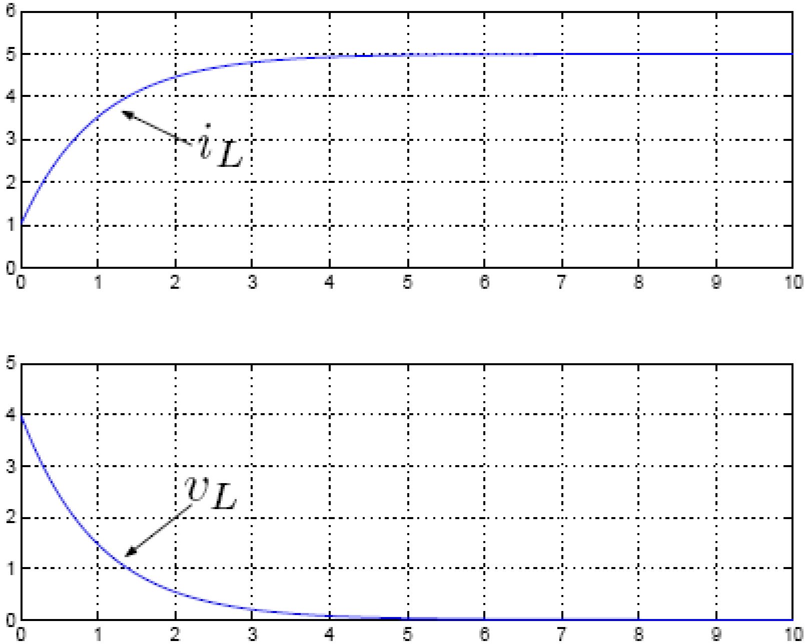
$i_{LP} = \frac{V_S}{R}$

Total solution :

$$i_L = i_{LP} + i_{LH} = \frac{V_S}{R} + Ae^{-\frac{t}{L/R}};\quad i_L|_{t=0} = \underline{\frac{V_S}{R} + A = I_{L0}} \rightarrow \underline{A = I_{L0} - \frac{V_S}{R}}$$

$$\rightarrow i_L = \underline{\frac{V_S}{R} + (I_{L0} - \frac{V_S}{R})}e^{-\frac{t}{L/R}}$$

Step Response; RL circuit



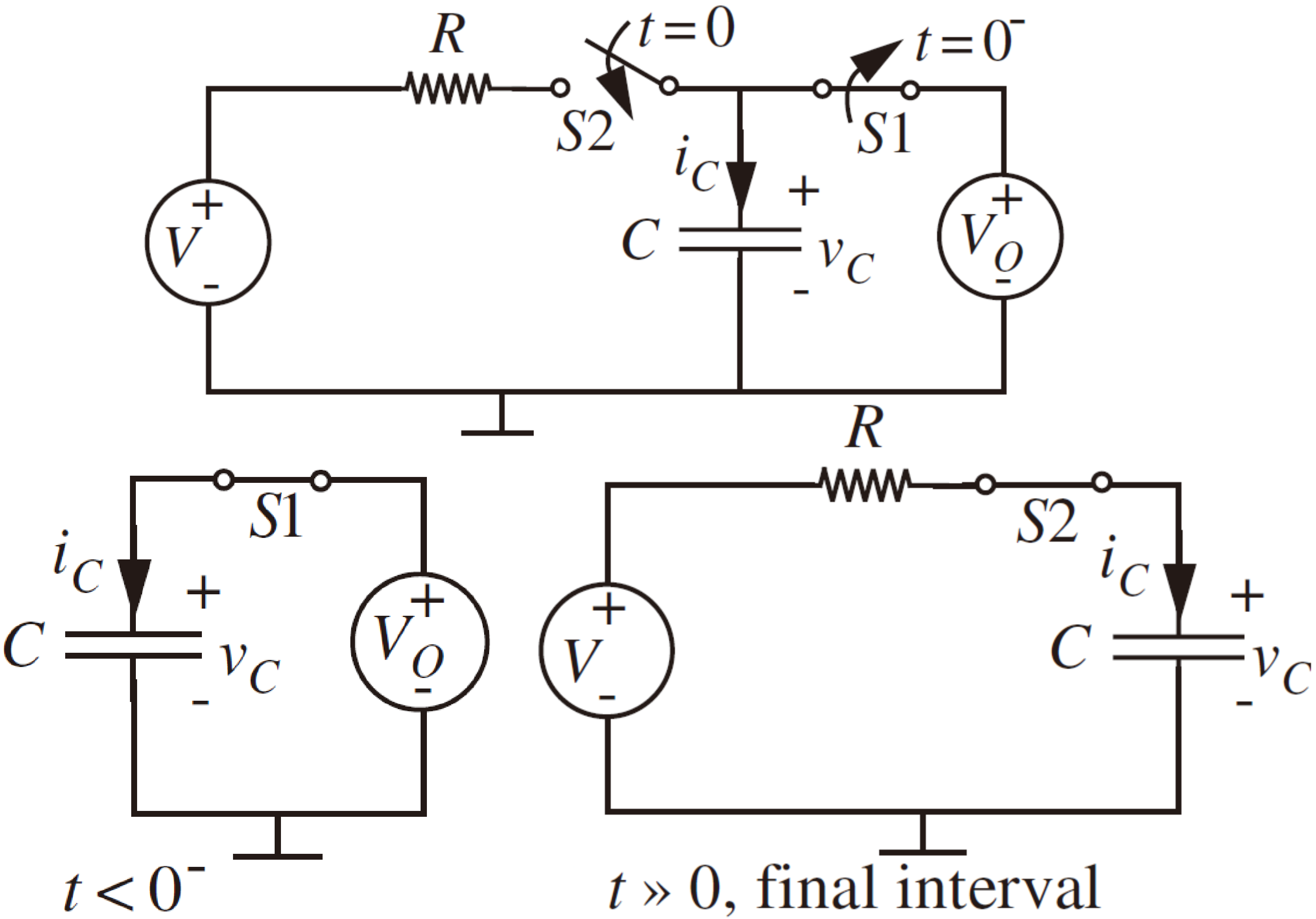
$$i_L = \frac{V}{R} + (I_{L0} - \frac{V}{R}) e^{-\frac{t}{L/R}}$$

$$v_L = L\frac{di_L}{dt} = (-R)(I_{L0} - \frac{V}{R}) e^{-\frac{t}{L/R}}$$

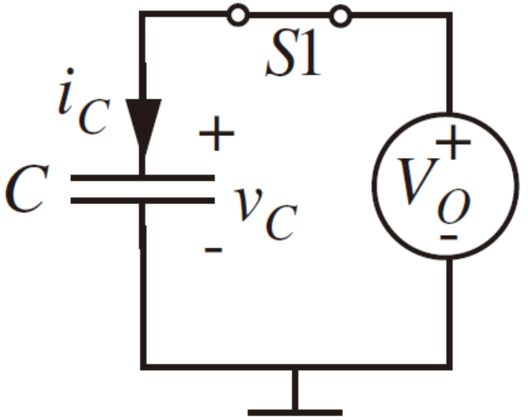
$$R = 1.0\,\Omega; \quad L = 1.0\,\mathrm{H};$$

$$i_L|_{t=0} = I_{L0} = 1.0\,\mathrm{A}; \quad V = 5\,\mathrm{V}$$

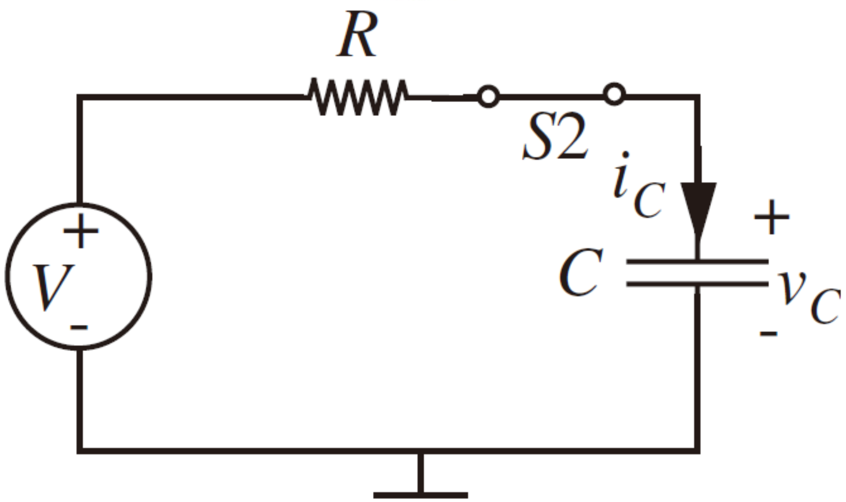
Intuitive Method



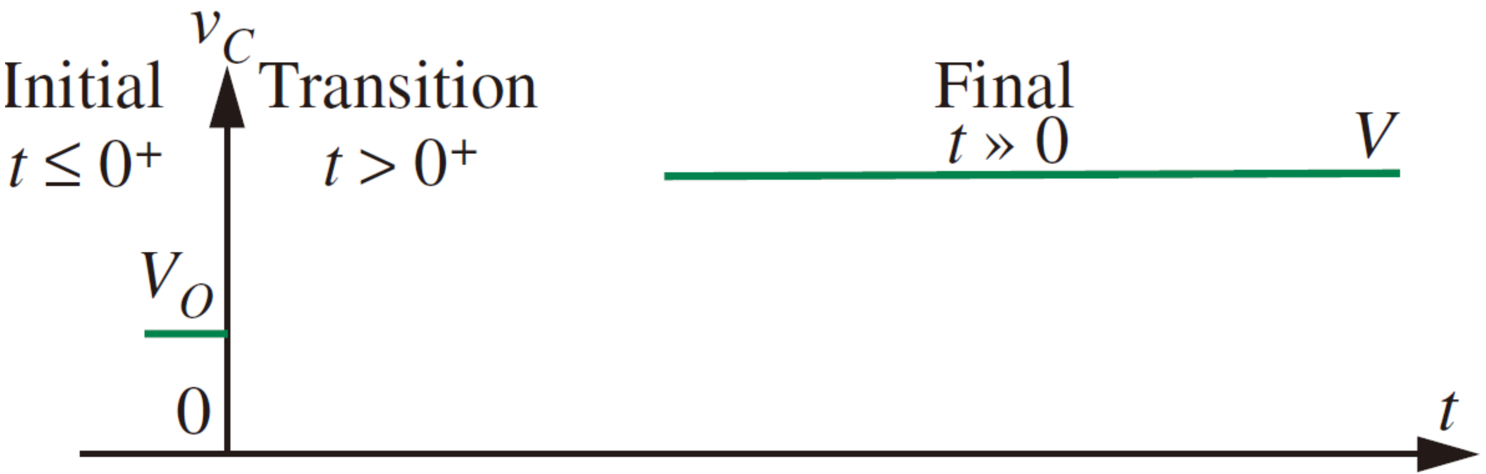
Intuitive Method



$t < 0^-$

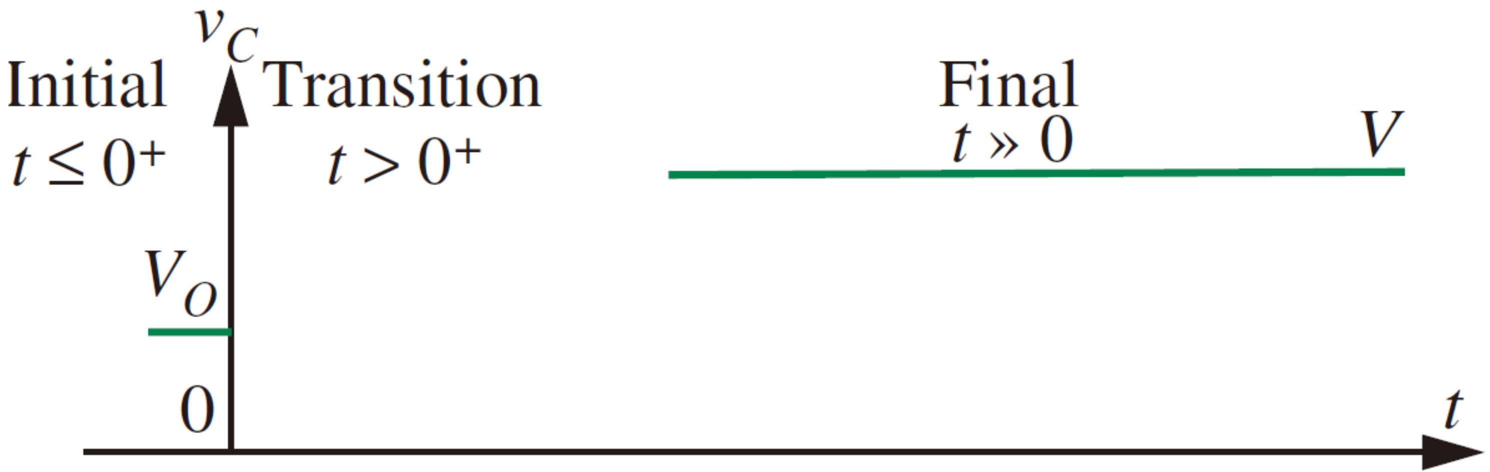


t » 0, final interval

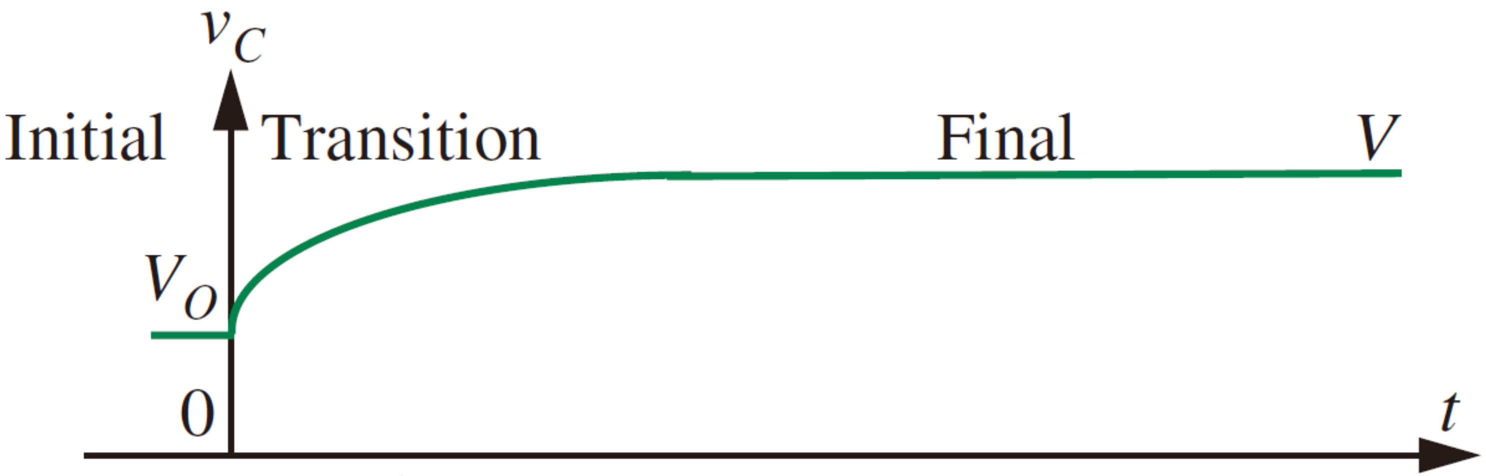


Initial, transition, and final intervals

Intuitive Method

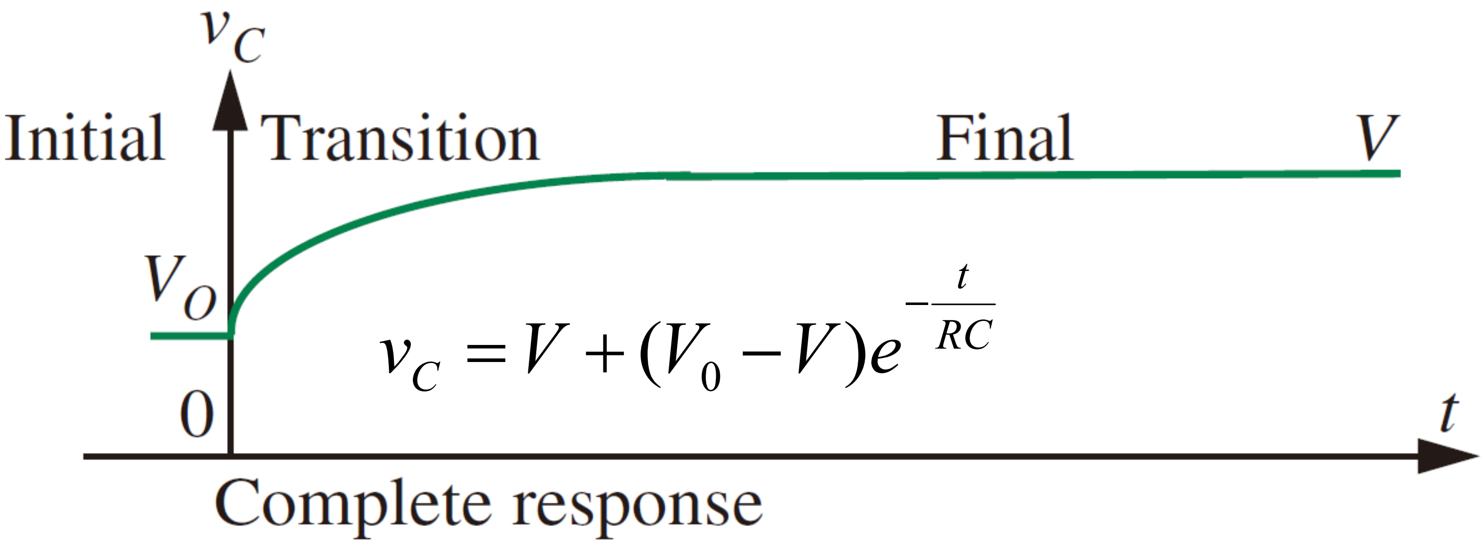


Initial, transition, and final intervals



Complete response

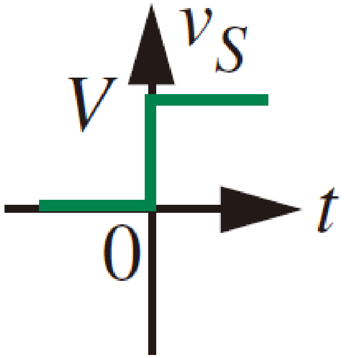
Intuitive Method

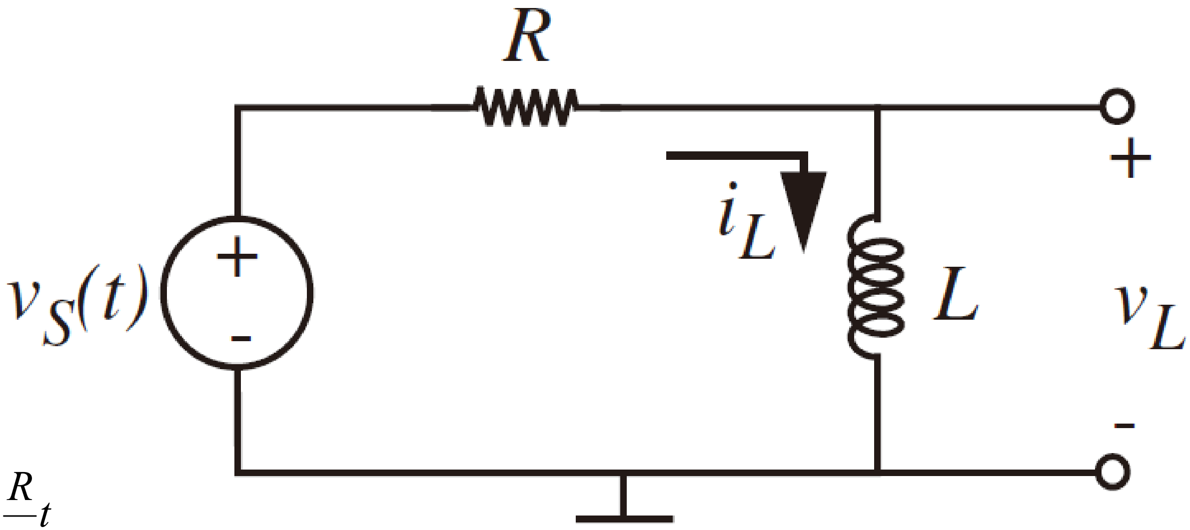


$$v = Final\,Value + (Initial\,Value - Final\,Value)e^{-\frac{t}{\tau}}$$

$$v = (Initial\,Value)e^{-\frac{t}{\tau}} + Final\,Value(1 - e^{-\frac{t}{\tau}})$$

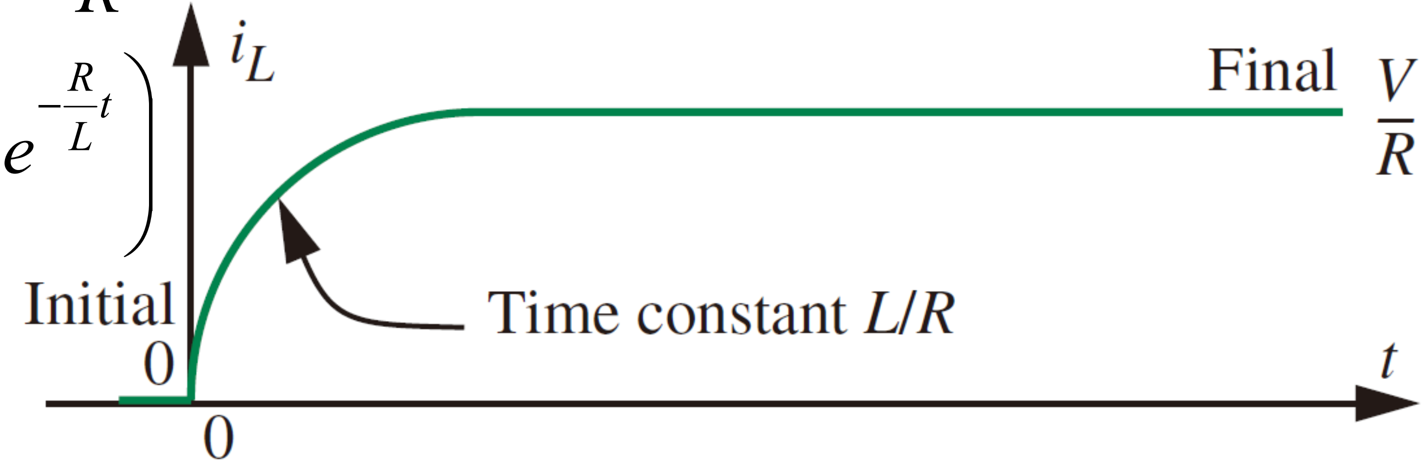
Example





$$i_L = \frac{V}{R} + (0 - \frac{V}{R})e^{-\frac{R}{L}t}$$

$$i_L = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$



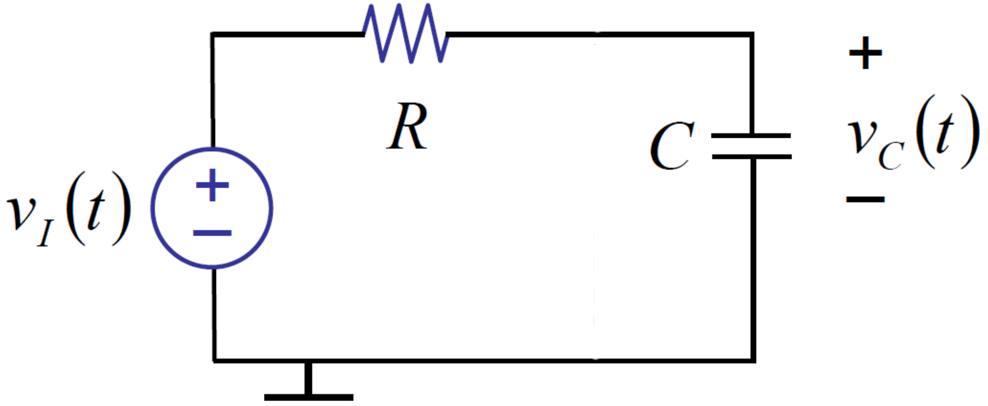
The RC Circuit

$$\frac{v_C - v_I}{R} + C\frac{dv_C}{dt} = 0$$

Given:

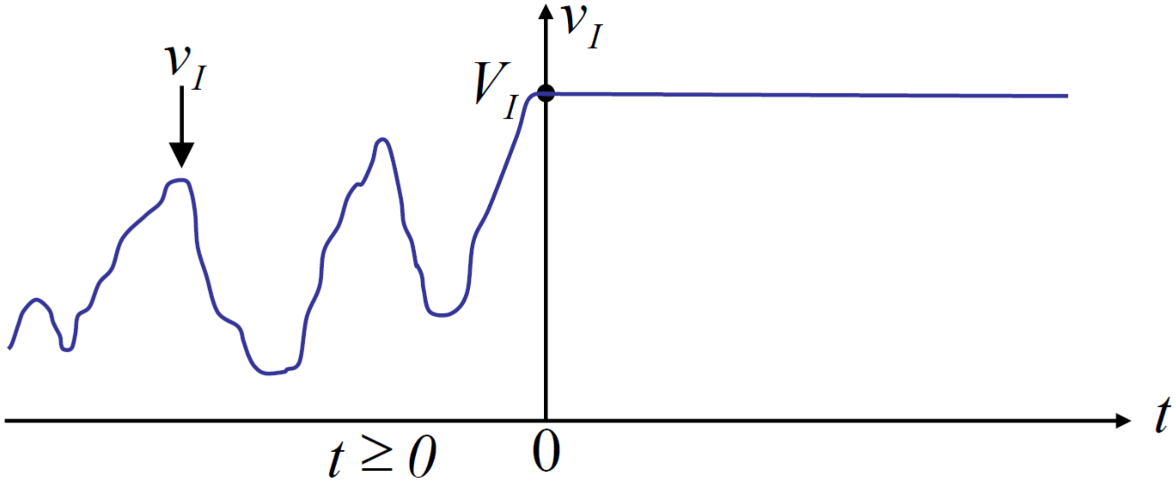
$$v_I(t) = V_I = 5\text{ V for } t \geq 0$$

$$v_C(0) = V_0 = 0\text{ V}$$

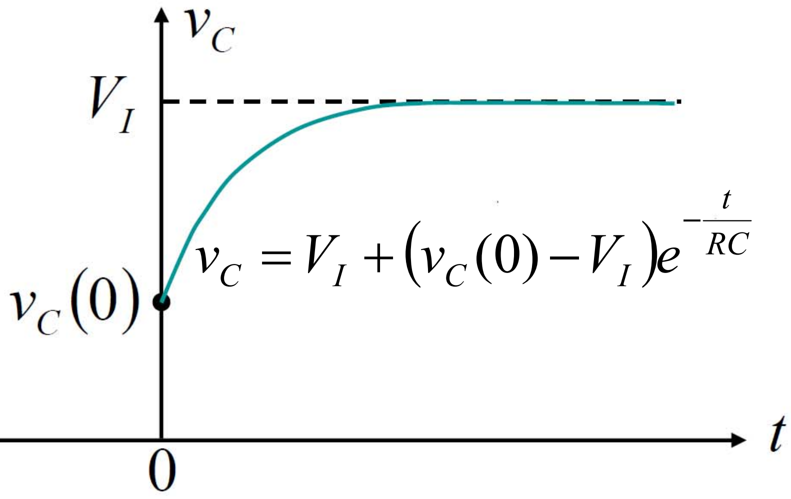


$$\boxed{v_C = V_I + (V_0 - V_I)e^{-\frac{t}{RC}}}$$

The RC Circuit



Notice that the capacitor voltage for $t \geq 0$ is independent of the form of the input voltage before $t = 0$.
Instead, it depends only on the capacitor voltage at $t = 0$, and the input voltage for $t \geq 0$.



$$v_C = V_I + \left(v_C(0) - V_I\right)e^{-\frac{t}{RC}}$$

State

- ***State***: summary of past inputs relevant to predicting the future State.

$$q = Cv$$

Actually, State variable is ***q***.

For linear capacitors, capacitor voltage ***v*** is also state variable

- Back to the simple RC circuit

$$v_C = f(v_C(0), v_I(t))$$

$$v_C = V_I + \left(v_C(0) - V_I\right)e^{-\frac{t}{RC}}$$

Summarizes the past input relevant to predicting future behavior.

State

- Let's Rearrange the total responses

$$\underset{\text{Forced response}}{\underset{\uparrow}{v_C = V_I}} + \left(v_C(0) \underset{\text{Natural response}}{\underset{\uparrow}{-}} V_I\right)e^{-\frac{t}{RC}}$$

- To the following form

$$v_C = V_I\left(1 \underset{\text{Zero state response (ZSR)}}{\underset{\uparrow}{-}} e^{-\frac{t}{RC}}\right) + \underset{\text{Zero input response (ZIR)}}{\underset{\uparrow}{v_C(0)e^{-\frac{t}{RC}}}}$$

- Zero state means $v_C(0) = 0$ and zero input means $v_I(t) = 0$.

$$v_C(t) = V_\mathrm{I}(1-\mathrm{e}^{-\frac{t}{RC}}) + v_C(0)\mathrm{e}^{-\frac{t}{\tau}} \qquad (t \geq 0)$$

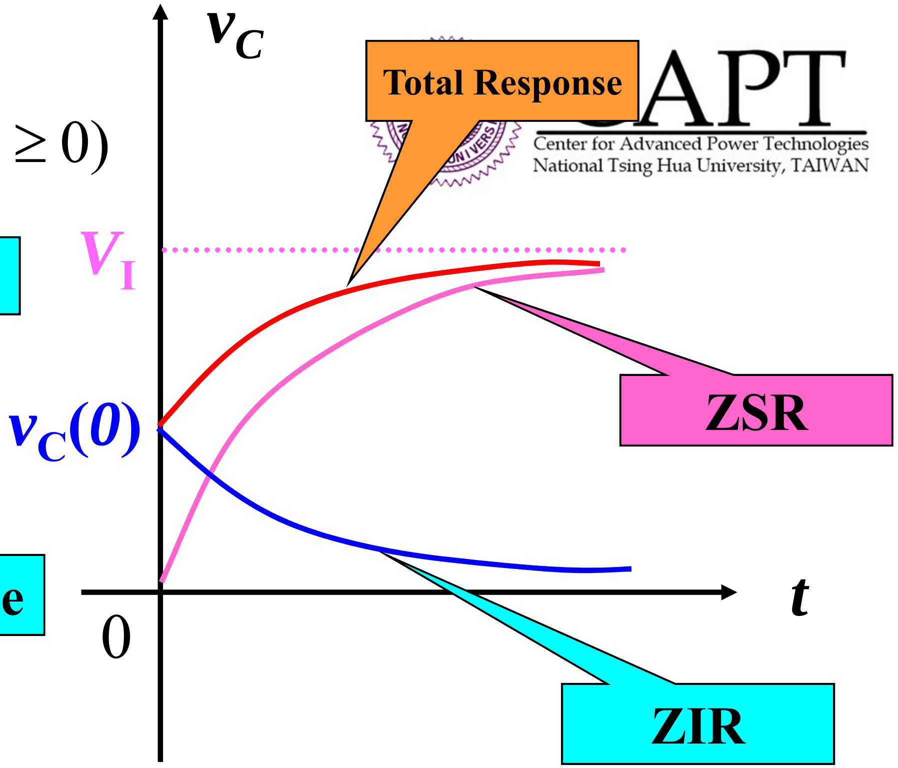
ZSR

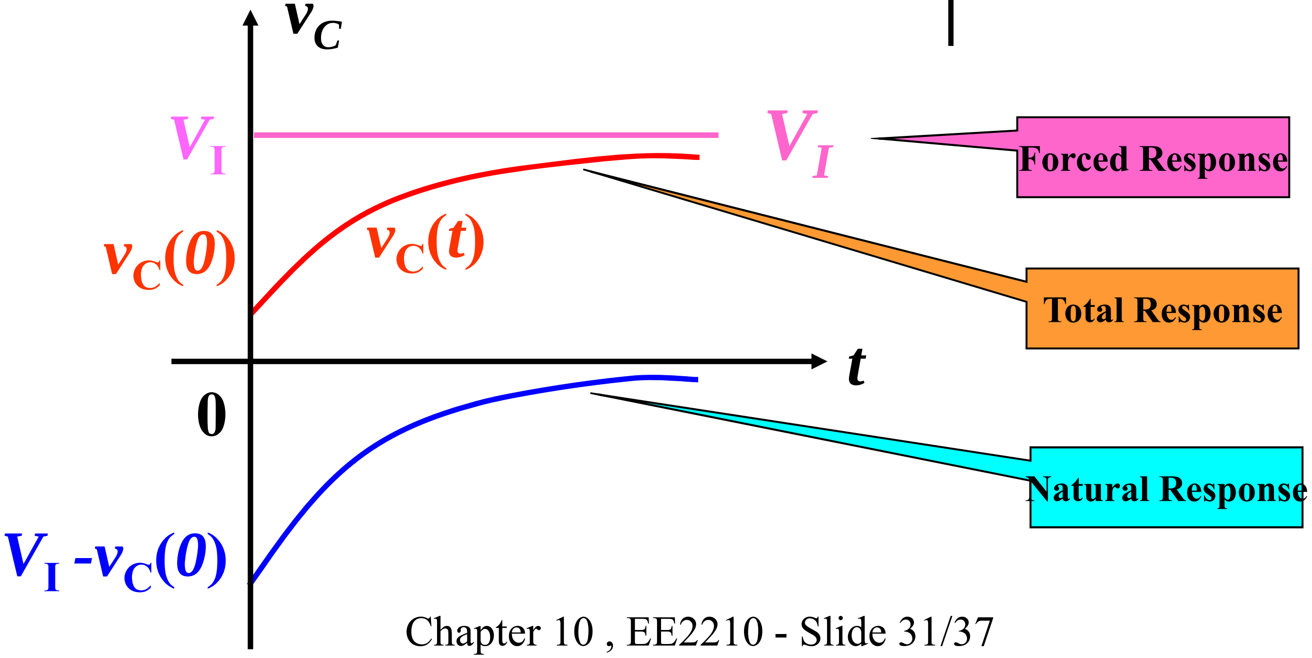
ZIR

$$v_C(t) = V_\mathrm{I} + (v_C(0) - V_\mathrm{I})\mathrm{e}^{-\frac{t}{RC}} \qquad (t \geq 0)$$

Forced Response

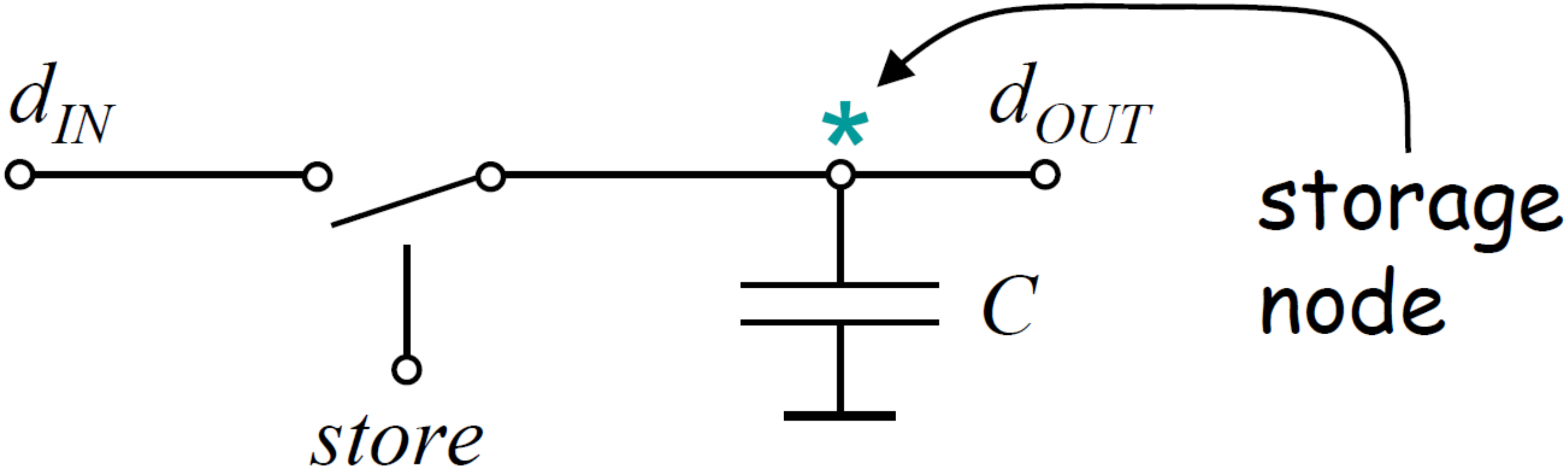
Natural Response



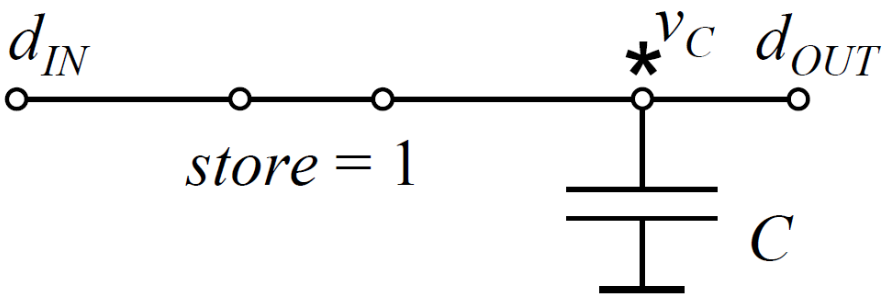


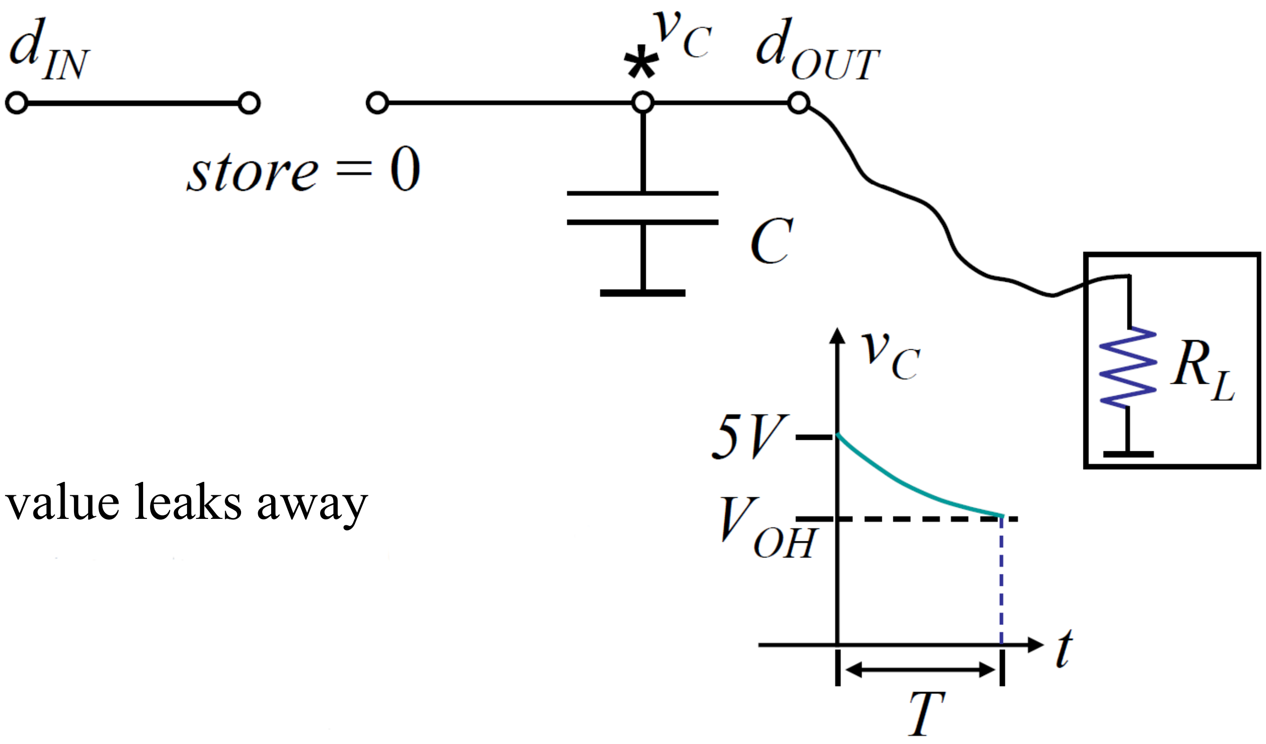
Memory

- Building a memory element (1st attemp)



Building Memory 1st attempt

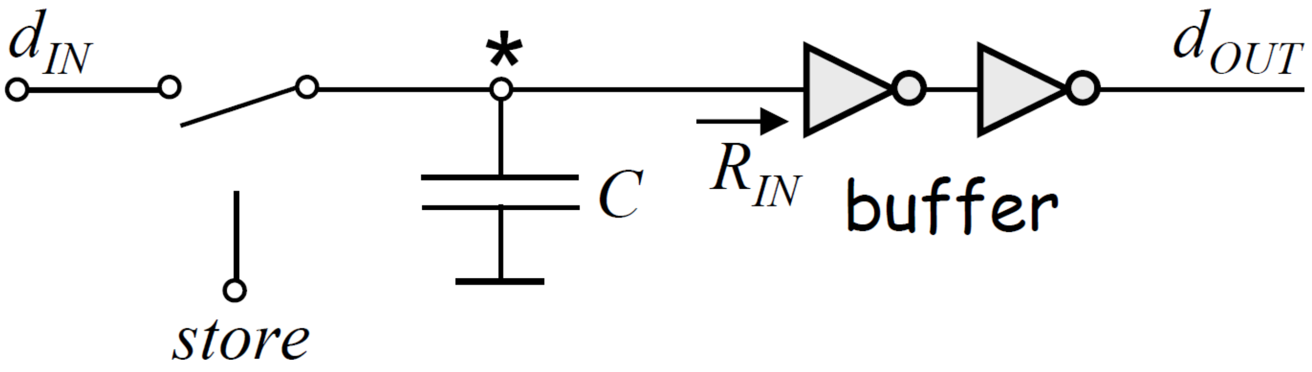




- Stored value leaks away

Building Memory 2nd attempt

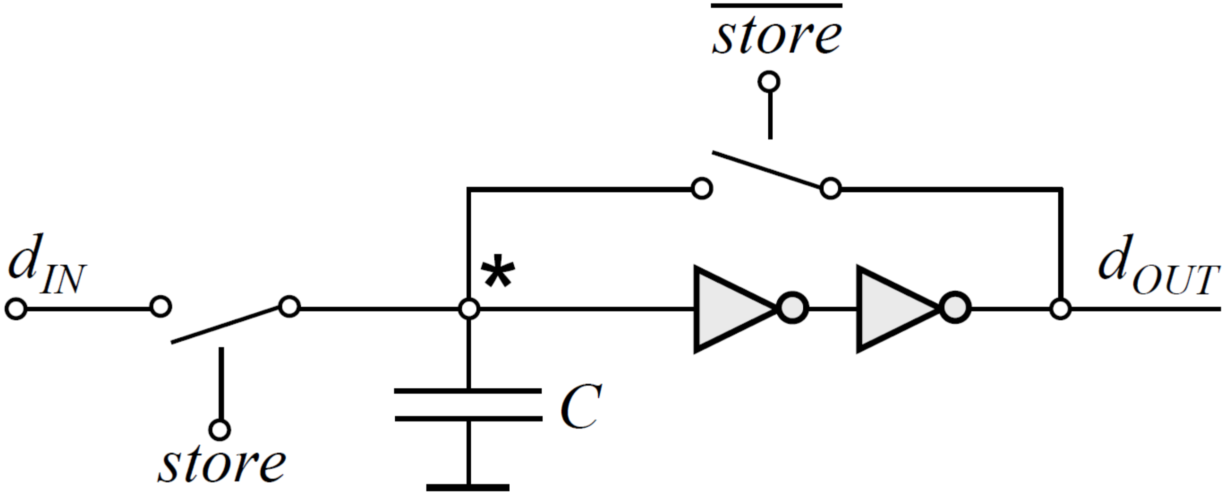
- Using a buffer stage to increase the input resistance seen from the storage node.



- Better, but still not perfect.

Building Memory 3rd attempt

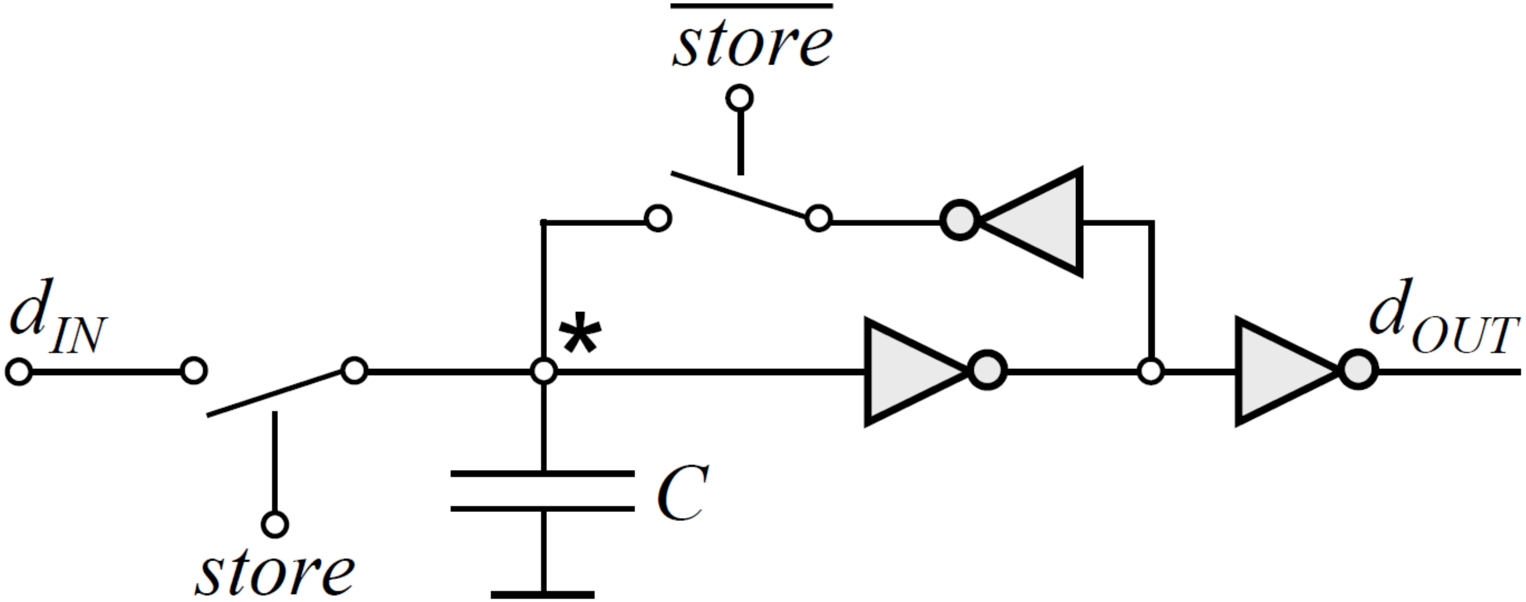
- Using a buffer stage to increase the input resistance seen from the storage node and a refresh stage to restore the data.



- Does this work?
- No, external value can still influence storage node.

Building Memory 4th attempt

- Using a buffer stage to increase the input resistance seen from the storage node and a decoupled refresh stage to restore the data.



- Works!!

Summary

- First Order Circuits are modeled by first order differential equations.
  - Homogeneous solution ⟺ Natural response;
  - Non-homogeneous solution ⟺ Forced response.
- Natural Response depends on circuit parameters and the initial conditions of energy storage elements;
- Forced Response depends circuit parameters, initial conditions, and forms of external excitations.
- The Intuitive Method
- State and Memory
  - ZSR;
  - ZIR.