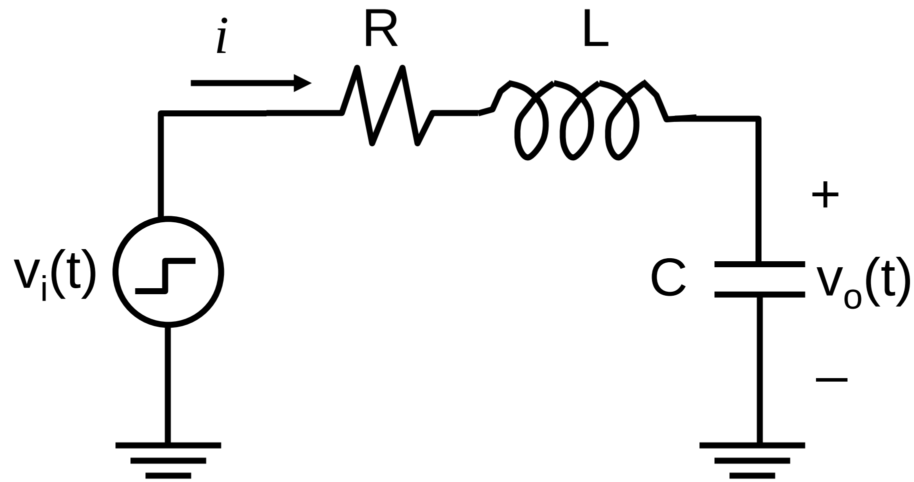
2nd-Order RLC Circuit

- By KVL:

$$v_i = iR + L\frac{di}{dt} + v_o$$



- Substituting i = C·dv$_o$/dt gives:

$$LC\frac{d^2v_o}{dt^2} + RC\frac{dv_o}{dt} + v_o = v_i$$

- By Laplace transform (assume $v_o(0)$ = $v_o'(0)$ = 0):

$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Introducing Damping Ratio and Natural Frequency

- 為便於分析二階系統的step response，the transfer function is re-written as:

$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$where \quad \omega_n = \frac{1}{\sqrt{LC}} \ (natural\ \ frequency) \quad and \quad \xi = \frac{R}{L}\cdot\frac{1}{2\omega_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

- ξ: damping ratio, dimensionless（無單位）；攸關於系統中能量的損失
  - $\xi < 1$, underdamped, producing two complex roots
  - $\xi = 1$, critically damped, producing two repeated real roots
  - $\xi > 1$, overdamped, producing two distinct real roots
- 二階系統的step response在damping ratio小於1時會有振盪行為
  - The roots of the denominator for $\zeta < 1$ are:

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Rise Time t_r

- Rise time t_r occurs at output $v_o(t) = 1$:

$$v_o(t_r) = 1 = 1 - e^{-\xi\omega_n t_r}\underbrace{\left(\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r\right)}_{=0}$$

$$\therefore t_r = \frac{\tan^{-1}\left(\frac{-\sqrt{1-\xi^2}}{\xi}\right)}{\omega_d}$$

Note: unit = radian, between $\pi/2$ and π

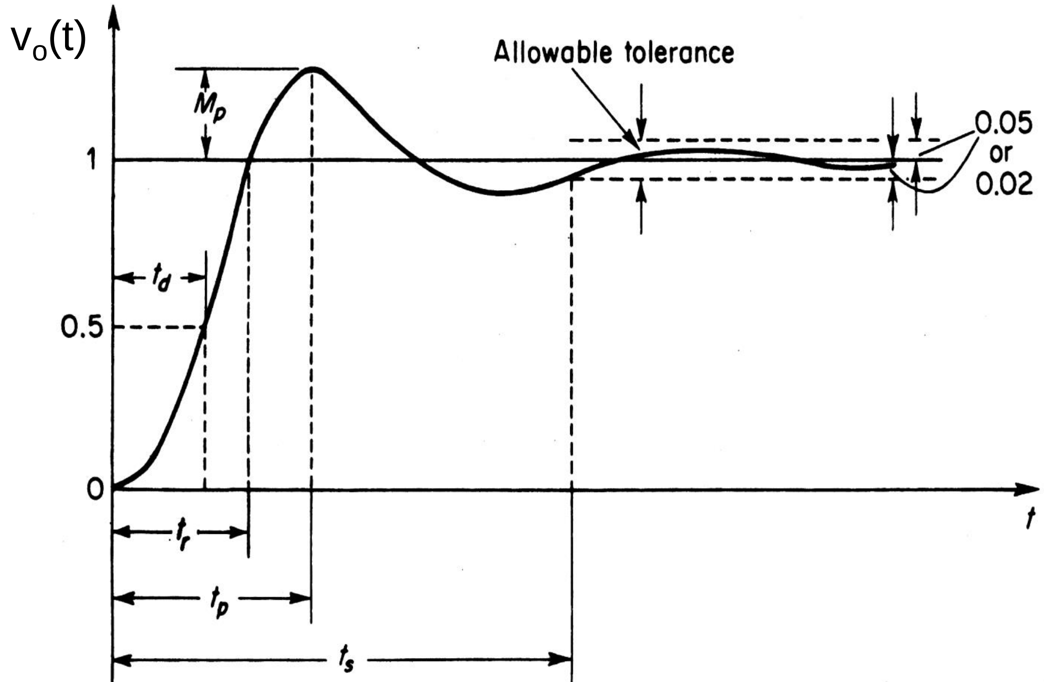
- 舉例： $\tan^{-1}(-1) = \frac{3}{4}\pi$

Peak Time t_p and Maximum Overshoot M_p

- t_p occurs at the first peak of $v_o(t)$:

$$\left.\frac{dv_o(t)}{dt}\right|_{t=t_p} = 0$$

$$\therefore t_p = \frac{\pi}{\omega_d}$$



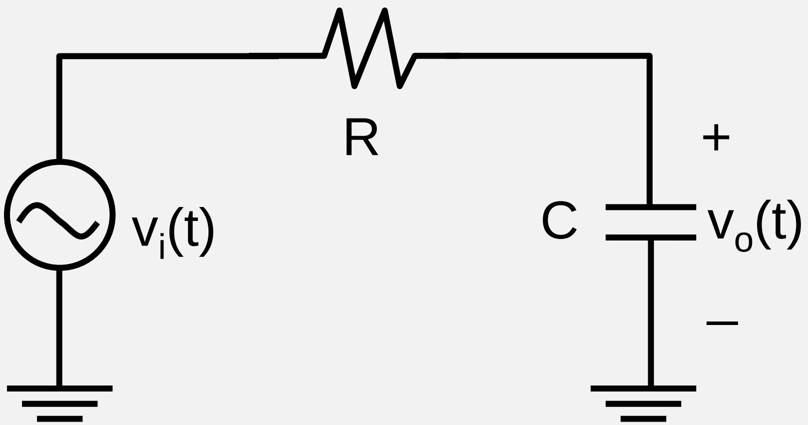
- So:

$$M_p = v_o(t_p) - 1 = e^{-\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\pi}$$

也請記得overshoot公式

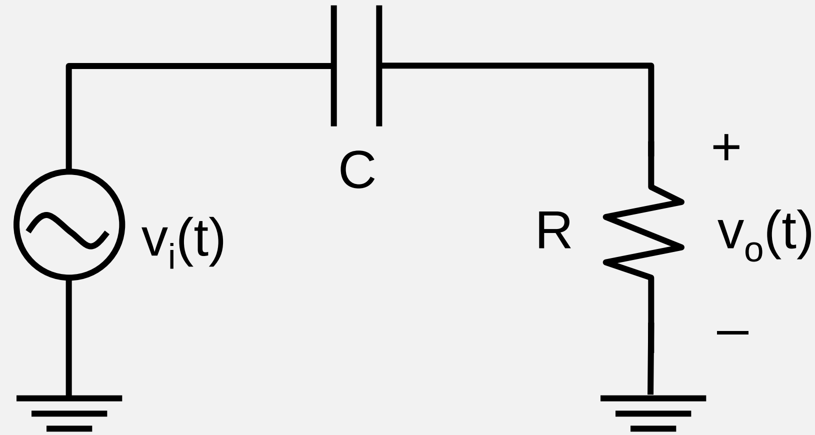
Low-Pass and High-Pass Filters

Low-Pass



$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1}$$

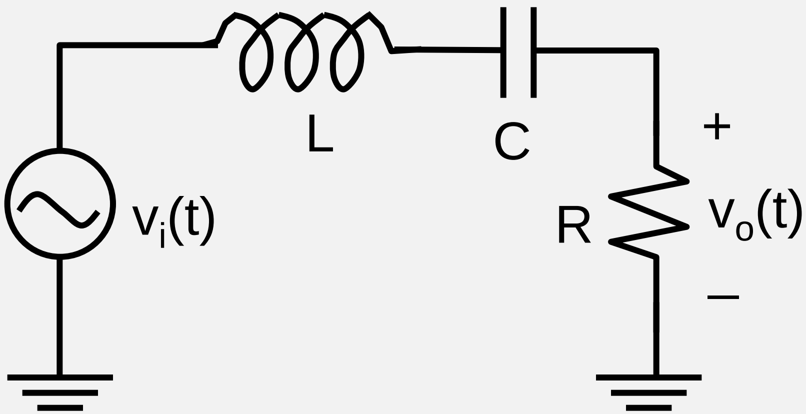
High-Pass

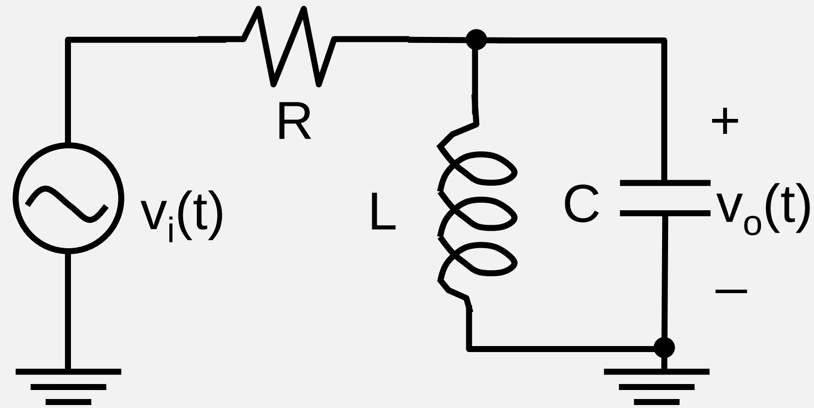


$$\frac{v_o(s)}{v_i(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{RCs + 1}$$

從impedance的觀點來導出不同的濾波電路

Band-Pass Filters



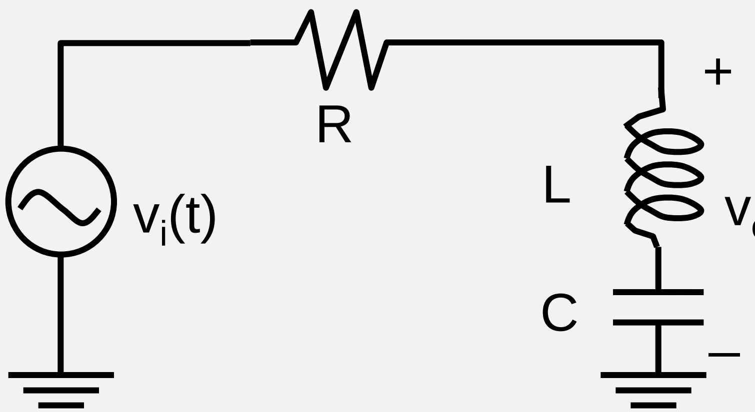


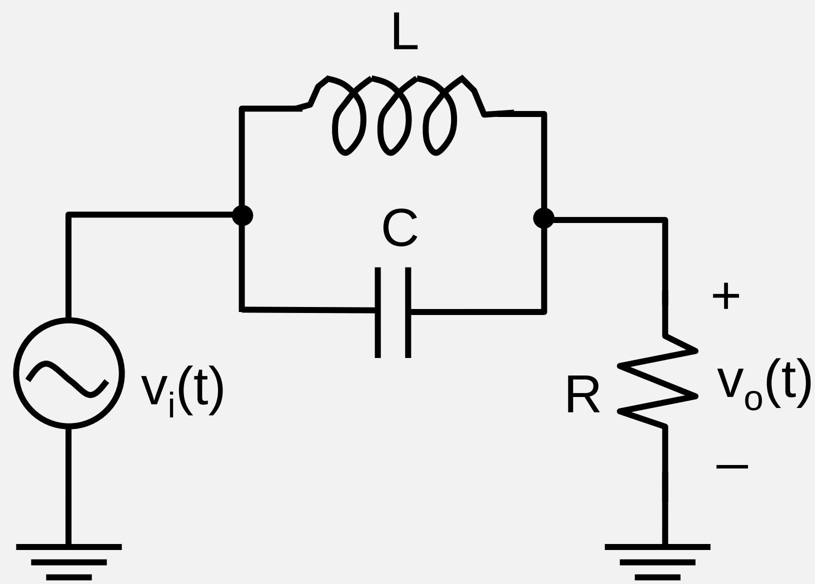
$$\frac{v_o(s)}{v_i(s)} = \frac{R}{R + \left(sL + \frac{1}{sC}\right)}$$

$$= \frac{RCs}{LCs^2 + RCs + 1}$$

$$\frac{v_o(s)}{v_i(s)} = \frac{Ls}{LRCs^2 + Ls + R}$$

Band-Reject Filters

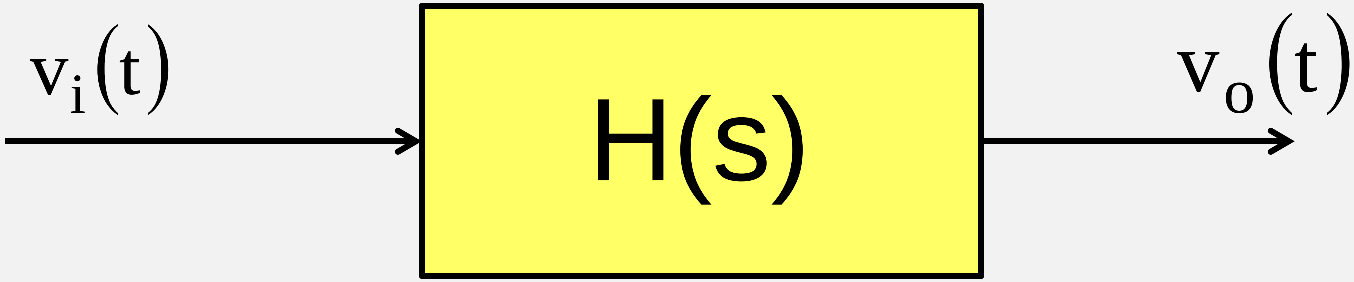




$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Summary：Establish System's Frequency Response



System

- 方法一、當你不知道系統表示式H(s)時：建議用剛剛例子所示範，輸入不同頻率的弦波，然後觀察相對應輸出弦波的值以及相角，來求得 $|H(j\omega)|$ 及 $\angle H(j\omega)$
- 方法二、當知道系統表示式H(s)時：以 $s = j\omega$ 代入H(s) (why? 如果有機會修控制系統時會提到)

$$H(j\omega) = |H(j\omega)| \cdot e^{j\angle H(j\omega)}$$

Cont'd: Phase of a LPF

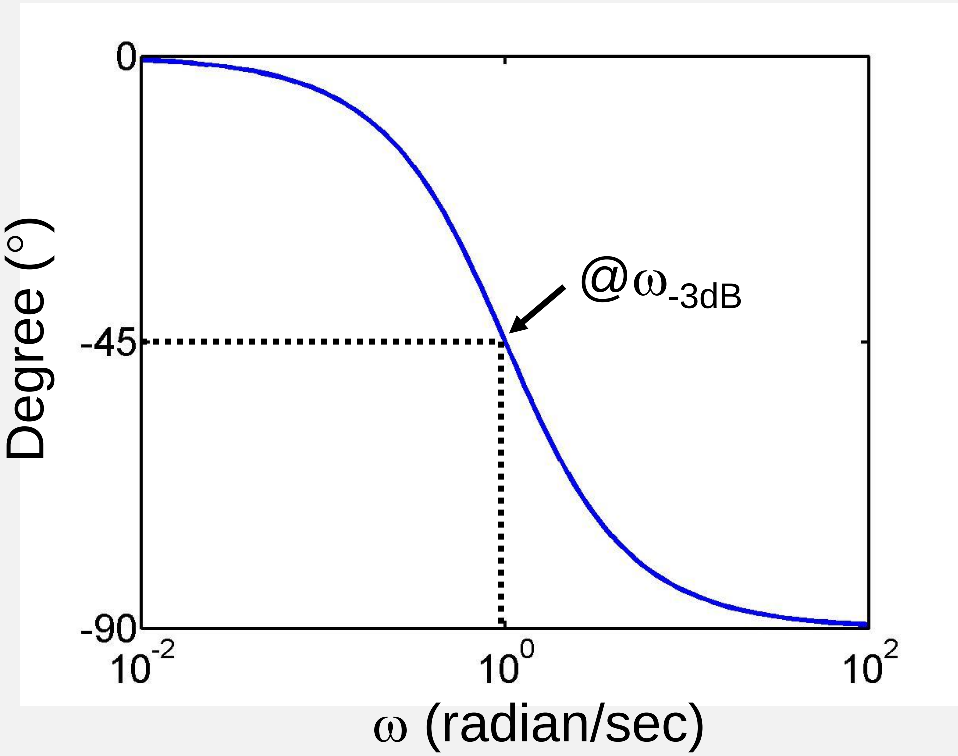
$$H(j\omega)=\frac{v_o(j\omega)}{v_i(j\omega)}=\frac{1}{j\omega T+1}$$

$$\angle H(j\omega)=\angle\frac{1}{j\omega T+1}=-tan^{-1}(\omega T)$$

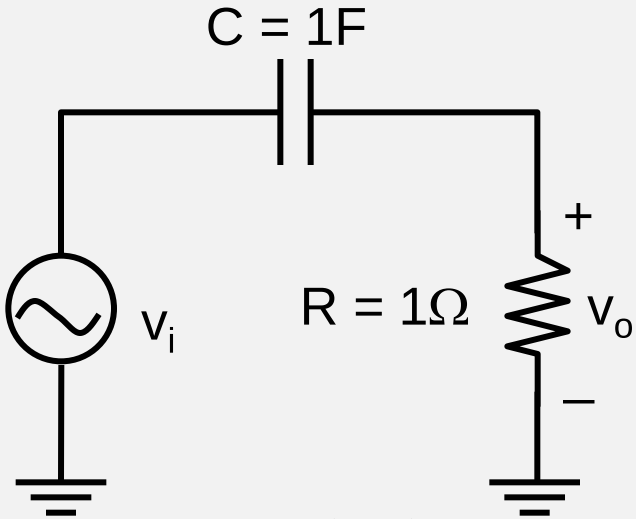
$$\angle H(j0)=-tan^{-1}(0T)=0^\circ$$

$$\angle H\left(j\frac{1}{T}\right)=-tan^{-1}(1)=-45^\circ$$

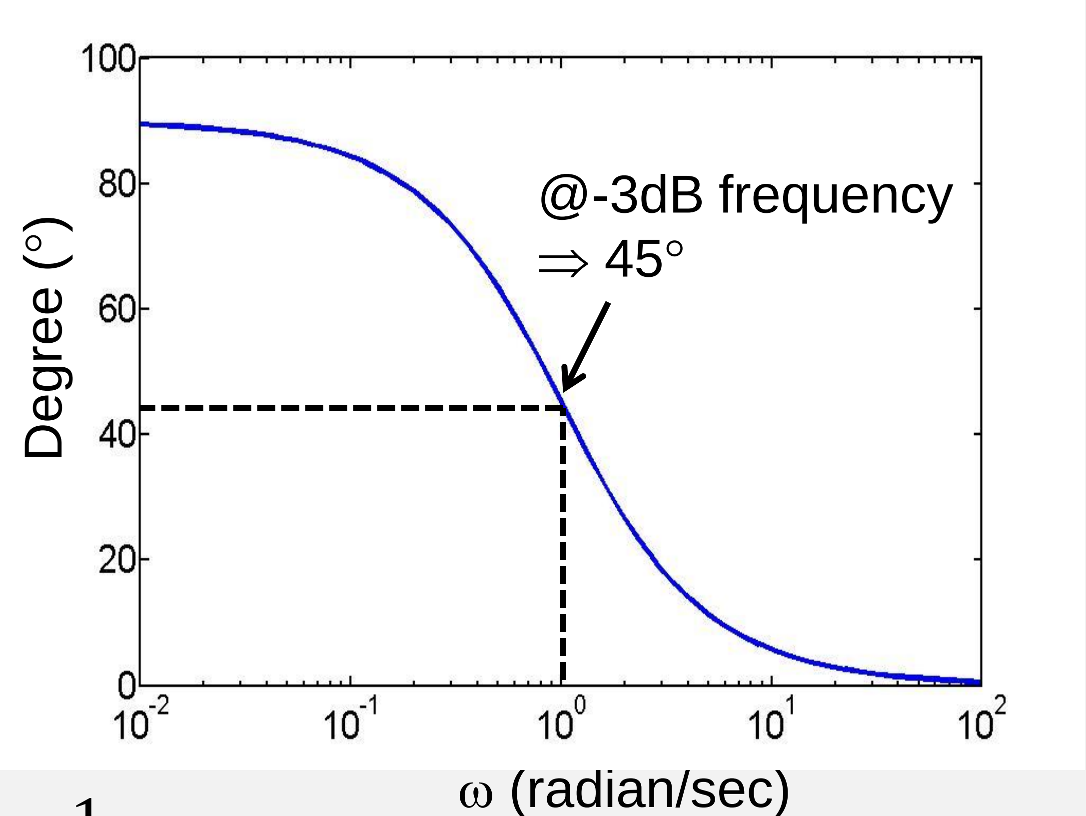
$$\angle H(j\infty)=-tan^{-1}(\infty)=-90^\circ$$



Cont'd: Phase of a HPF



$$H(j\omega)=\frac{v_o(j\omega)}{v_i(j\omega)}=\frac{j\omega}{j\omega+1}$$



- Phase:

$$\angle H(j\omega)=90^\circ-\tan^{-1}\omega$$

$$\angle H(j0)=90^\circ;\angle H\left(j\frac{1}{T}\right)=45^\circ;\angle H(j\infty)=0^\circ$$

Cont'd:假設R = 1 Ω, L = 1 H, C = 1 F

- 想想看：low-pass, band-pass及band-reject filters在自然振頻 ($\omega_n = 1/\sqrt{LC}$) 的增益各是多少？

Band-Pass:

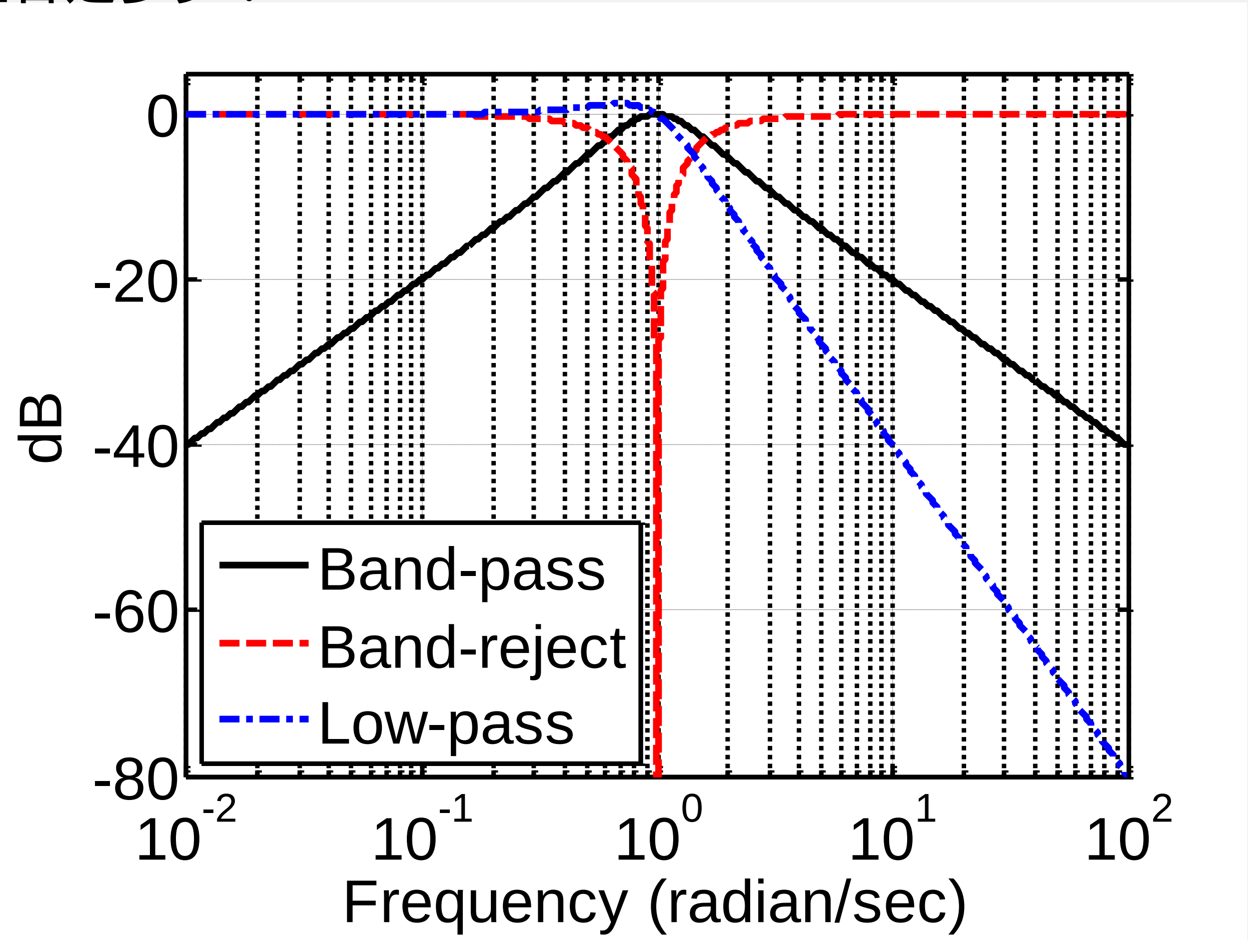
$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Band-Reject:

$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Low-Pass:

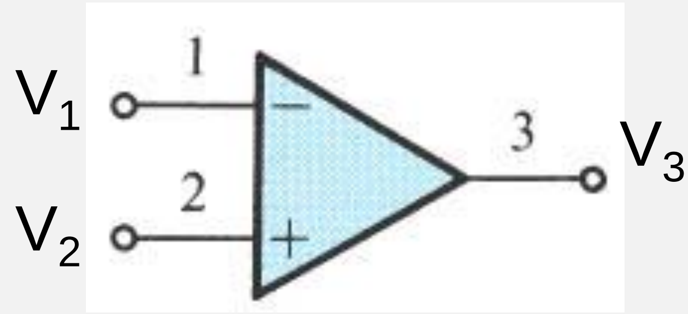
$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



The Ideal Op-amp

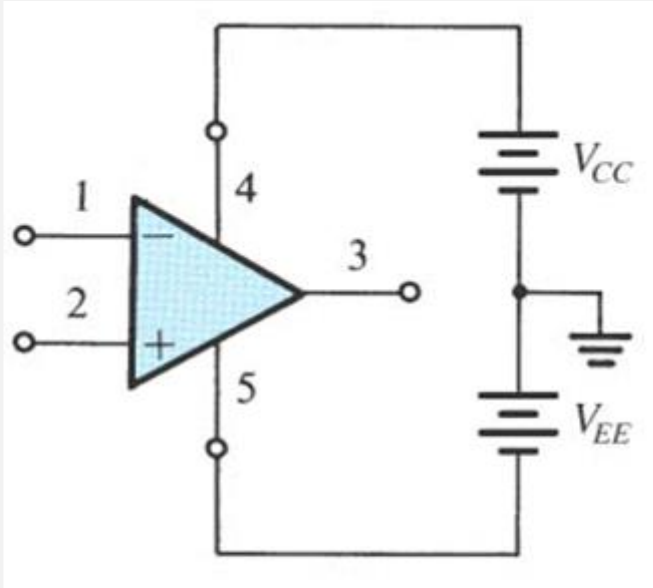
- An op-amp has
  - Differential inputs: V_1 and V_2
  - Single-ended output : V_3
  - So:

$$V_3 = A \cdot (V_2 - V_1)$$



Gain = *A*

- For an ideal op-amp: $A = \propto$ ($A = 10^3$ to 10^5 for typical op-amps)
  - What is the large gain for?
- An op-amp needs DC power supplies, either positive V_{CC} and negative V_{EE}, or V_{dd} and 0



Cont'd：Inverting Amplifier

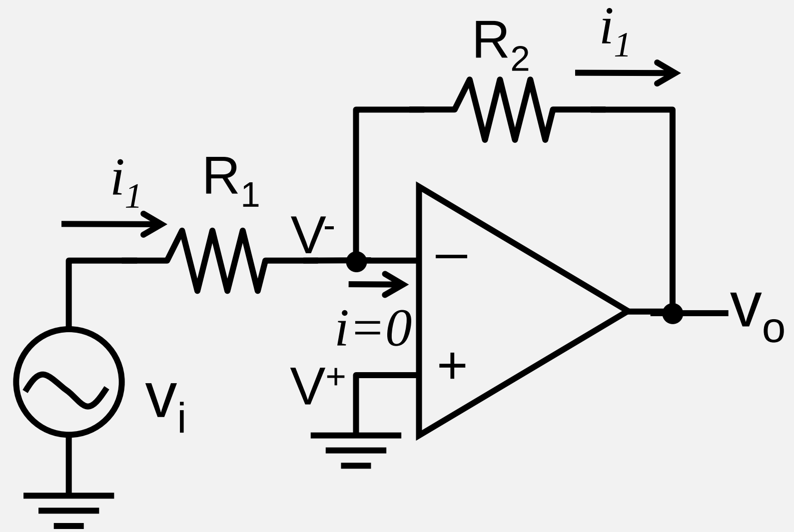
- Virtual short, so:

$$V^- = V^+ = 0$$

$$i_1 = \frac{V_i - V^-}{R_1} = \frac{V_i}{R_1}$$

$$V_o = V^- - i_1 R_2 = -\frac{R_2}{R_1} V_i$$

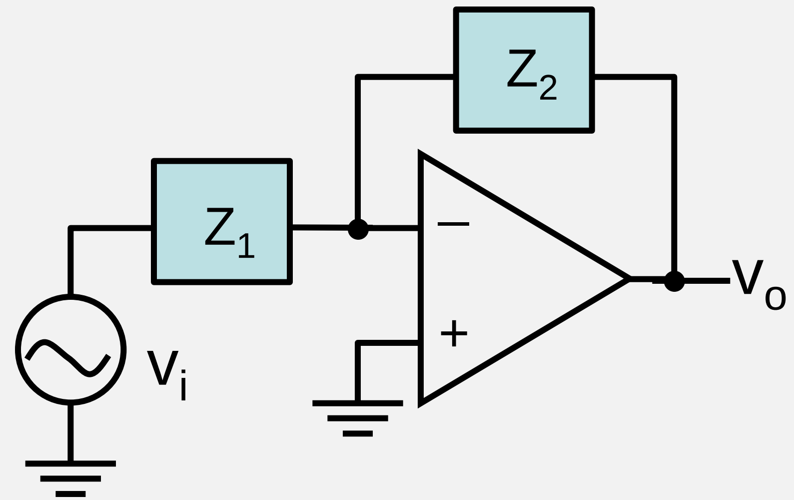
$$\Rightarrow \frac{V_o}{V_i} = -\frac{R_2}{R_1}$$



$R_{in} = R_1$

（重要）Inverting Configuration

- In general: $$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1}$$



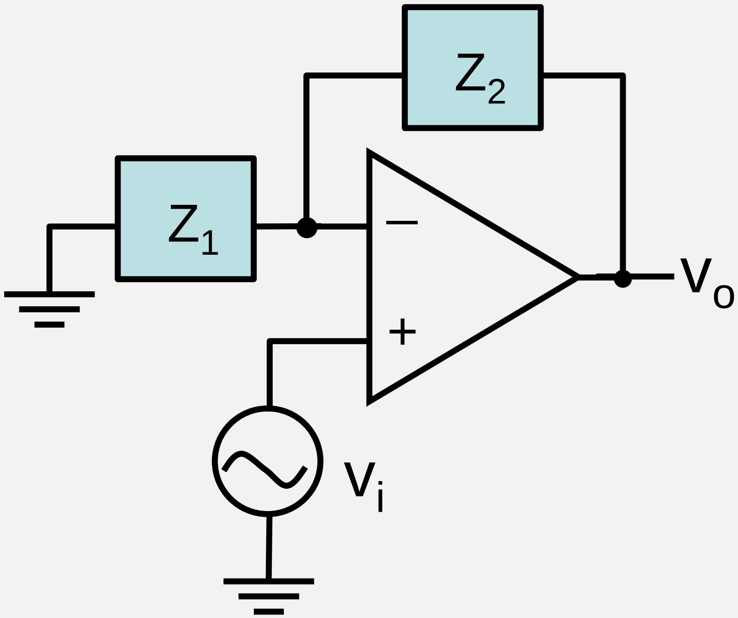
- For example: Z_1 = R, Z_2 = 1/(sC):

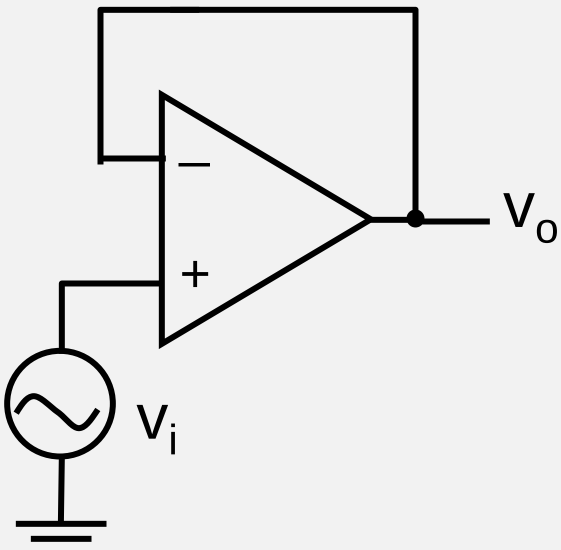
$$\frac{V_o(s)}{V_i(s)} = -\frac{1}{sRC}$$ **(integrator)**

$$\frac{V_o(s)}{V_i(s)} = -\frac{1}{sRC} \quad \Rightarrow V_o(s) = -\frac{1}{sRC} V_i(s)$$

$$\Rightarrow V_o(t) = -\frac{1}{RC} \int V_i(t) \cdot dt$$

（重要）Non-Inverting Configuration





- Similarly:

$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2}{Z_1}$$

- Voltage follower ($Z_1 = \infty$, $Z_2 = 0$) with a gain of 1 and better driving capability

$$\frac{V_o(s)}{V_i(s)} = 1$$

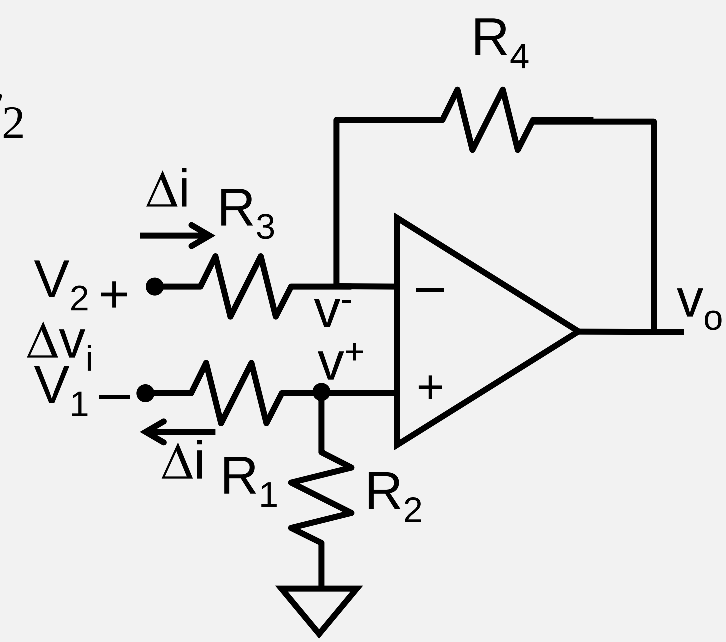
Difference Amplifier

- It can remove the common-mode signal, and subtract and amplify the differential signal
- By superposition:

$$v_o = \left(1+\frac{R_4}{R_3}\right)\cdot\left(\frac{R_2}{R_1+R_2}\right)\cdot v_1 - \frac{R_4}{R_3}\cdot v_2$$

  - For $R_1 = R_2 = R_3 = R_4$

$$v_o = v_1 - v_2$$



$$\Delta v_i - (R_1 + R_3)\cdot \Delta i + \underbrace{\left(v^+ - v^-\right)}_{\approx 0} = 0$$

$$R_{in} = \frac{\Delta v_i}{\Delta i} = R_1 + R_3$$ (Drawback)

High-Input-Impedance Differential-Mode Instrumentation Amplifier

- 提供很大的輸入阻抗

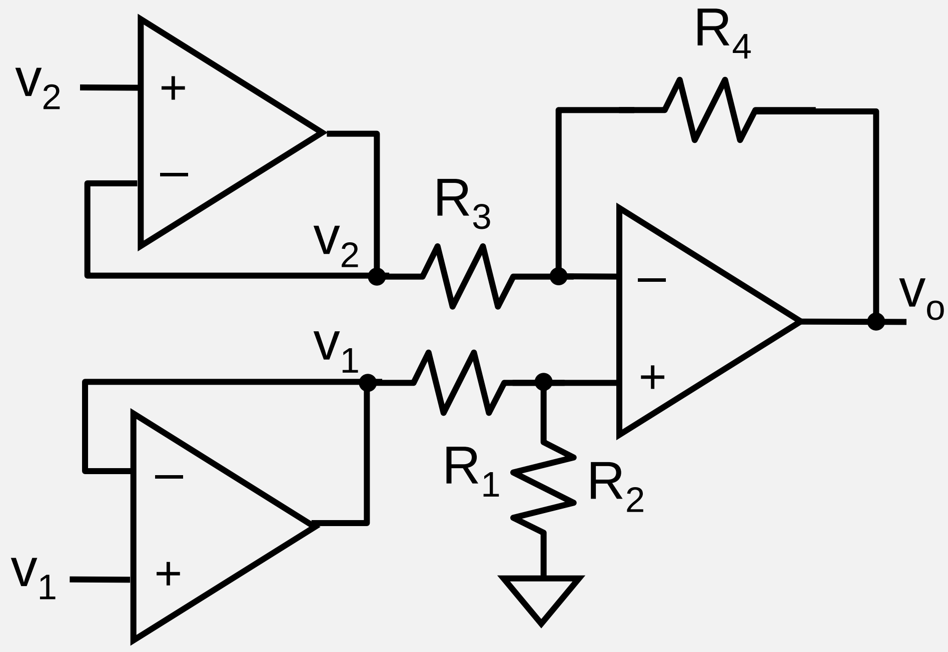
$$v_o = \left(1 + \frac{R_4}{R_3}\right) \cdot \left(\frac{R_2}{R_1 + R_2}\right) \cdot v_1 - \frac{R_4}{R_3} \cdot v_2$$

$$when\ \frac{R_4}{R_3} = \frac{R_2}{R_1}$$

$$v_o = \frac{R_2}{R_1} \cdot (v_1 - v_2)$$

$$when\ R_1 = R_2 = R_3 = R_4$$

$$v_o = v_1 - v_2$$



Variable-Gain Differential-Mode Instrumentation Amplifier (Design Problem IV)

The gain can be conveniently adjusted by tuning R_G

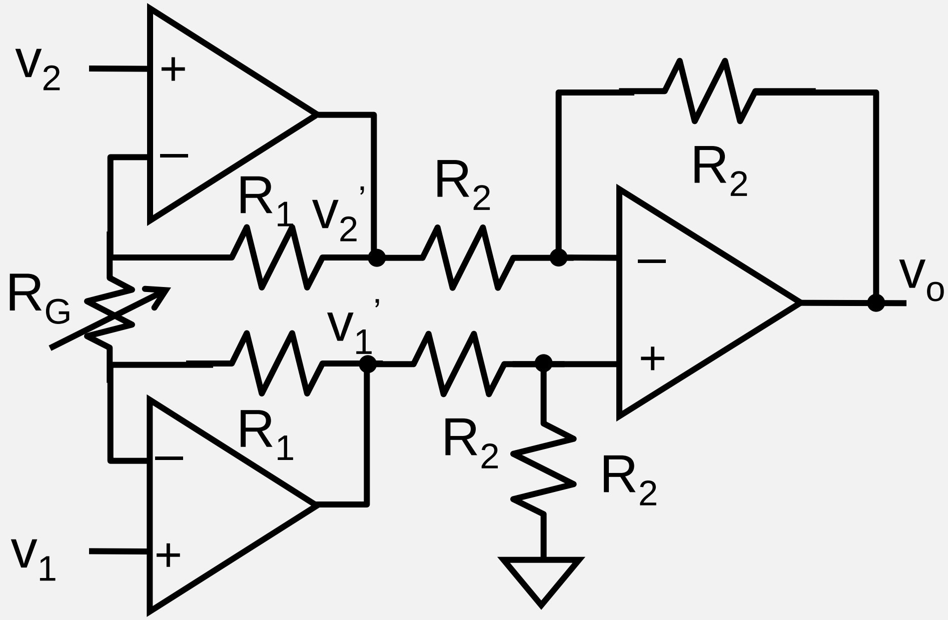
$$v_o = v_1' - v_2' \quad (1)$$

$$v_1' = \left(1 + \frac{R_1}{R_G}\right) \cdot v_1 - \frac{R_1}{R_G} \cdot v_2 \quad (2)$$

$$v_2' = \left(1 + \frac{R_1}{R_G}\right) \cdot v_2 - \frac{R_1}{R_G} \cdot v_1 \quad (3)$$

Substitute (2)(3) into (1):

$$v_o = \left(1 + \frac{2R_1}{R_G}\right) \cdot (v_1 - v_2)$$



Definition：Quality Factor (Q)

- The resonant peak magnitude is related to energy dissipation
- Definition:

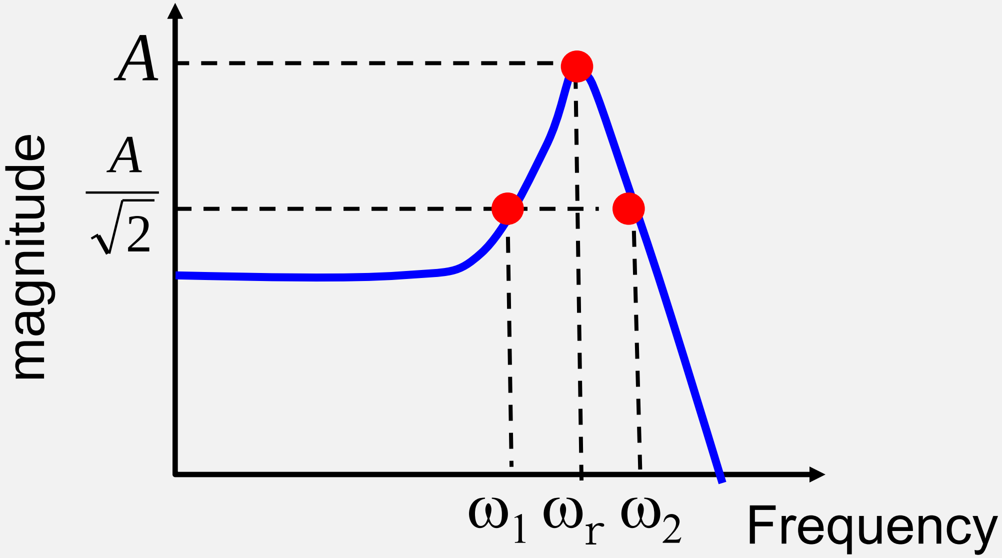
$$Q \equiv \frac{\text{Energy stored}}{\text{Average energy dissipated per radian}}$$

  - Q is a dimensionless parameter and related to the damping ratio by $Q \approx 1/(2\xi)$
  - $Q = \infty$ implies there is no energy dissipation when a circuit oscillates

Obtain Q by Analysis or Experiment

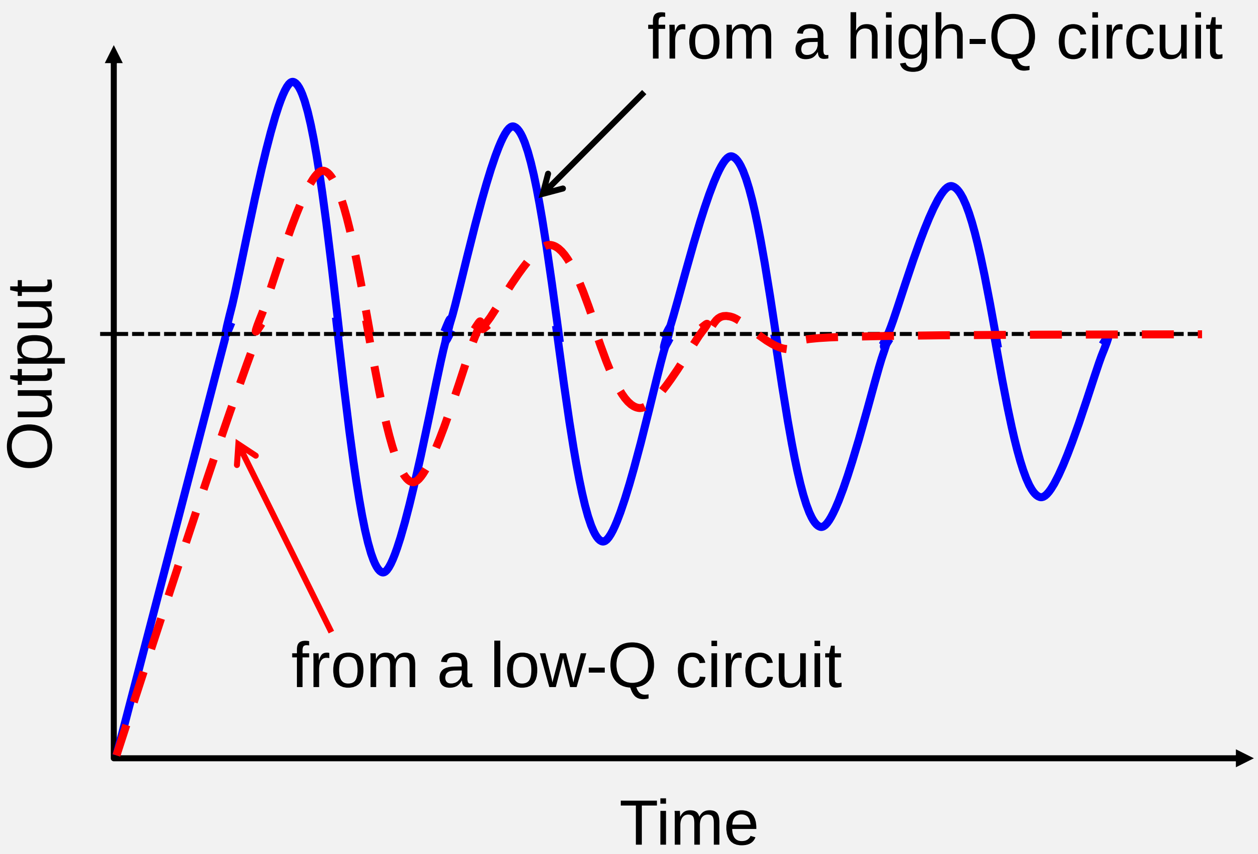
- Q can be obtained by analysis:
  - For $\xi << 1$, Q is related to the damping ratio by $Q \approx 1/(2\xi)$。
    Remember: Q and ξ are both dimensionless
- Q can be obtained by experiment using the half-power bandwidth (defined by ω_1 and ω_2):

$$Q = \frac{\omega_r}{\omega_2 - \omega_1}$$ （重要！你實驗demo Q值要用到）

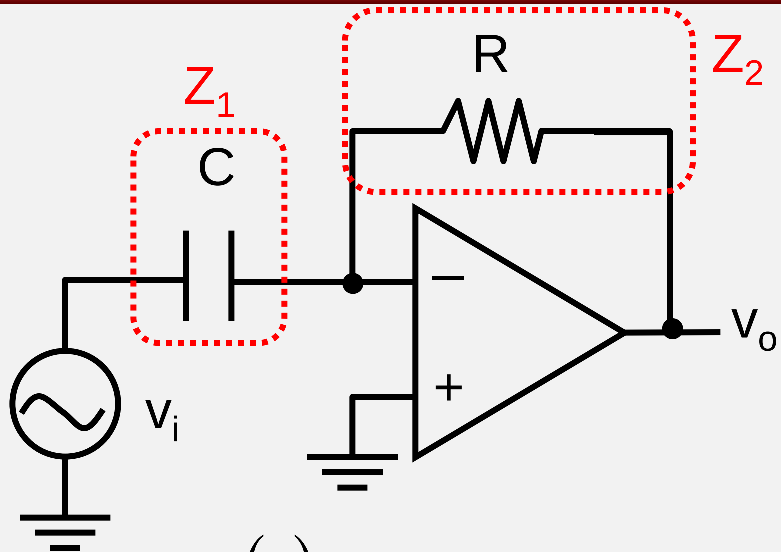


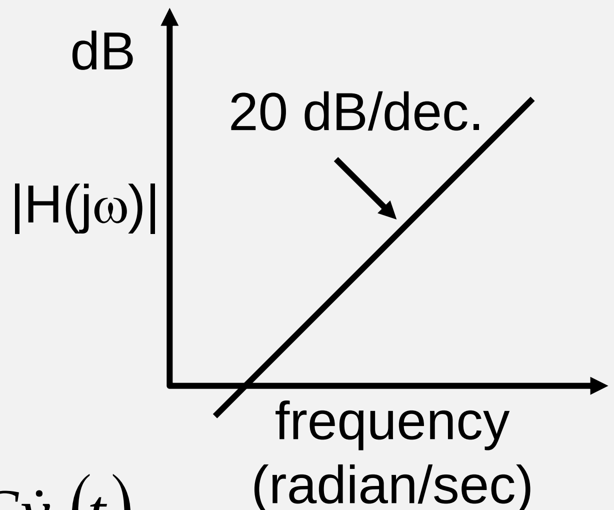
Step Response vs. Q

- A high-Q circuit has low energy dissipation, and long oscillation time in its step response



First-Order Circuit: Differentiator





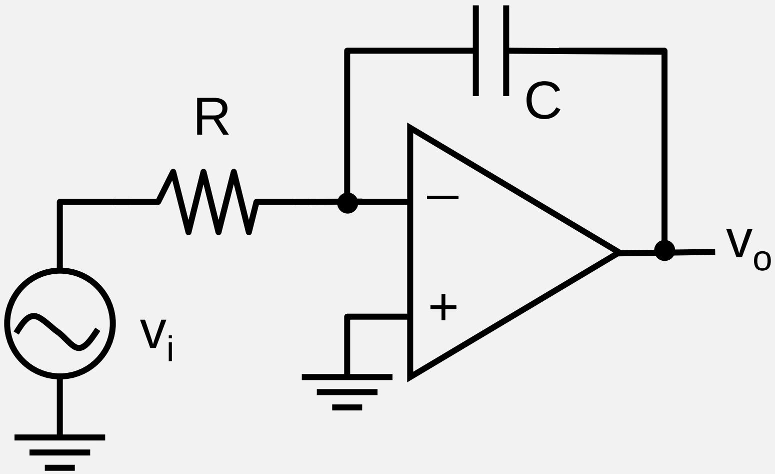
$$H(s)=\frac{v_o(s)}{v_i(s)}=-\frac{Z_2}{Z_1}=-sRC\,,v_o(t)=-RC\dot{v}_i(t)$$

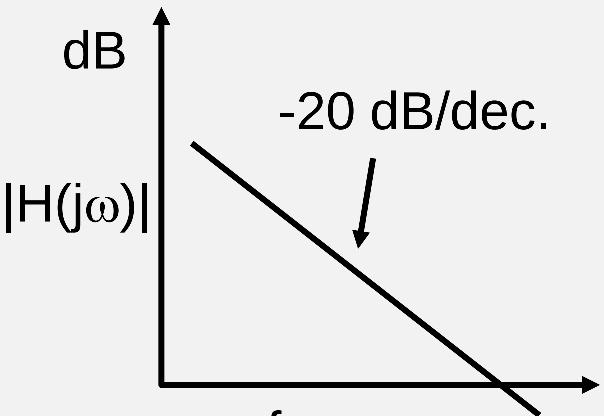
$$\Rightarrow H(j\omega)=\frac{v_o(j\omega)}{v_i(j\omega)}=-j\omega RC$$

$$\therefore |H(j\omega)|=\omega RC$$

1. 頻率每增加十倍，增益也增為十倍（即20 dB/dec）
2. H(j0) = 0 = -∝ dB

Integrator



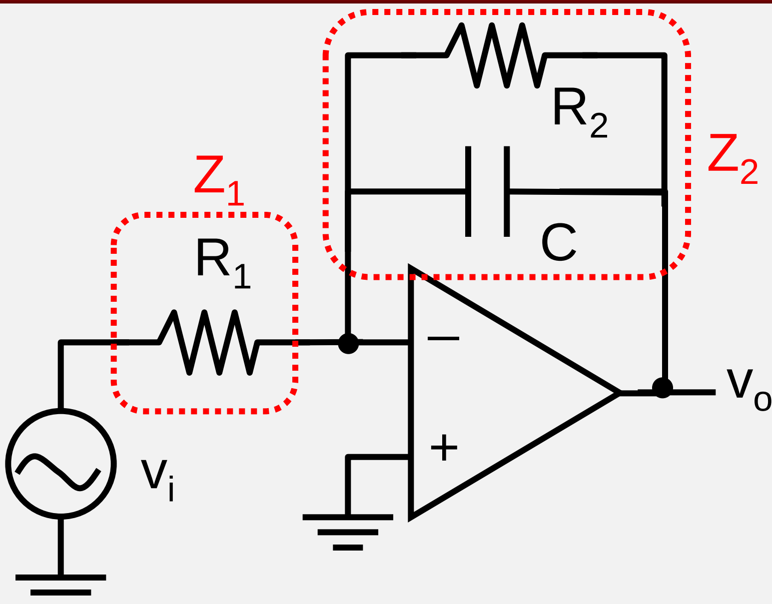


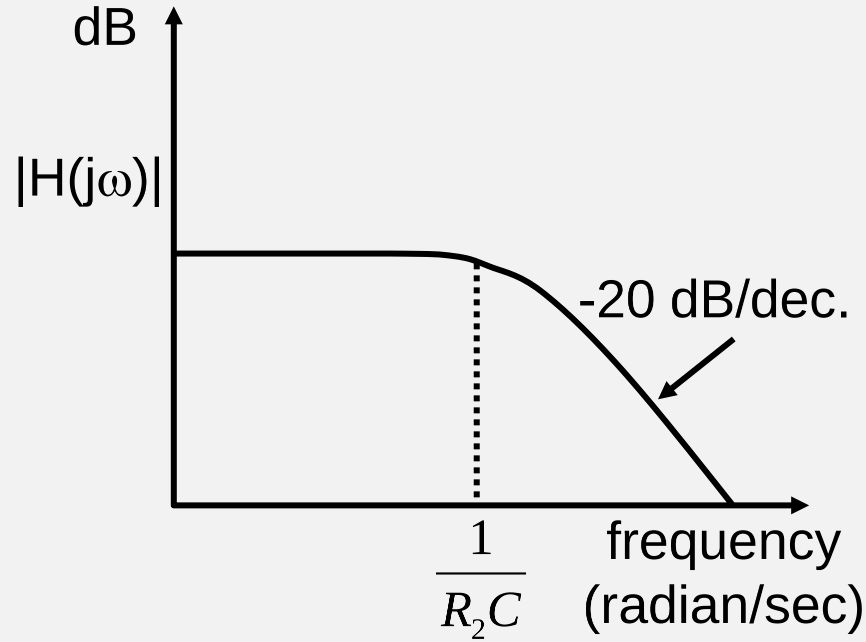
$$H(s)=\frac{v_o(s)}{v_i(s)}=-\frac{1}{sRC},\ v_o(t)=-\frac{\int_0^t v_i(t)dt}{RC}$$

$$\therefore \left|H(j\omega)\right|=\frac{1}{\omega RC}$$

1. 頻率每增加十倍，增益減為十分之一（即-20 dB）
2. |H(j0)|=∝

Low-Pass Filter with Gain





$$H(s)=\frac{v_o(s)}{v_i(s)}=-\frac{Z_2}{Z_1}=-\frac{R_2}{R_1}\frac{1}{sR_2C+1}$$

$$\therefore |H(j\omega)|=\frac{R_2}{R_1}\frac{1}{\sqrt{\omega^2R_2^2C^2+1}}$$

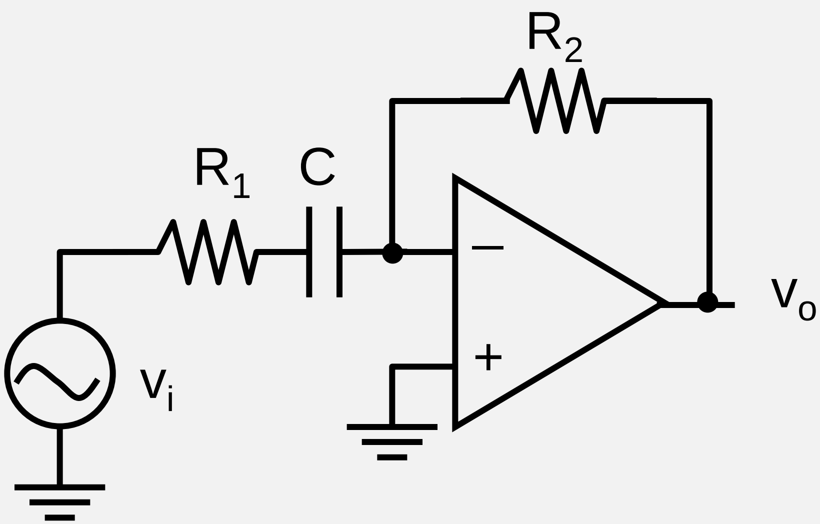
$\frac{1}{R_2C}$ is the -3dB frequency

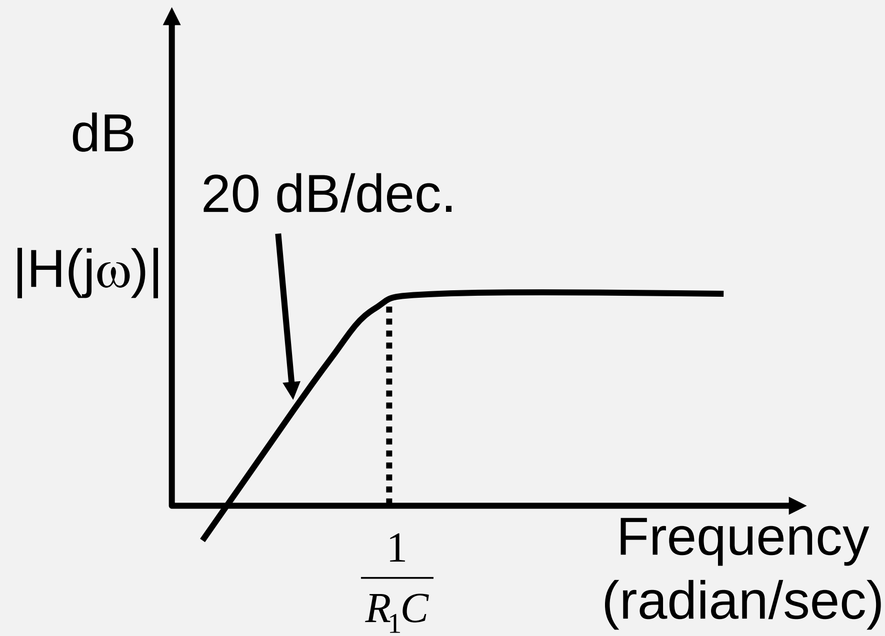
$$|H(j\omega)|_{\omega=0}=\frac{R_2}{R_1}$$

$$|H(j\omega)|_{\omega=\frac{1}{R_2C}}=0.707\frac{R_2}{R_1}$$

$$|H(j\omega)|_{\omega=\infty}=0$$

High-Pass Filter with Gain





$$H(s)=\frac{v_o(s)}{v_i(s)}=-\frac{R_2}{R_1}\frac{sR_1C}{sR_1C+1}$$

$$\therefore \left|H(j\omega)\right|=\frac{R_2}{R_1}\frac{\omega R_1C}{\sqrt{\omega^2R_1^2C^2+1}}$$

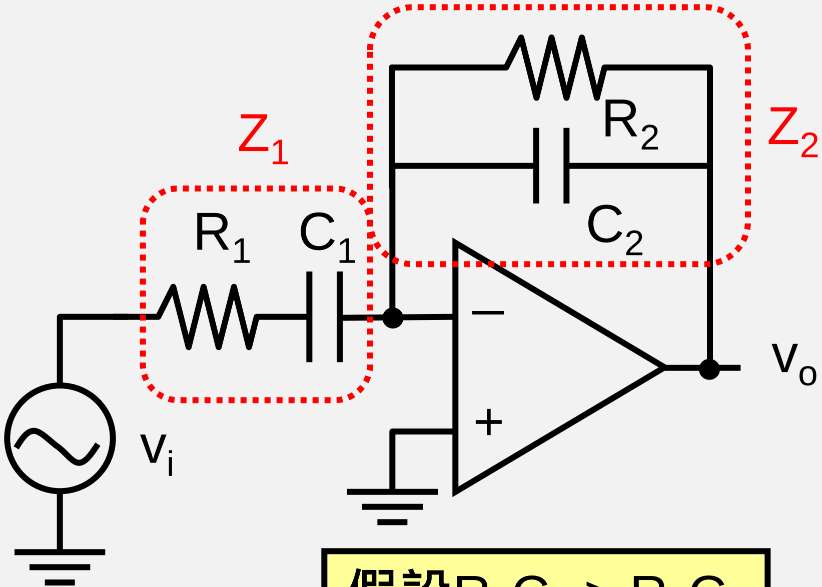
$\frac{1}{R_1C}$ is the -3dB frequency

$$\left|H(j\omega)\right|_{\omega=0}=0$$

$$\left|H(j\omega)\right|_{\omega=\frac{1}{R_1C}}=0.707\frac{R_2}{R_1}$$

$$\left|H(j\omega)\right|_{\omega=\infty}=\frac{R_2}{R_1}$$

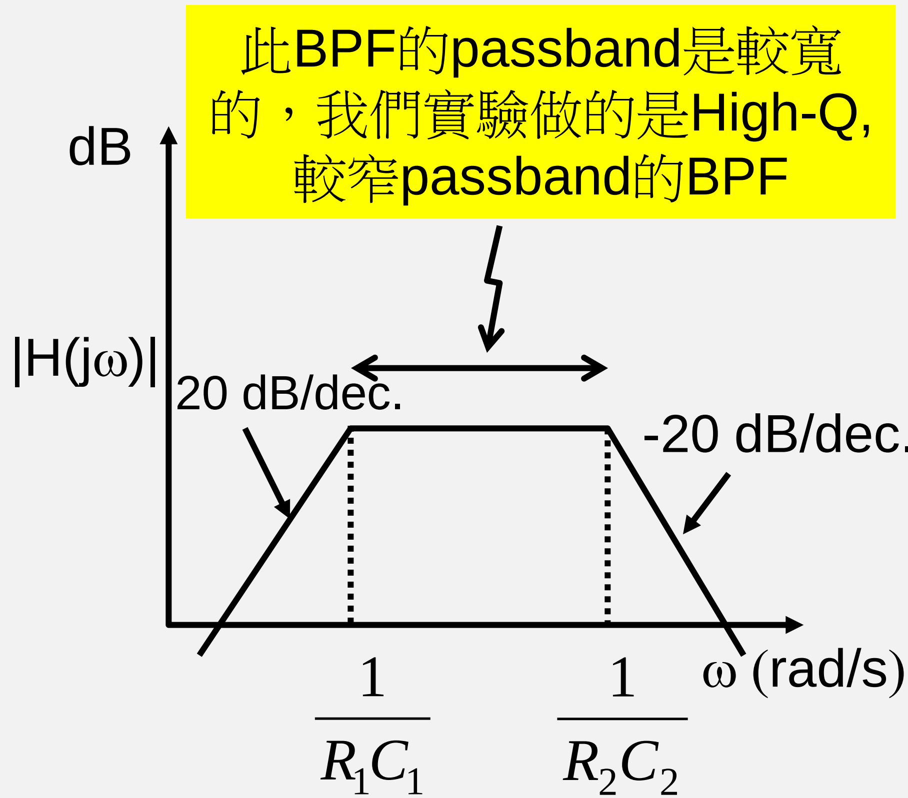
Band-Pass Filter



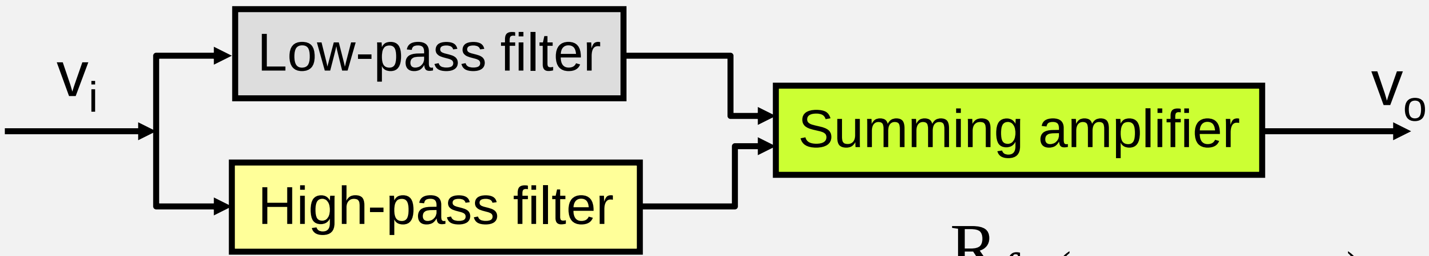
假設$R_1C_1 > R_2C_2$

$$H(s) = \frac{v_o(s)}{v_i(s)} = -\frac{Z_2}{Z_1}$$

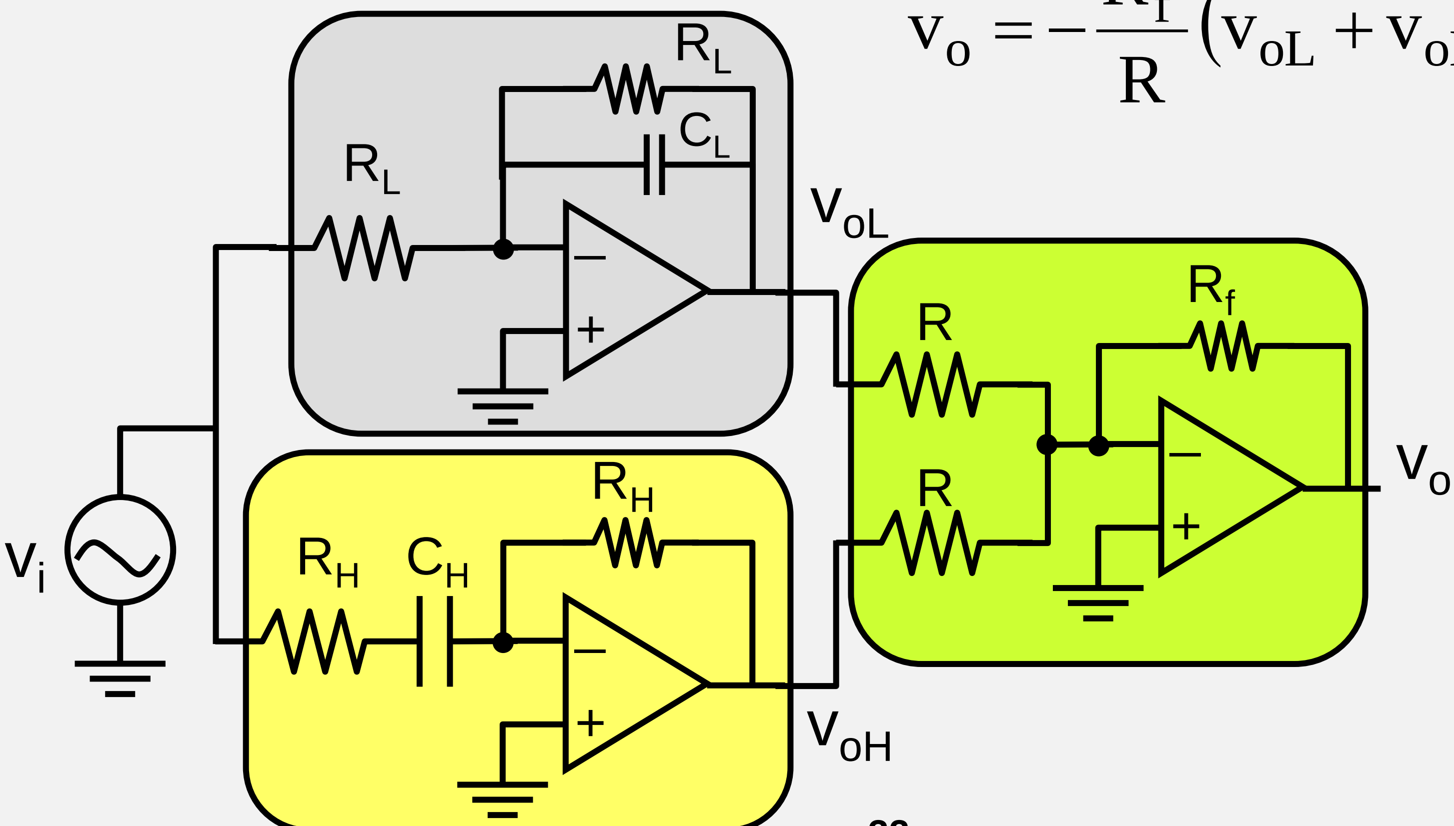
$$= -\frac{R_2}{R_1}\frac{sR_1C_1}{sR_1C_1+1}\cdot\frac{1}{sR_2C_2+1}$$



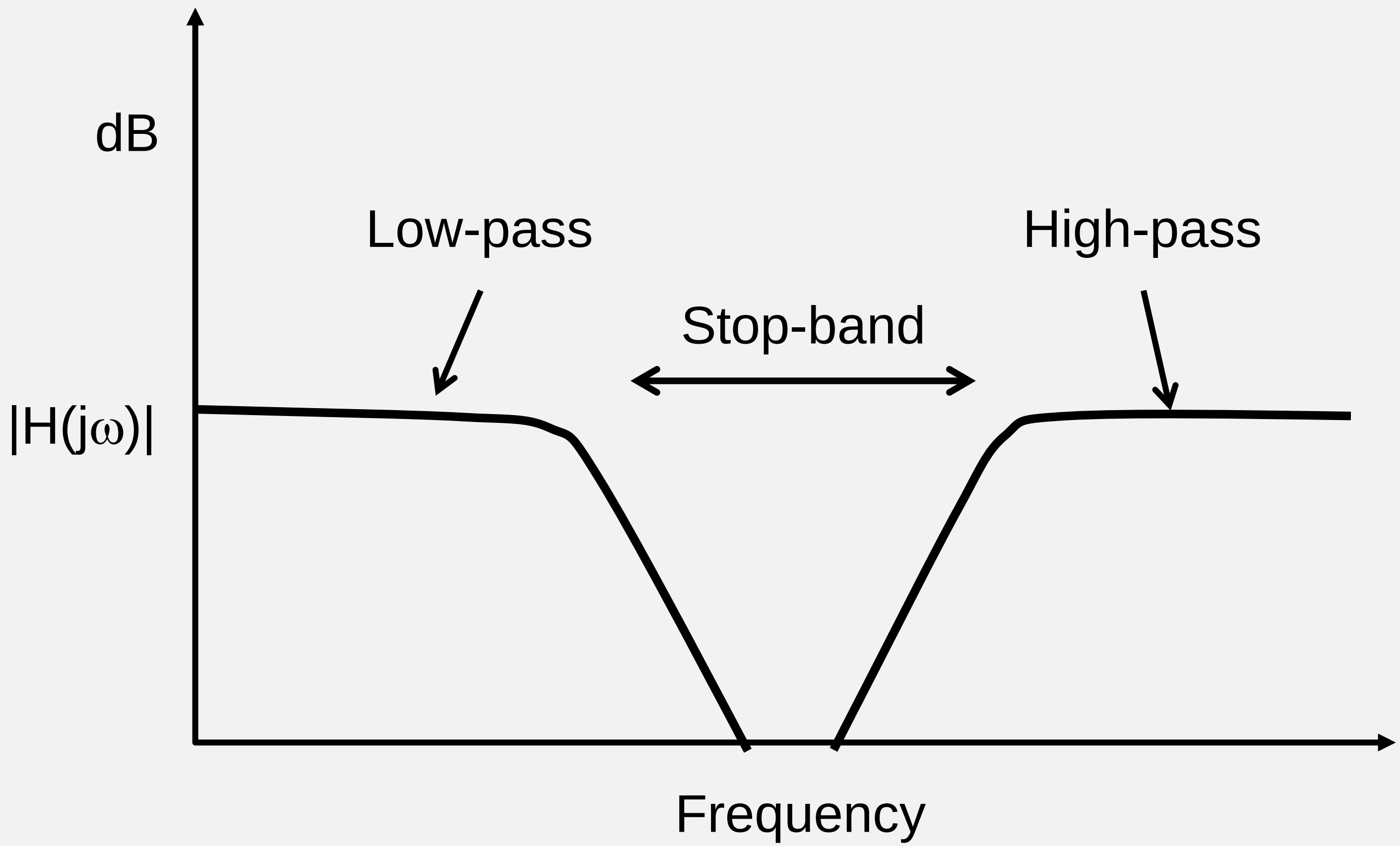
Band-Reject Filter



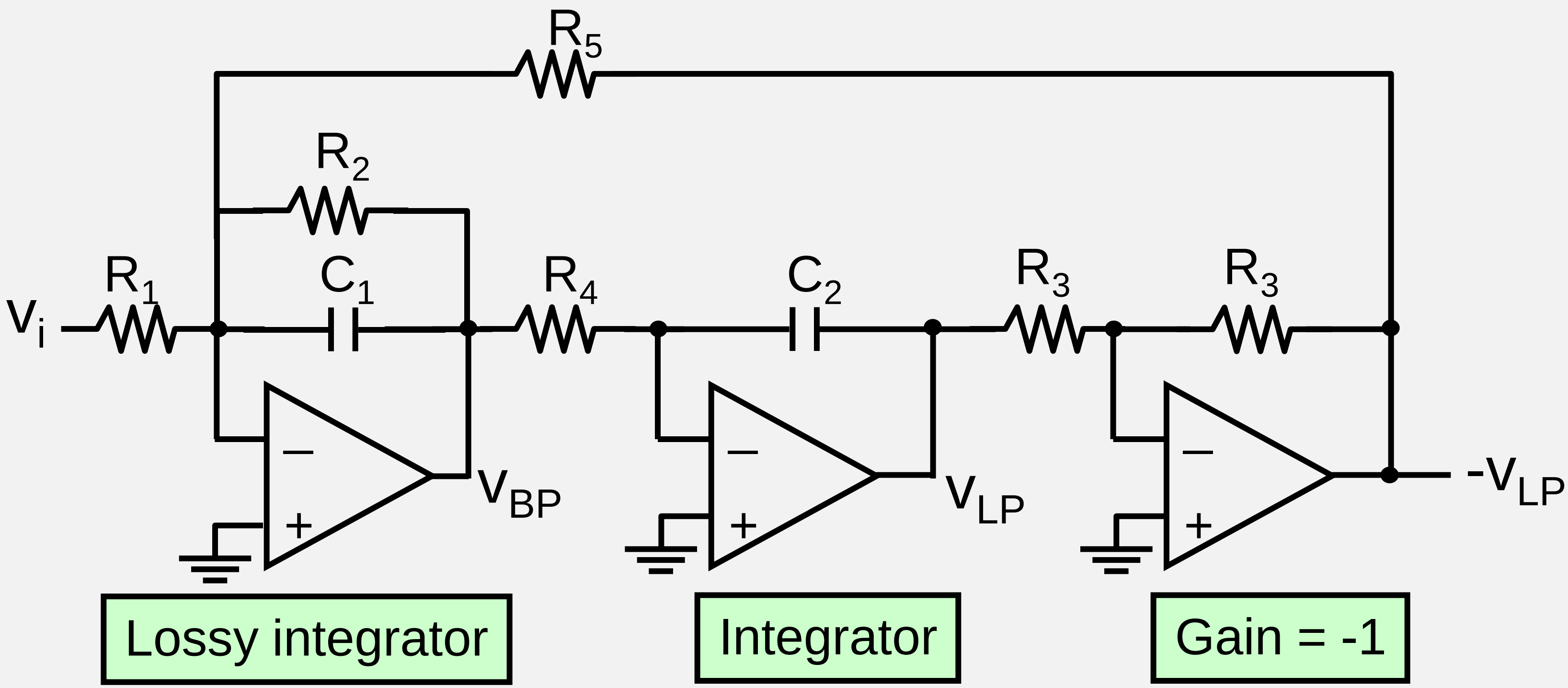
$$v_o = -\frac{R_f}{R}\left(v_{oL} + v_{oH}\right)$$



Cont'd: Frequency Response



Biquad Filter (Tow-Thomas Filter) (Design Problem I)



- It simultaneously realizes the band-pass (v_{BP}) and low-pass (v_{LP}) frequency responses

Cont'd: Please Derive $\frac{v_{LP}(s)}{v_i(s)}$ on Your Own

At the inverting node of first op amp, by KCL:

$$\frac{v_i}{R_1} - \frac{v_{LP}}{R_5} + \frac{v_{BP}}{R_2} + \frac{v_{BP}}{1/sC_1} = 0 \qquad (1)$$

$$\text{and } v_{LP} = -\frac{v_{BP}}{R_4 C_2 s} \qquad (2)$$

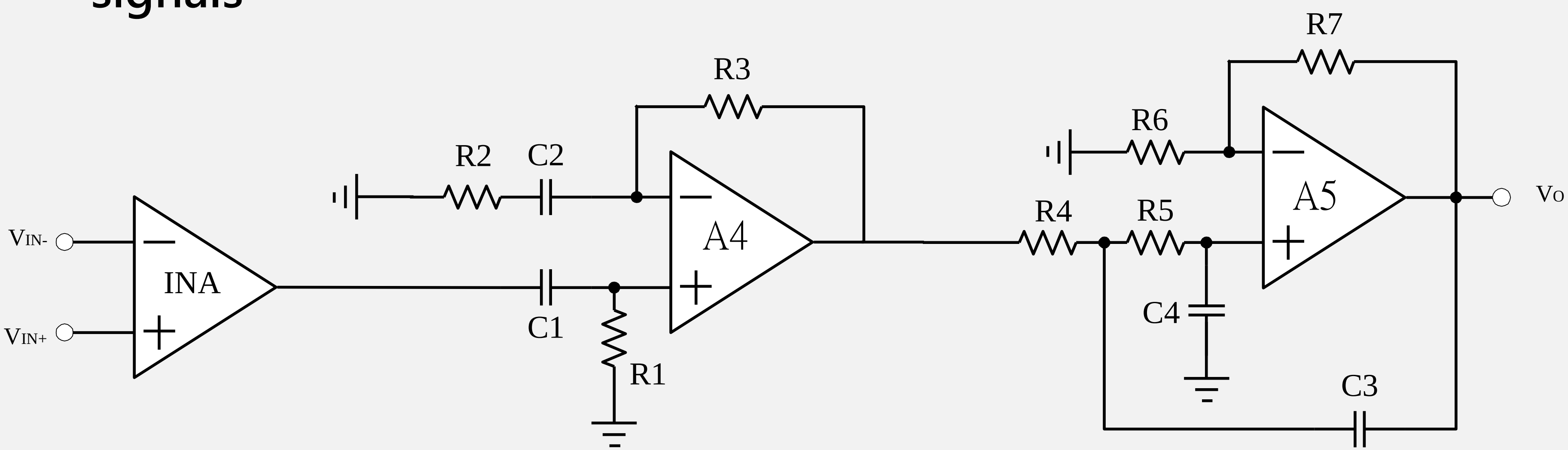
Solve (1) and (2):

$$\therefore \frac{v_{LP}(s)}{v_i(s)} = \frac{R_5}{R_1} \cdot \frac{1}{R_4 R_5 C_1 C_2 s^2 + \frac{R_4 R_5 C_2}{R_2} s + 1}$$

$$\therefore \omega_n = \frac{1}{\sqrt{R_4 C_1 R_5 C_2}} \qquad Q = \frac{R_2 \sqrt{C_1}}{\sqrt{R_4 R_5 C_2}}$$

ECG Measurement (Design Problem II to IV)

- 針對ECG量測，要慎選filter的-3dB frequency,才能濾掉interference signals



Instrumentation Amplifier

High-Pass Filter

Low-Pass Filter