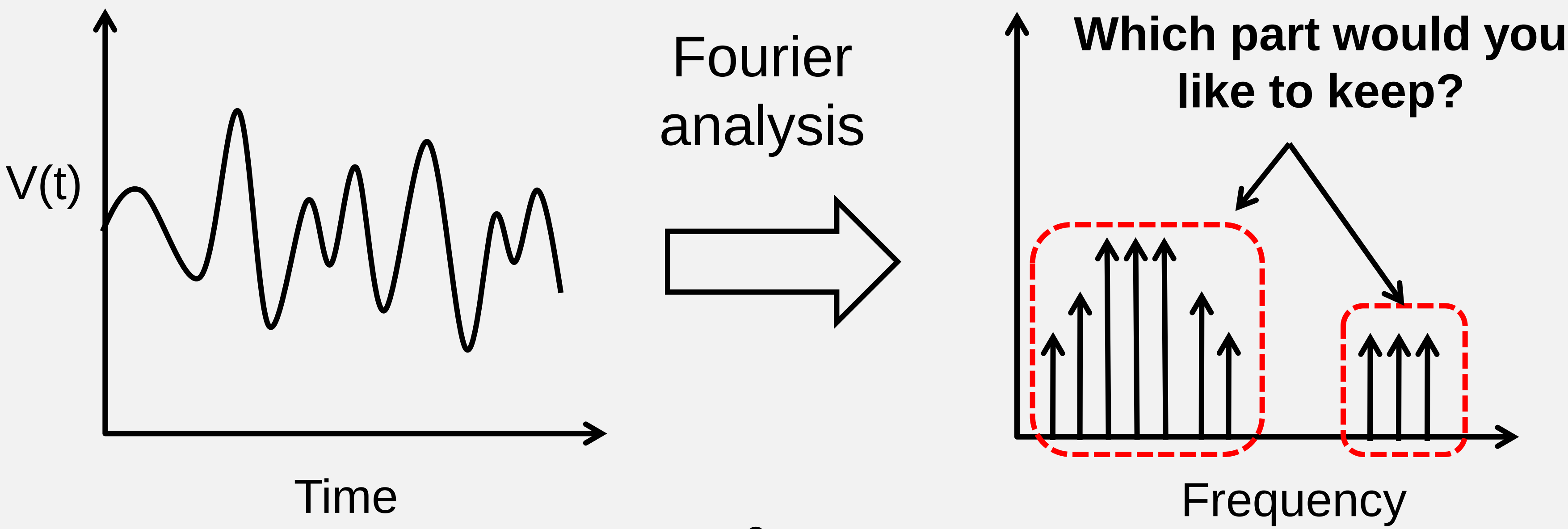
Lab 3: Passive Filters（被動式濾波電路）

- Filter types
- Frequency Response: Bode Diagram

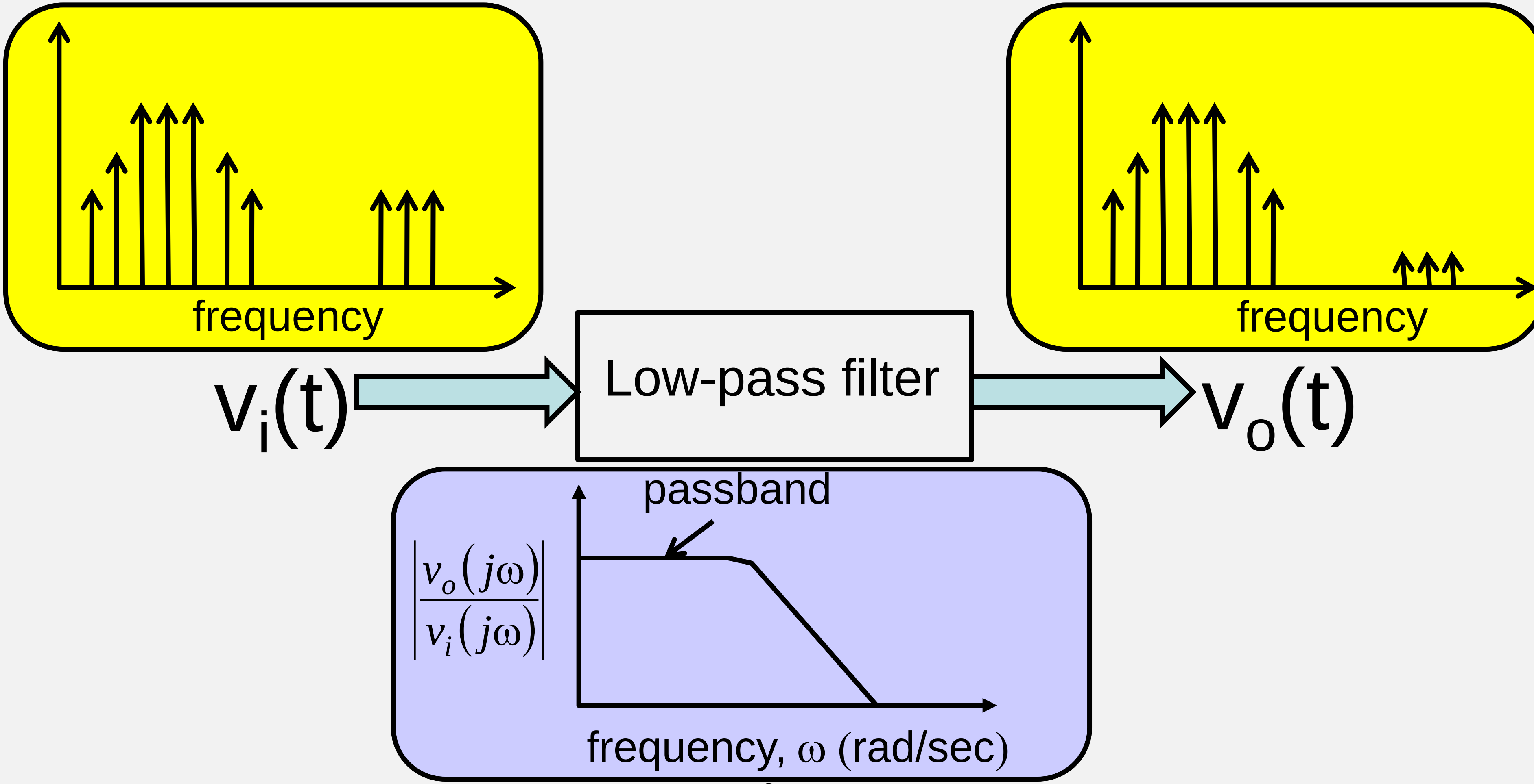
Signals: Time Domain（時域）vs. Frequency Domain（頻域）

- The spectrum（頻譜）of a time-domain signal can be obtained by Fourier analysis
- Filters of various types can keep or attenuate signals at different frequency ranges
- What are passive and active filters?



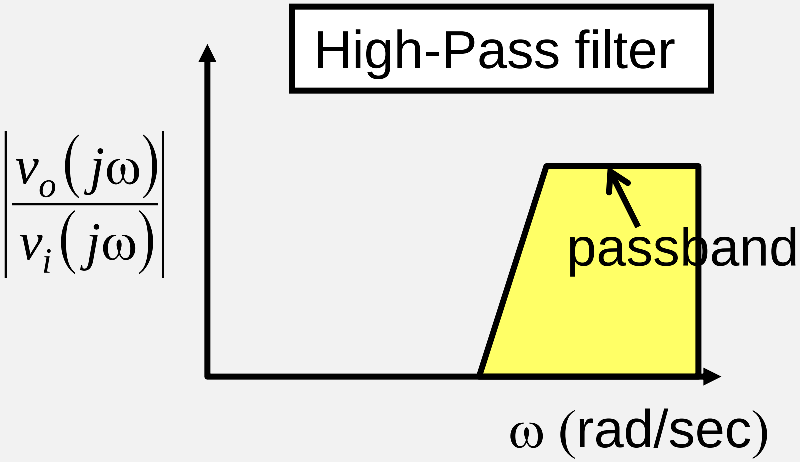
Filter Types: First, a Low-Pass Filter（低通濾波器）

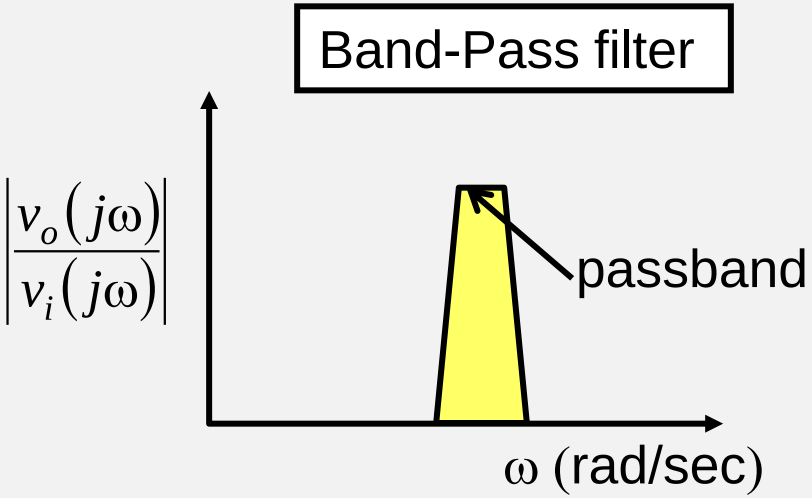
- The frequency response of a low-pass filter as shown attenuates high-frequency components of the signal

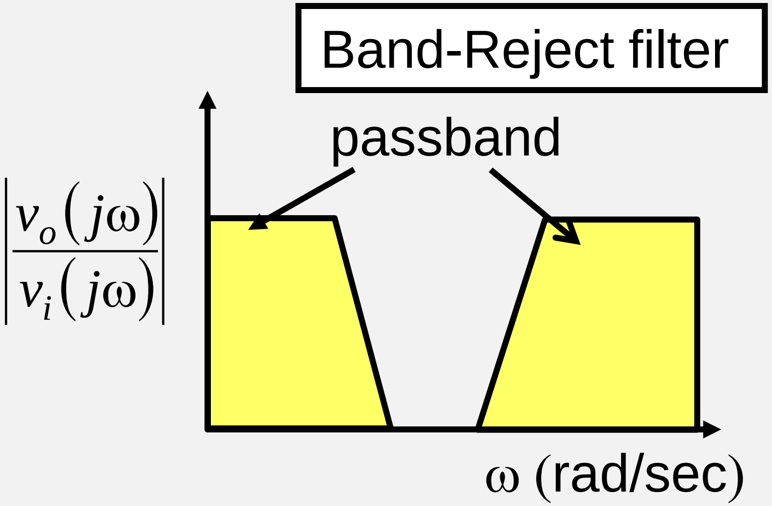


Cont'd: Frequency Responses of Other Types of Filters

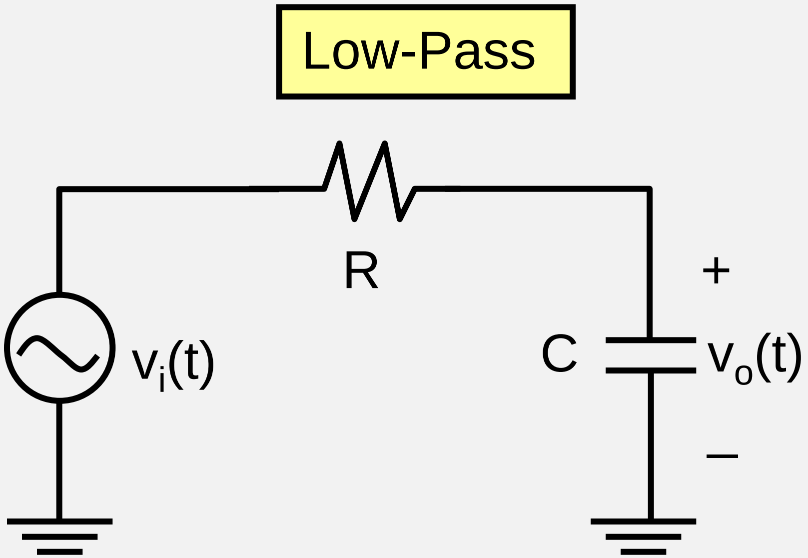
- High-pass filter
- Band-pass filter
- Band-reject filter
- 我們稍後會講解何謂 frequency response，以下先講解如何以LRC來製作filter

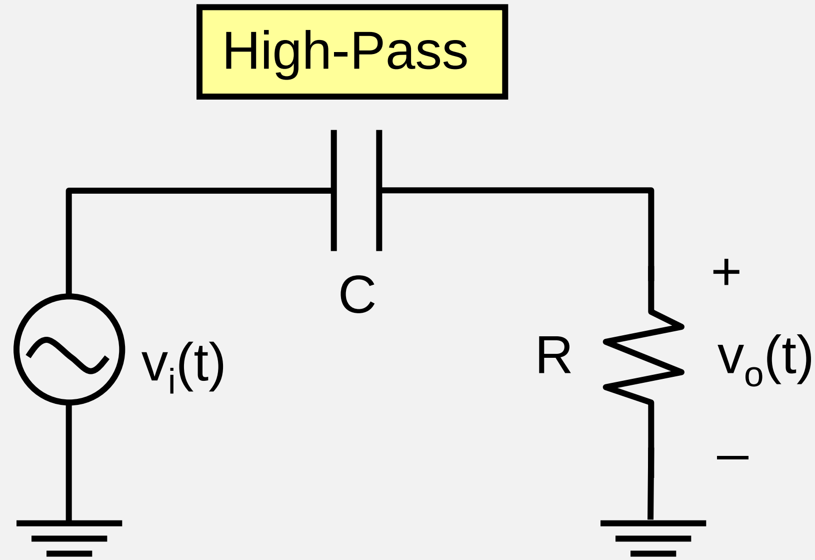






Low-Pass and High-Pass Filters



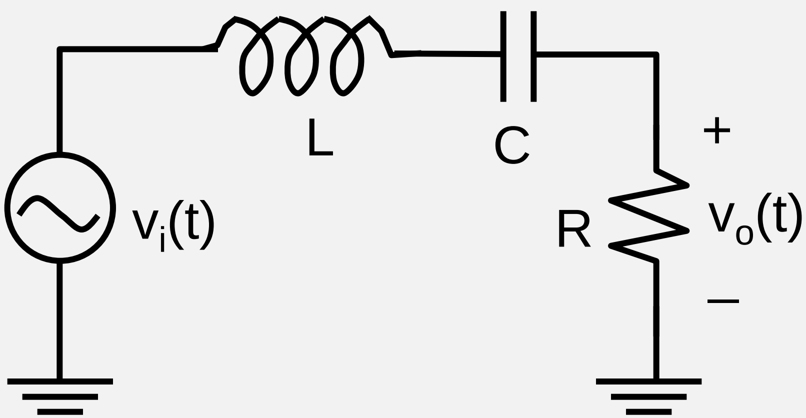


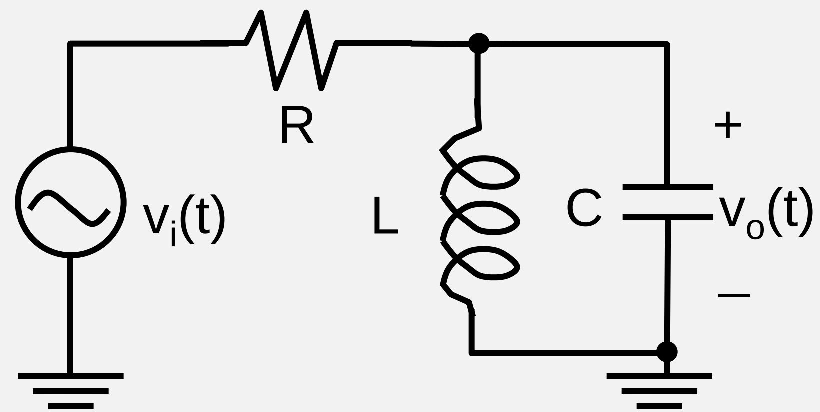
$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1}$$

$$\frac{v_o(s)}{v_i(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{RCs + 1}$$

從impedance的觀點來導出不同的濾波電路

Band-Pass Filters



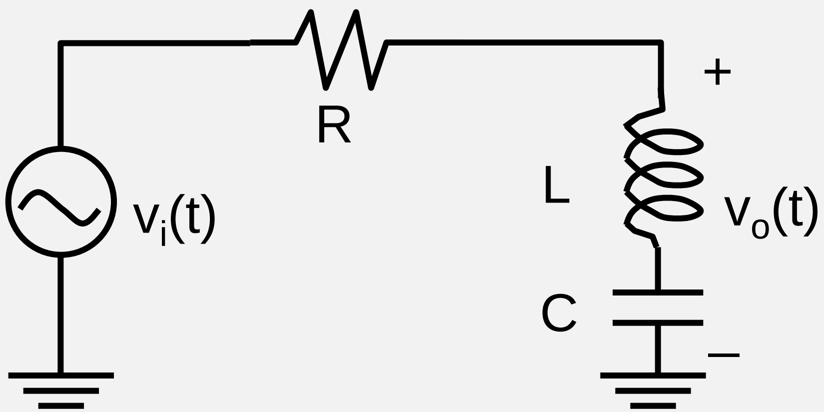


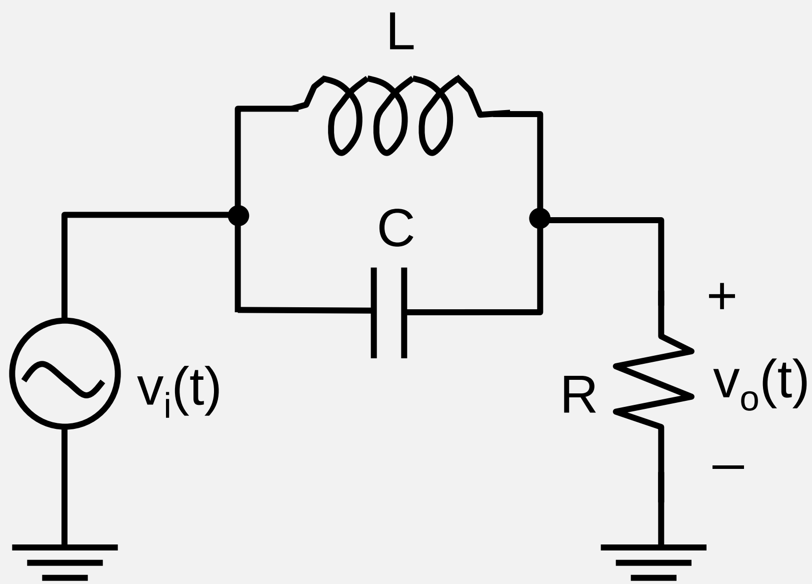
$$\frac{v_o(s)}{v_i(s)} = \frac{R}{R + \left(sL + \frac{1}{sC}\right)}$$

$$= \frac{RCs}{LCs^2 + RCs + 1}$$

$$\frac{v_o(s)}{v_i(s)} = \frac{Ls}{LRCs^2 + Ls + R}$$

Band-Reject Filters



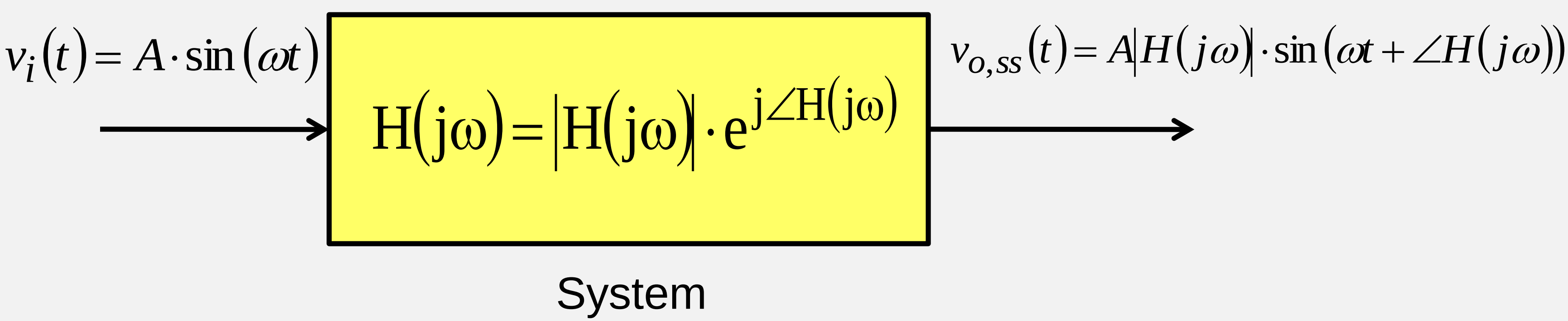


$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

What is the Frequency Response of a Circuit/System?

- Given a sinusoidal input to a system (for example, a circuit), the steady-state output（記得，強調的是穩態輸出） is modified by the gain and phase of **system's** frequency response
- Frequency response of a system can be represented by Bode diagram, and used in filtering problems (this lab) and control system problems (大三下控制系統)



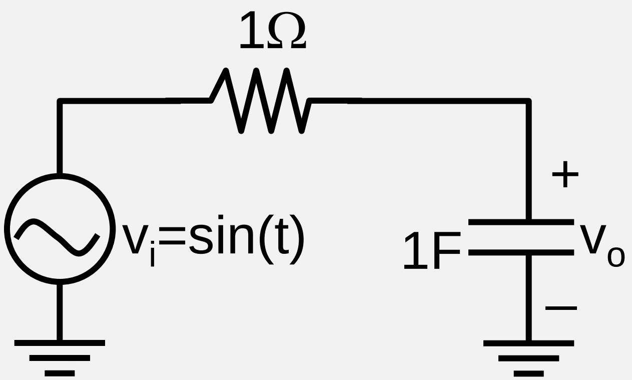
Example: a Low-Pass Filter

- Here is a RC circuit: v_i = sin(t)

$$\frac{v_o(s)}{v_i(s)} = \frac{1}{s+1} \qquad v_i(s) = \frac{1^2}{s^2+1^2}$$

$$v_o(s) = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{a}{s+1} + \frac{bs+c}{s^2+1}$$

$$= \frac{0.5}{s+1} + \frac{-0.5s+0.5}{s^2+1}$$



$$v_o(t) = 0.5e^{-t} - 0.5\cos t + 0.5\sin t$$

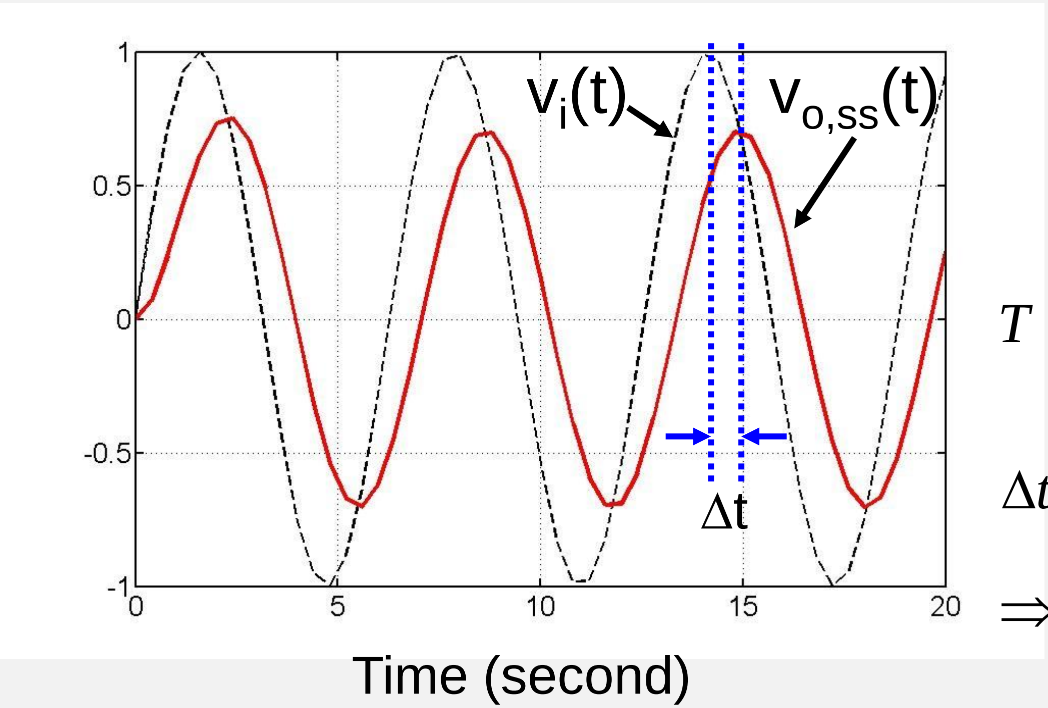
$$= 0.5e^{-t} + \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\sin t - \frac{1}{\sqrt{2}}\cos t\right)$$

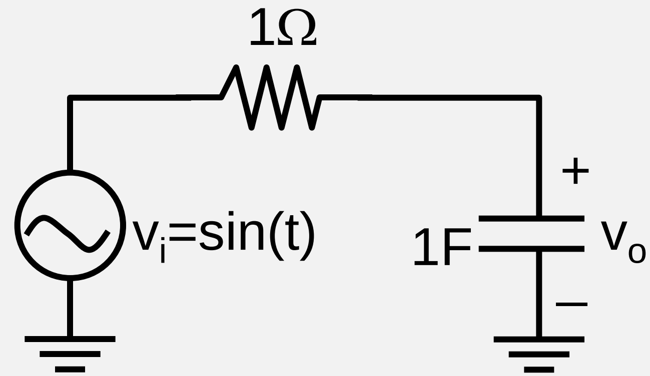
$$= 0.5e^{-t} + 0.707\sin(t-45°)$$

$$v_{o,ss}(t) = 0.707\sin(t-45°)$$

0.707 and -45° are the magnitude and phase of the frequency response at ω = 1 rad/sec

Cont'd: Obtain the Phase from Waveforms





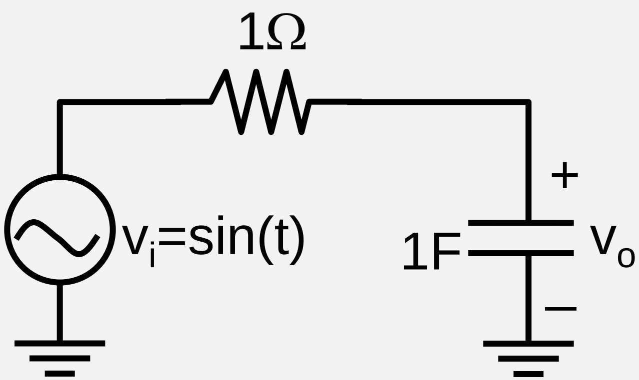
$$T = \frac{1}{f} = \frac{2}{\omega/2\pi} = 2\pi \;\; for\, \omega = 1\, rad/\sec$$

$$\Delta t = \frac{\theta}{360°} \cdot T = 0.785 \text{ sec}$$

$$\Rightarrow \theta = 45° \left(+ or - ?\right)$$

-45° means $v_{o,ss}(t)$ lags $v_i(t)$ by 0.785 sec.

Cont'd : Verify $|H(j\omega)|$ and $\angle H(j\omega)$ at $\omega = 1$ rad/sec



$$v_{o,ss}(t) = 0.707\sin(t - 45°)$$

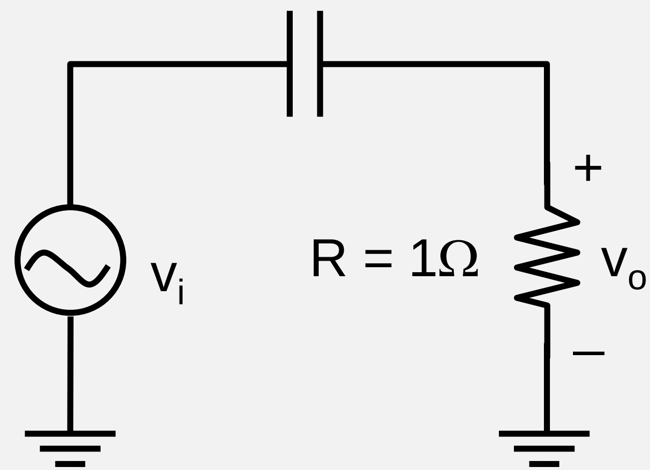
$$H(s) = \frac{v_o(s)}{v_i(s)} = \frac{1}{s+1}$$

$$H(j\omega)\Big|_{\omega=1} = \frac{1}{j+1} = 0.707\,e^{-j\frac{\pi}{4}}$$

Example: a High-Pass Filter

- Input v_i = sin(t)

$$\frac{v_o(s)}{v_i(s)} = \frac{s}{s+1} \quad \Rightarrow v_o(s) = \frac{s}{s+1} \cdot \frac{1^2}{s^2+1^2}$$



- 類似前例，以下請自行驗證：

$$v_o(t) = 0.5(\sin t + \cos t) - 0.5e^{-t}$$

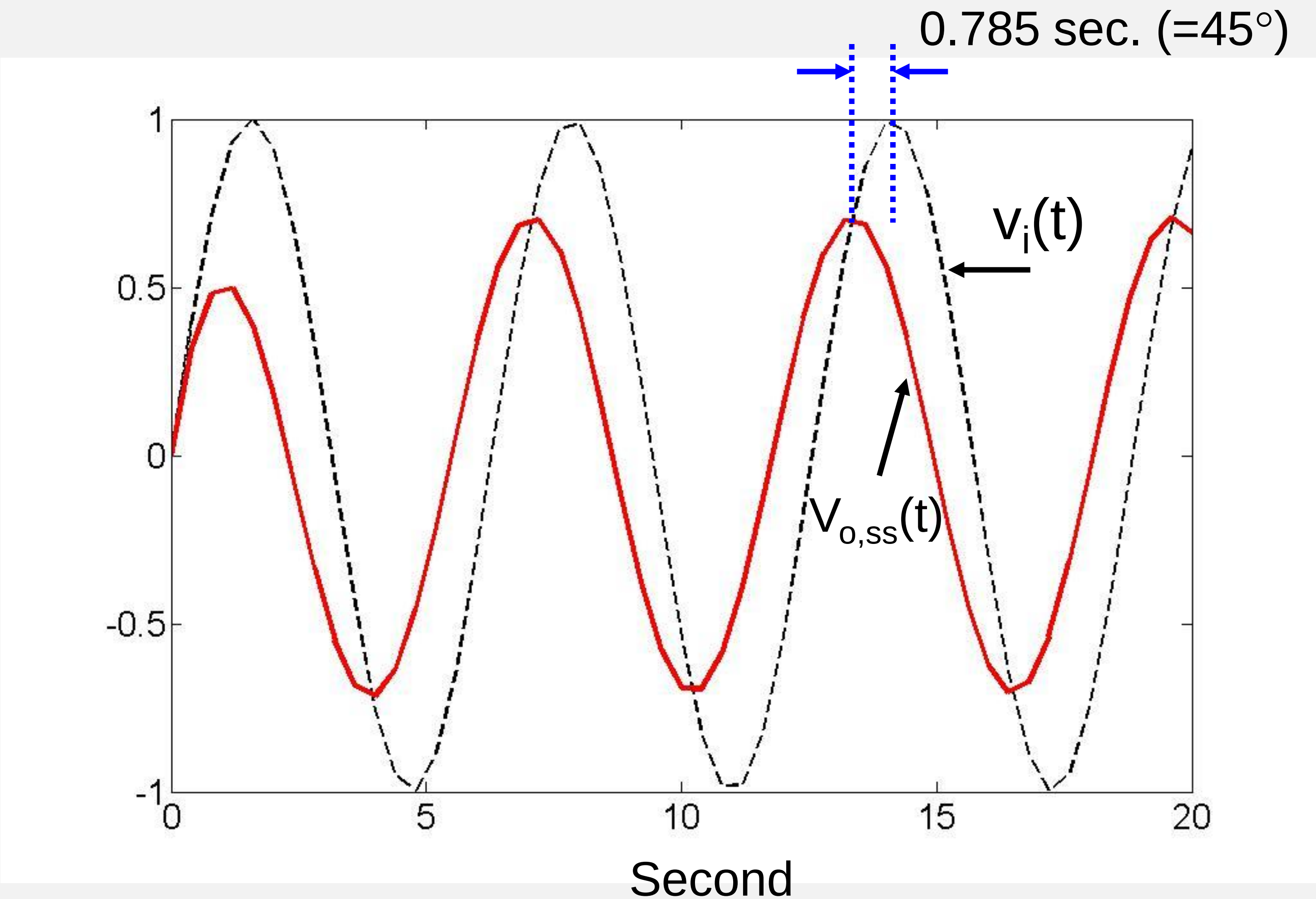
$$= 0.707 \sin(t + 45°) - 0.5e^{-t}$$

Steady-state part

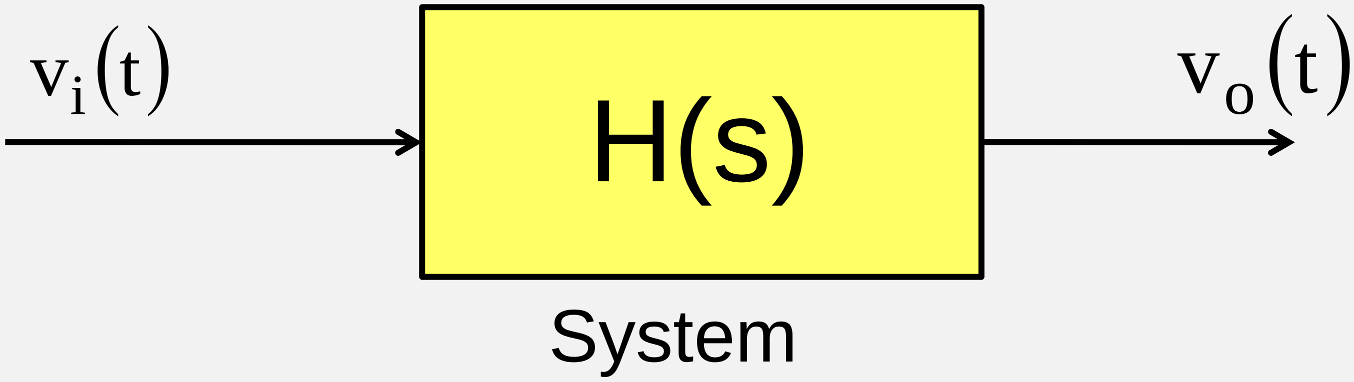
Transient part

0.707 and +45° are the magnitude and phase of the frequency response at ω = 1 rad/sec

Cont'd：觀察波形及相角 -- The Steady-State Output $V_{o,ss}$ Leads by 45°



Summary：Establish System's Frequency Response

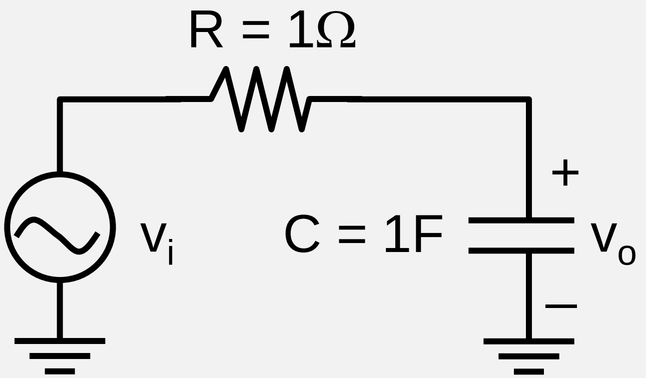


- 方法一、當你不知道系統表示式H(s)時：建議用剛剛例子所示範，輸入不同頻率的弦波，然後觀察相對應輸出弦波的值以及相角，來求得|H(jω)|及∠H(jω)
- 方法二、當知道系統表示式H(s)時：以s = jω代入H(s) (why? 如果有機會修控制系統時會提到)

$$H(j\omega) = |H(j\omega)| \cdot e^{j\angle H(j\omega)}$$

Frequency Response Represented by the Bode Diagram

- Bode diagram contains gain and phase plots with frequency represented by logarithmic scale
- By replacing s = jω to get frequency-domain representation
- Example: a low-pass filter



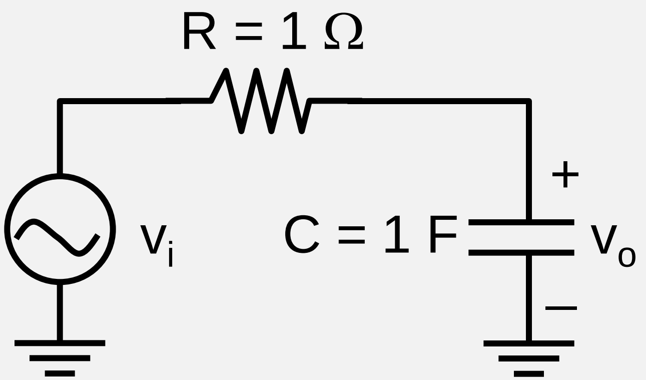
$$H(s) = \frac{v_o(s)}{v_i(s)} = \frac{1}{RCs + 1}$$

$$= \frac{1}{Ts + 1} \quad \text{where} \quad T = RC = 1\,\text{sec}$$

$$s = j\omega \Rightarrow H(j\omega) = \frac{v_o(j\omega)}{v_i(j\omega)} = \frac{1}{j\omega T + 1}$$

Cont'd: Magnitude of a LPF

$$\left|H(j\omega)\right| = \left|\frac{v_o(j\omega)}{v_i(j\omega)}\right| = \frac{1}{\sqrt{\omega^2 T^2 + 1}}$$



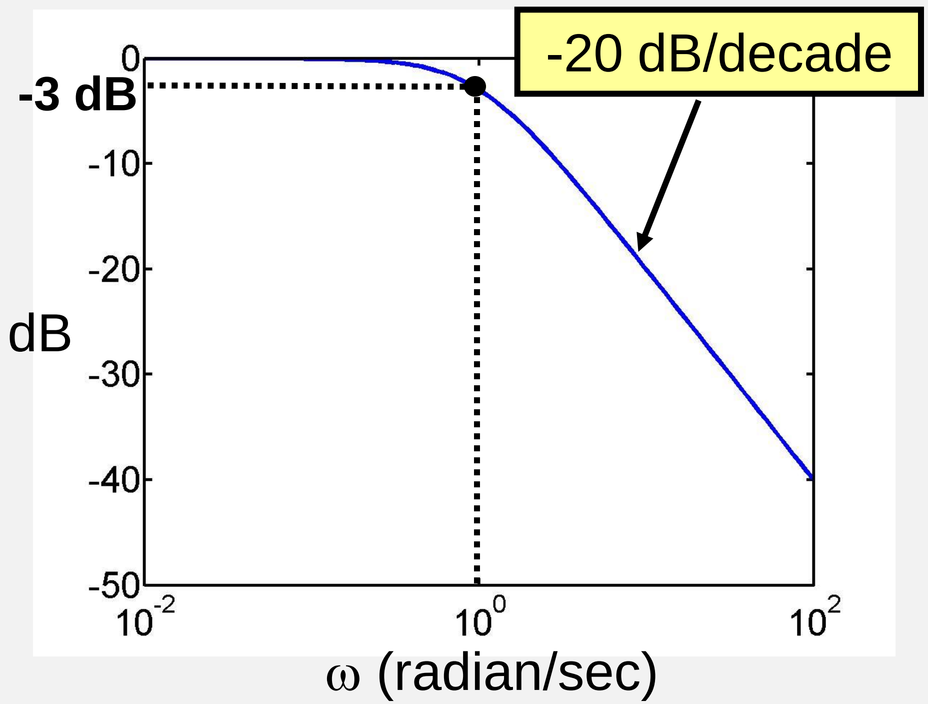
- Unit of magnitude: Decibel (dB)

$$20Log_{10}\left|H(j\omega)\right|$$

$$\left|H(j\omega)\right|_{\omega=0} = 1 = 0\ dB$$

$$\left|H(j\omega)\right|_{\omega=1/T} = \frac{1}{\sqrt{2}} = 0.707 \ = -3\ dB$$

$$\left|H(j\omega)\right|_{\omega=\infty} = 0 = -\infty dB$$



Cont'd: Phase of a LPF

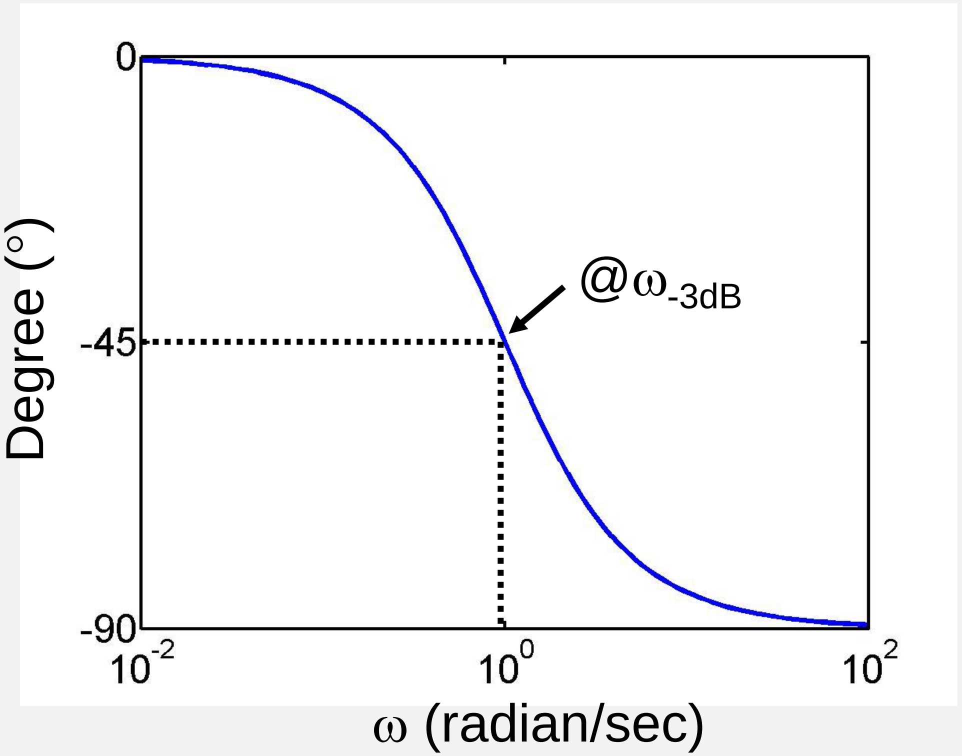
$$H(j\omega)=\frac{v_o(j\omega)}{v_i(j\omega)}=\frac{1}{j\omega T+1}$$

$$\angle H(j\omega)=\angle\frac{1}{j\omega T+1}=-tan^{-1}(\omega T)$$

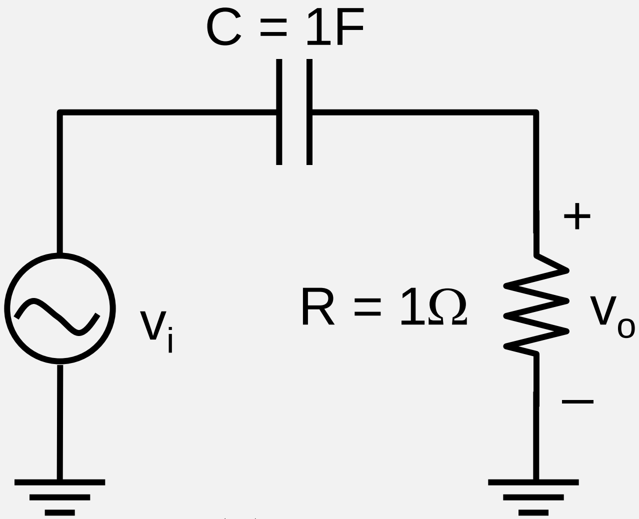
$$\angle H(j0)=-tan^{-1}(0T)=0^\circ$$

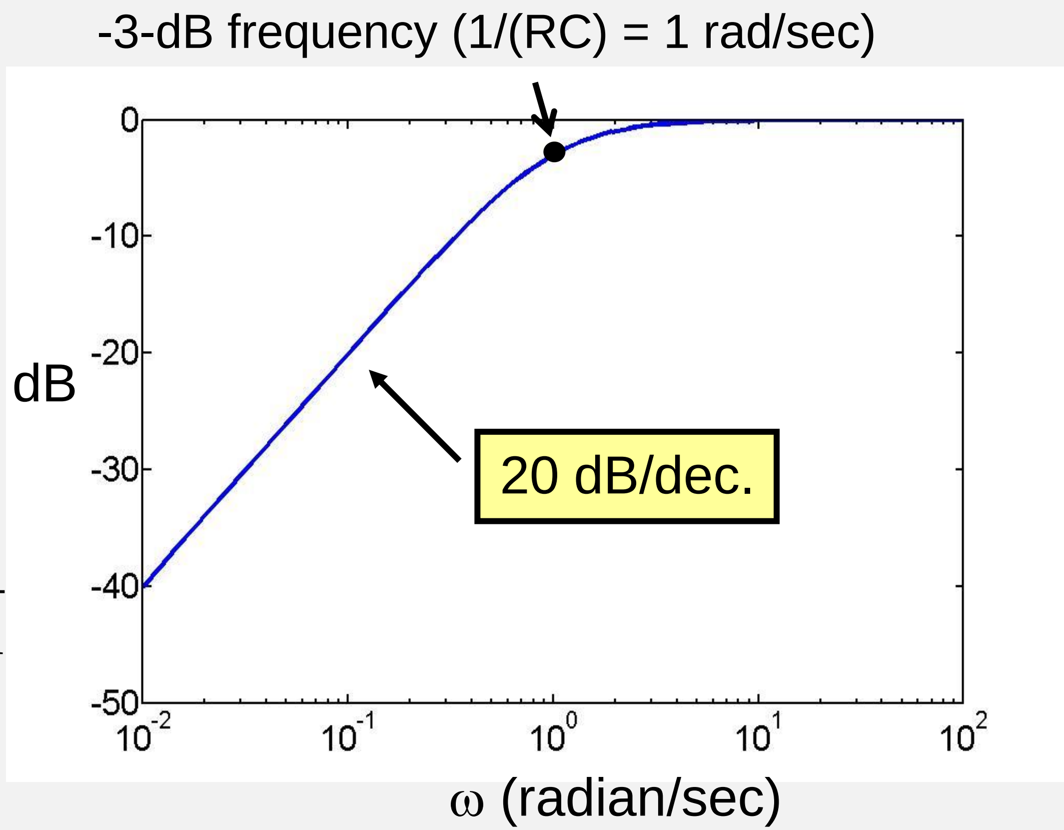
$$\angle H\left(j\frac{1}{T}\right)=-tan^{-1}(1)=-45^\circ$$

$$\angle H(j\infty)=-tan^{-1}(\infty)=-90^\circ$$



Example: A High-Pass Filter





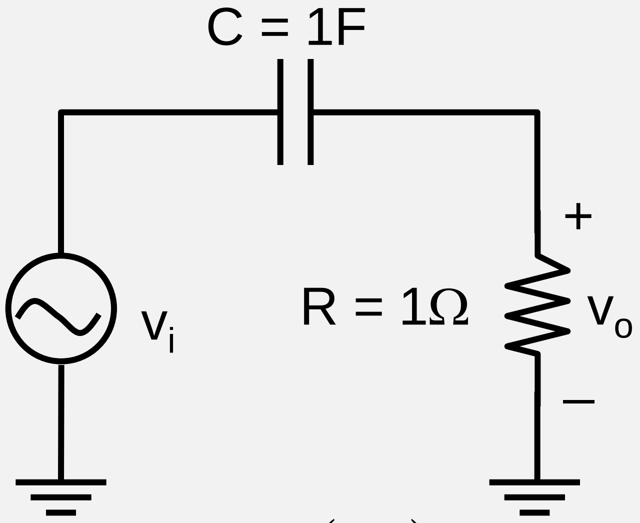
$$H(s)=\frac{v_o(s)}{v_i(s)}=\frac{sRC}{sRC+1}=\frac{s}{s+1}$$

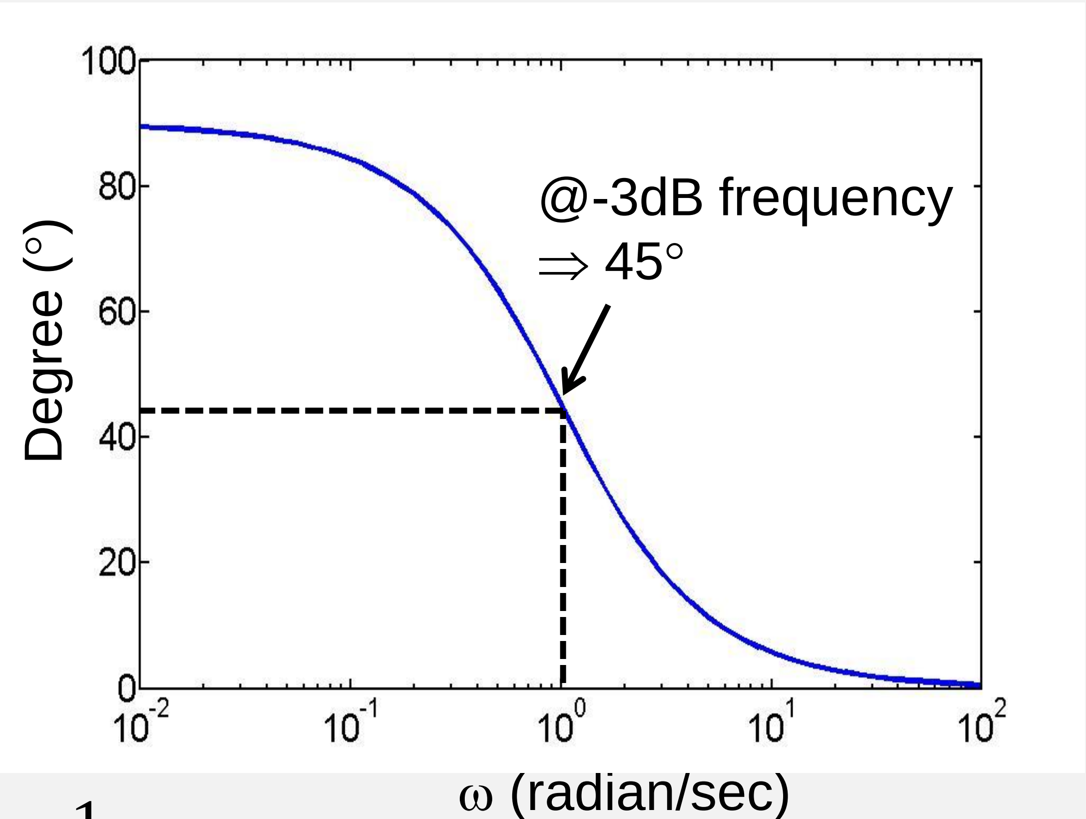
- **Magnitude:**

$$\left|H(j\omega)\right|=\frac{\omega}{\sqrt{\omega^2+1}}$$

$$\left|H(j0)\right|=0=-\infty dB;\left|H\left(j\frac{1}{T}\right)\right|=0.707=-3dB;\left|H(j\infty)\right|=1=0dB;$$

Cont'd: Phase of a HPF





$$H(j\omega)=\frac{v_o(j\omega)}{v_i(j\omega)}=\frac{j\omega}{j\omega+1}$$

- Phase:

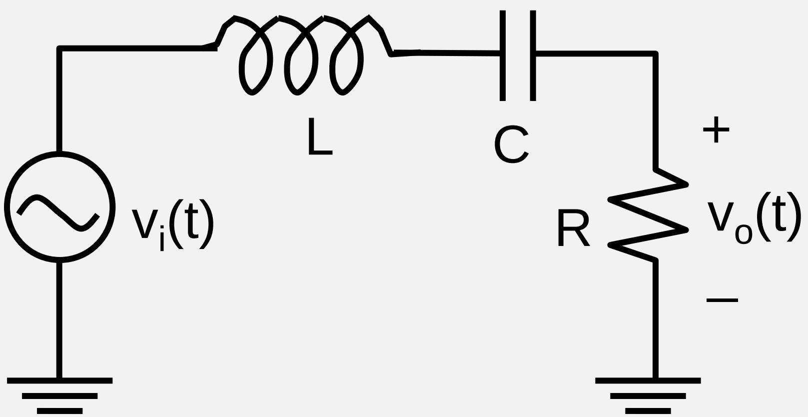
$$\angle H(j\omega)=90^\circ-\tan^{-1}\omega$$

$$\angle H(j0)=90^\circ;\angle H\left(j\frac{1}{T}\right)=45^\circ;\angle H(j\infty)=0^\circ$$

Bode Diagram: 2nd-Order Systems

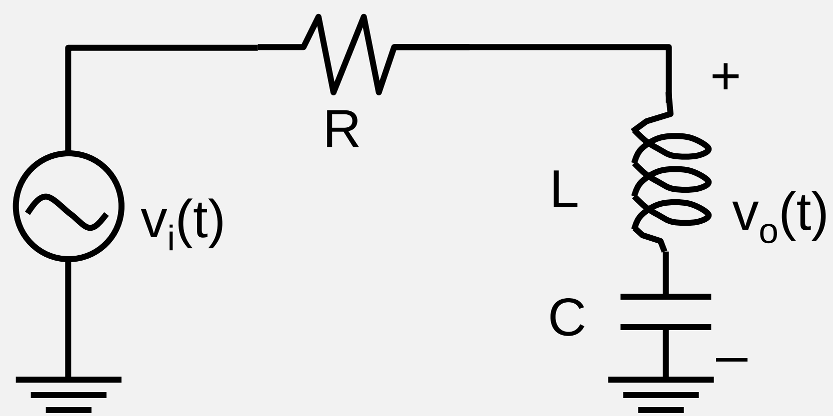
- 相較於RC組成的一階電路，經LRC組成的電路其分母可達到二階，如之前提到的band-pass及band-reject filters

Band-Pass



$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Band-Reject

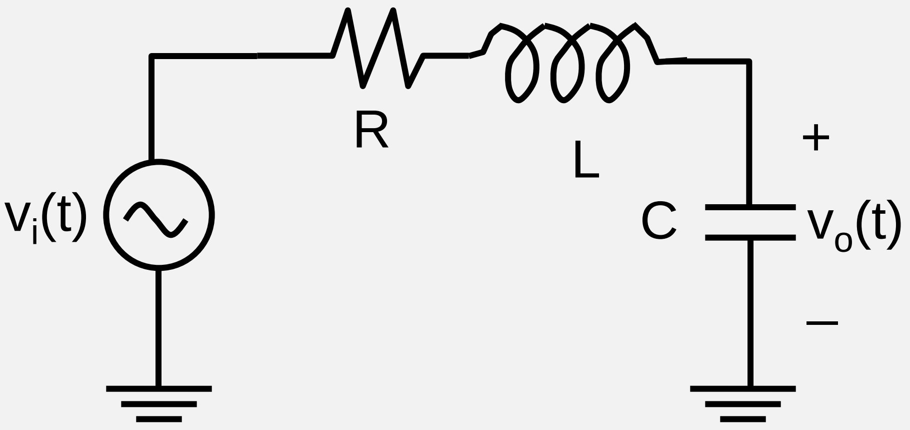


$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

2^{nd}-Order Low-Pass Filter

- 如圖，這是一個二階的low-pass filter，瞭解它的frequency response將有助於瞭解band-pass及band-reject filters之frequency responses

$$H(s)=\frac{v_o(s)}{v_i(s)}=\frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



$$=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\qquad \omega_n=\frac{1}{\sqrt{LC}}$$

- Replace s = jω:

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\xi\omega_n(j\omega)+\omega_n^2}=\frac{\omega_n^2}{\left(\omega_n^2-\omega^2\right)+j2\xi\omega_n\omega}$$

$$=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\xi\frac{\omega}{\omega_n}}$$

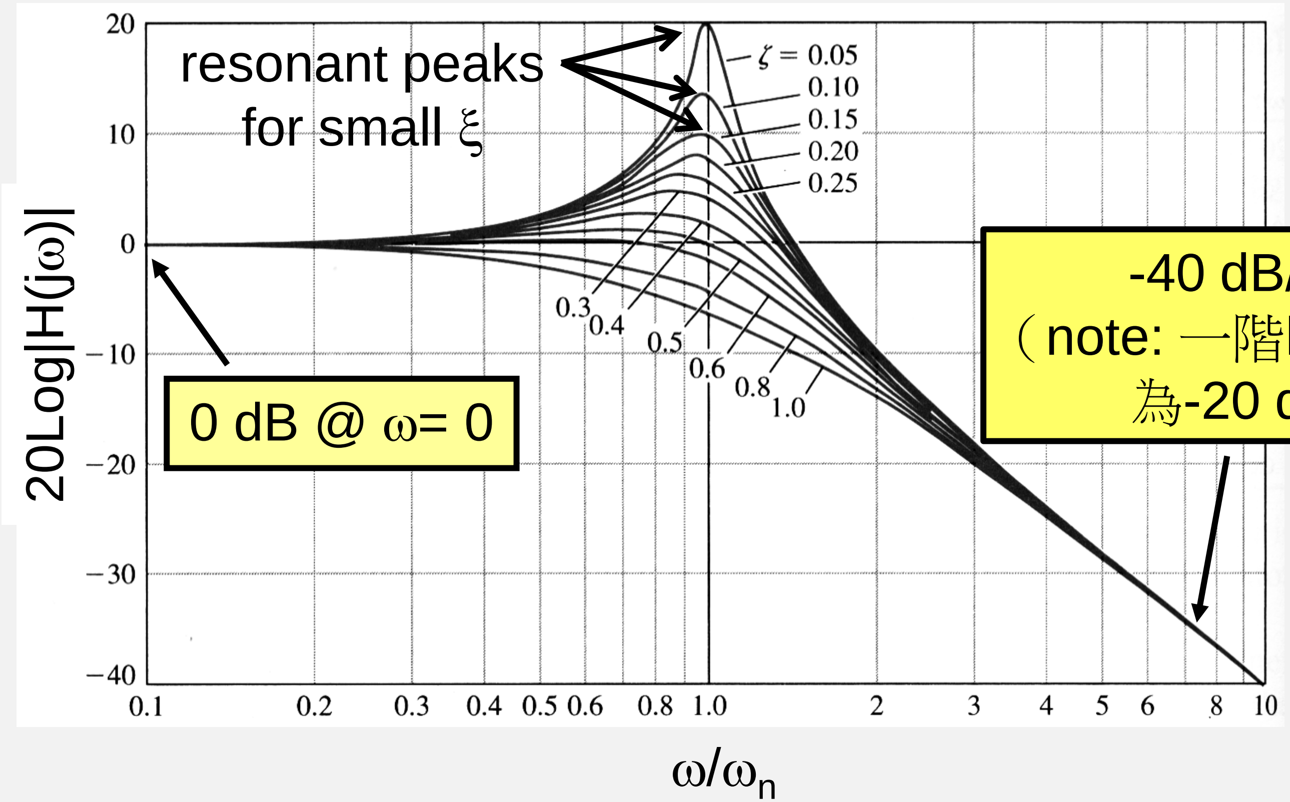
Cont'd: Magnitudes for Different Damping Ratios ξ

- Magnitude: $\left|H(j\omega)\right| = \dfrac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\dfrac{\omega}{\omega_n}\right)^2}}$

$$\left|H(j\omega)\right|_{\omega=0} = 1$$

$$\left|H(j\omega)\right|_{\omega=\omega_n} = \frac{1}{2\xi}$$

$$\left|H(j\omega)\right|_{\omega=\infty} = 0$$



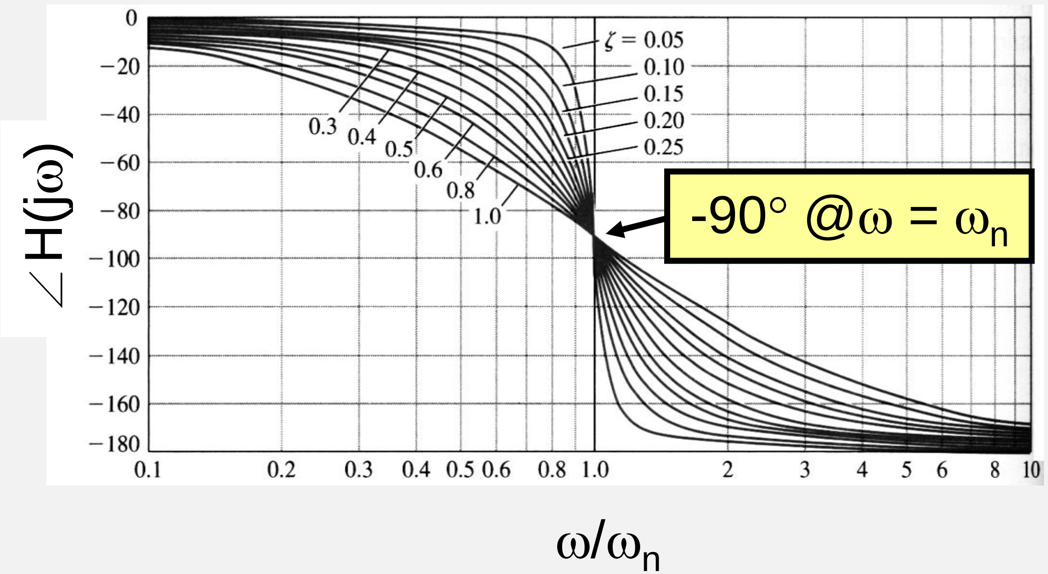
Cont'd: Phases for Different Damping Ratios ξ

- The transfer function:

$$H(j\omega)=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\xi\frac{\omega}{\omega_n}} \Rightarrow \angle H(j\omega)=-tan^{-1}\underbrace{\left[\frac{2\xi\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)}\right]}_{0\rightarrow 180^\circ}$$

- Note:
  - $\angle H(j\omega)|_{\omega=0} = 0°$;
  - $\angle H(j\omega)|_{\omega=\infty} = -180°$

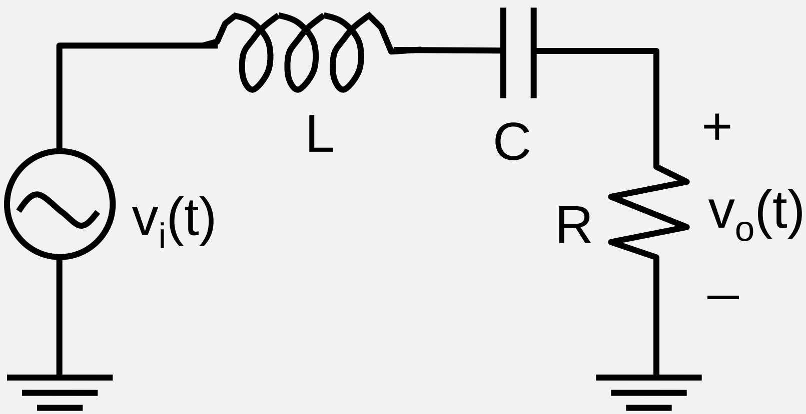
- Note: $\angle H(j\omega_n) = -90°$ for all ξ values



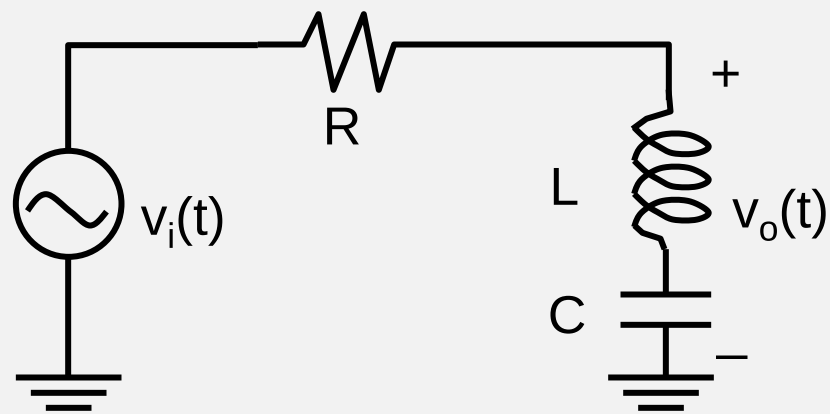
Bode Diagrams for Band-Pass, Band-Reject, and Low-Pass Filters

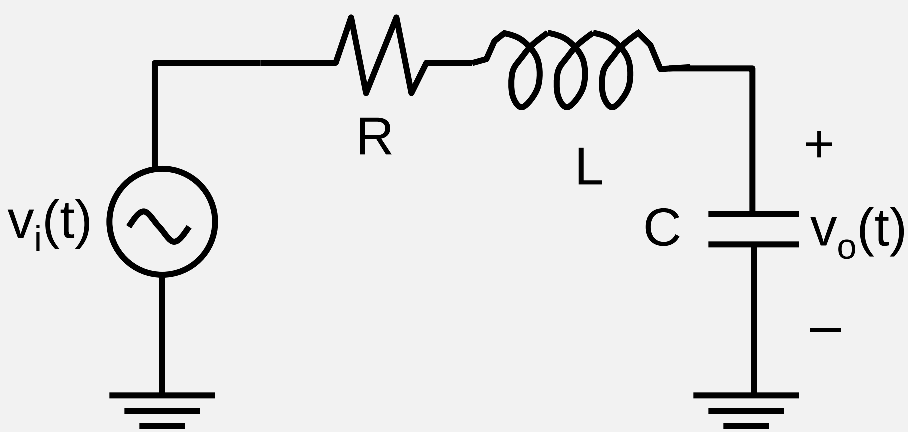
- 假設R = 1 Ω, L = 1 H, C = 1 F, 比較此三種filters的Bode diagrams

Band-Pass



Band-Reject





Low-Pass

Cont’d:假設R = 1 Ω, L = 1 H, C = 1 F

- 想想看：low-pass, band-pass及band-reject filters在自然振頻 ($\omega_n = 1/\sqrt{LC}$) 的增益各是多少？

Band-Pass:

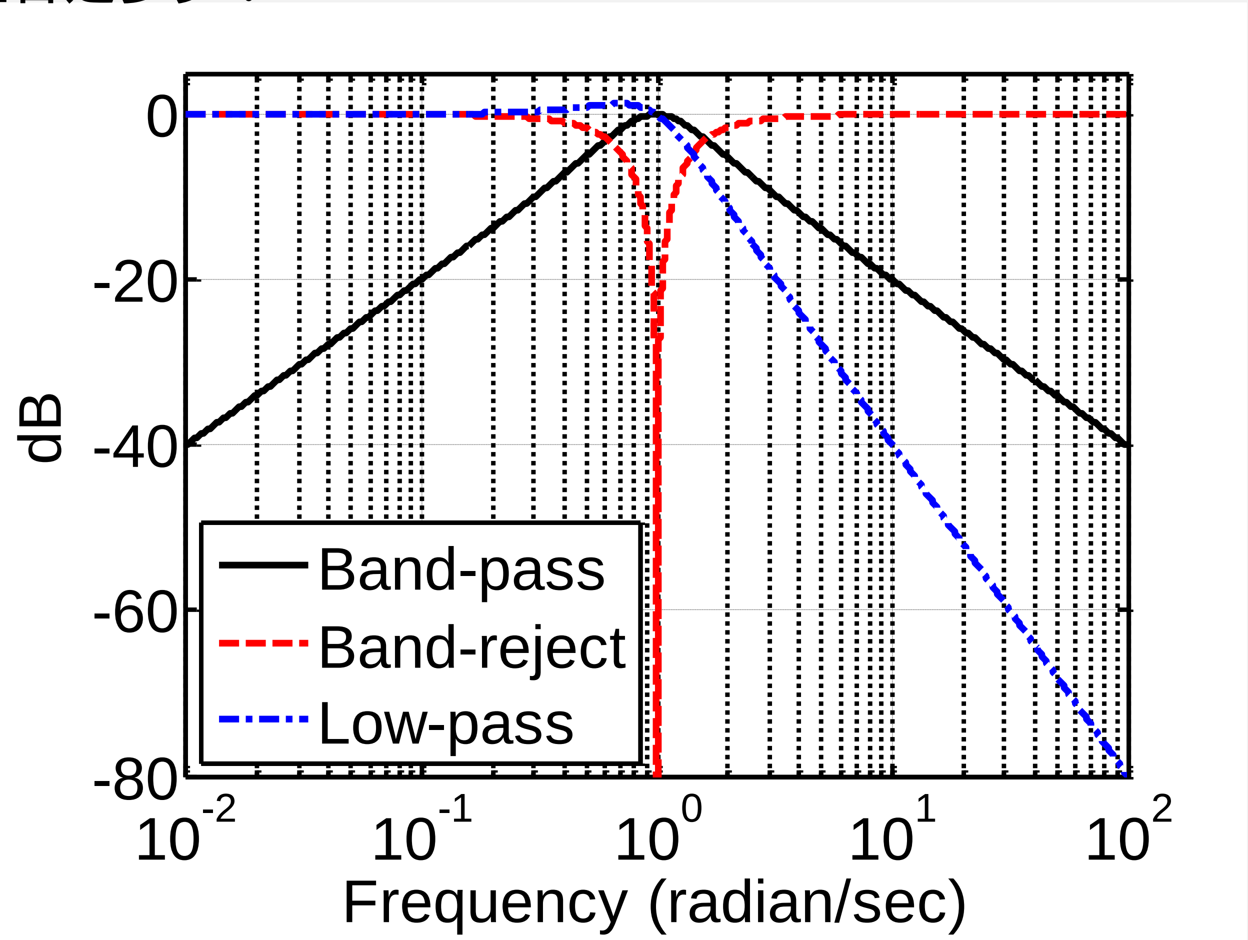
$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Band-Reject:

$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

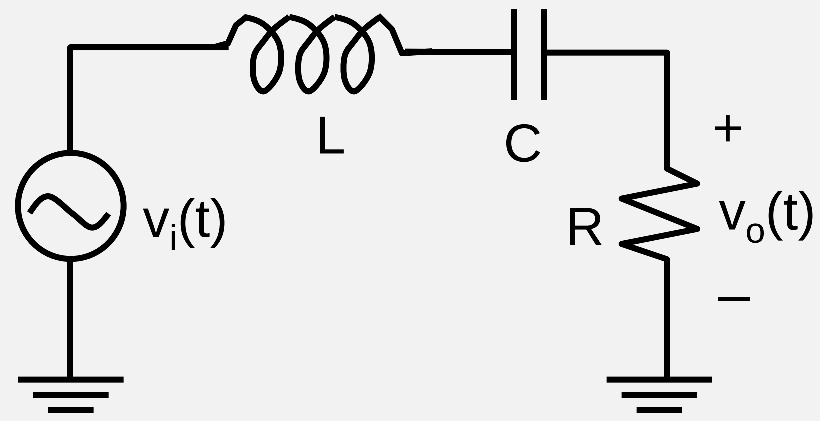
Low-Pass:

$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



Band-Pass Filter: the Effect of Adjusting the Value of R

- Let L = 1 H, C = 1 F
- In this case, the damping ratio reduces when the resistor value decreases

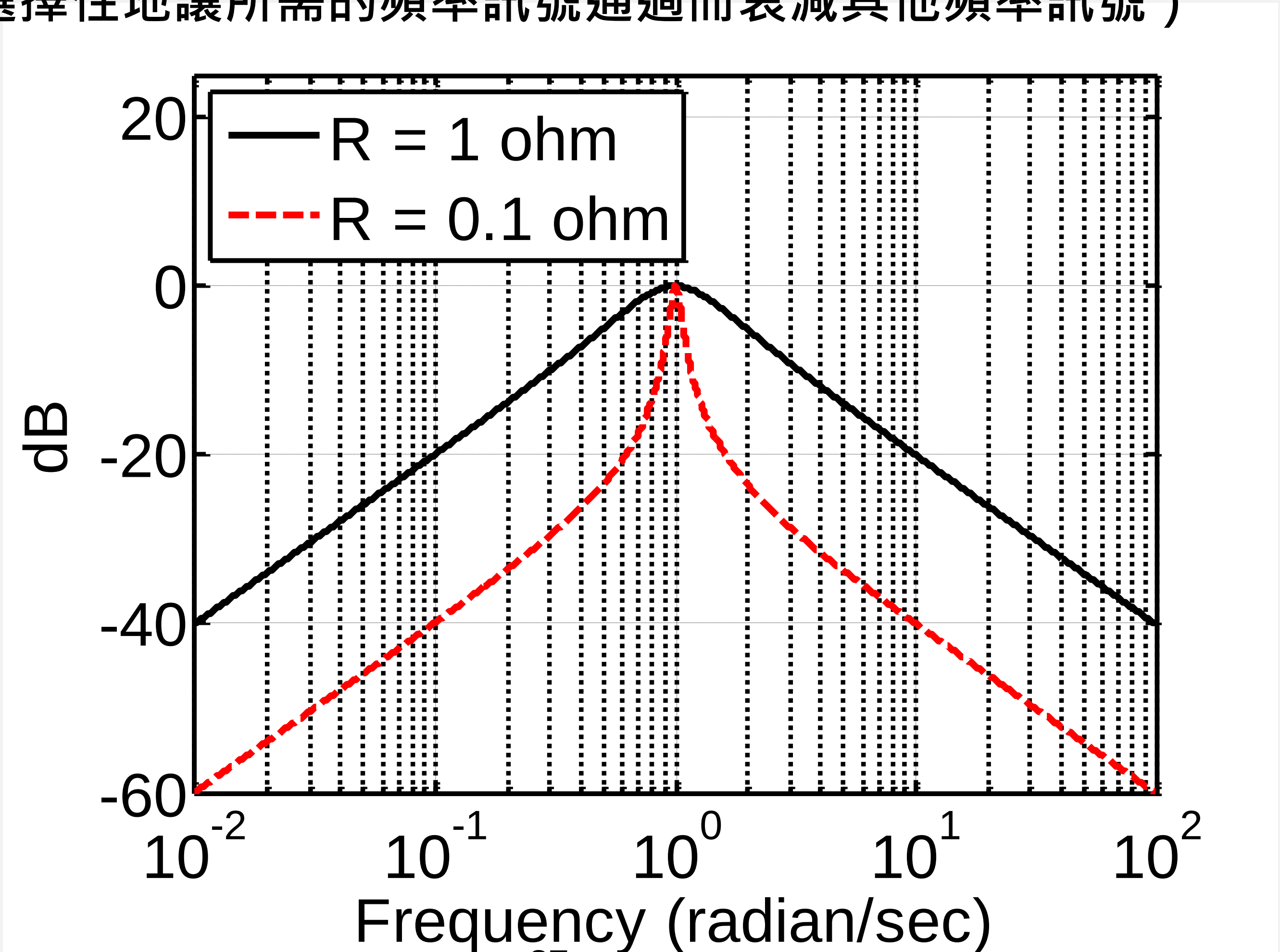


$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}, \omega_n = \frac{1}{\sqrt{LC}}$$

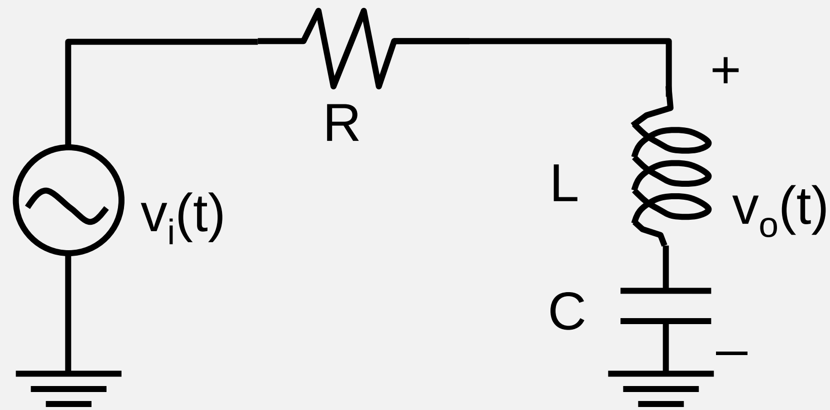
Cont'd: Frequency Responses for Different Values of R

- Damping ratio降低使得passband 的頻寬變窄，濾波的效果變好（可以更選擇性地讓所需的頻率訊號通過而衰減其他頻率訊號）



Band-Reject Filter: the Effect of Adjusting the Value of R

- 假設L = 1 H, C = 1 F
- In this case, the damping ratio reduces when the resistor value decreases



$$\frac{v_o(s)}{v_i(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}, \omega_n = \frac{1}{\sqrt{LC}}$$

Cont'd: Frequency Responses for Different Values of R

- Damping ratio降低使得濾波的效果變好 ，可以 更選擇性地衰減不要的頻率訊號而留下想要的部分