Time-Domain Responses of 1st- and 2nd-Order Circuits

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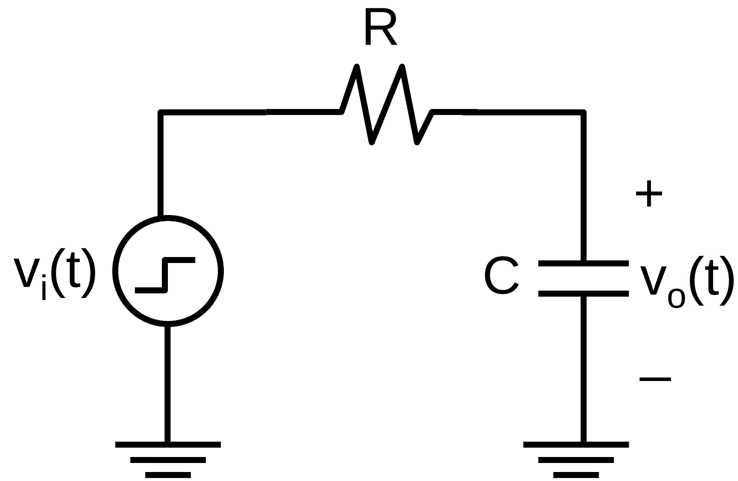
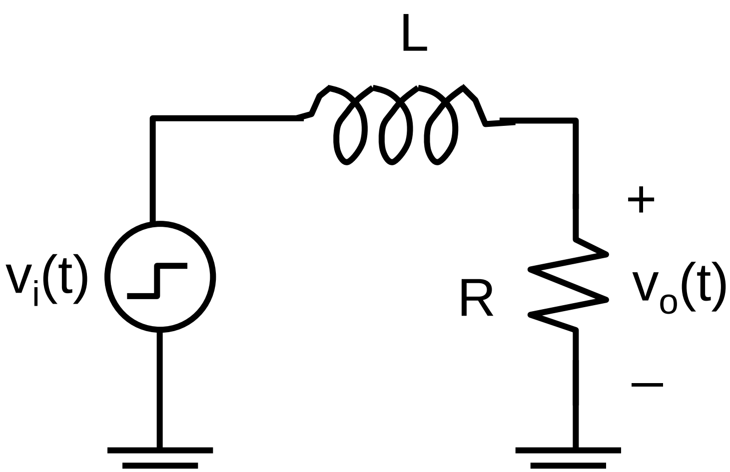
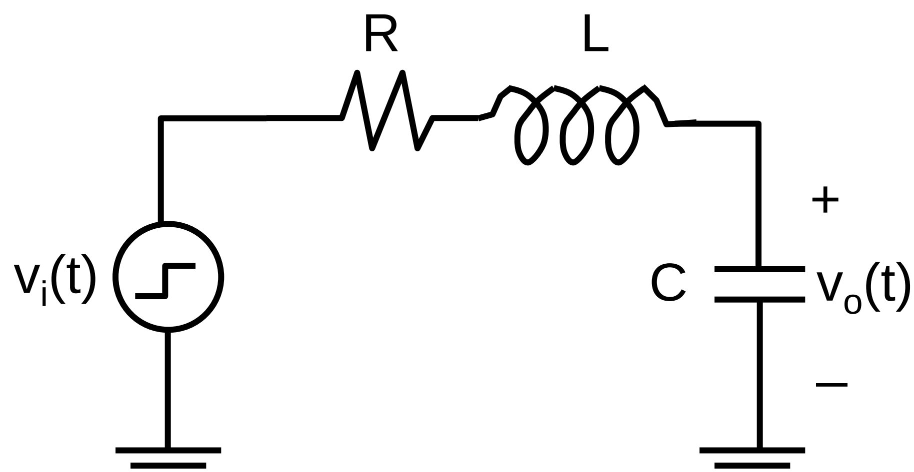
幾個RLC組成的1st and 2nd-Order Circuits

- 如何判斷1st or 2nd-order?
  - 將$v_o(s)/v_i(s)$寫出來，觀察分母的次數
  - 也可以由$v_o(s)/v_i(s)$得回原本的differential equation



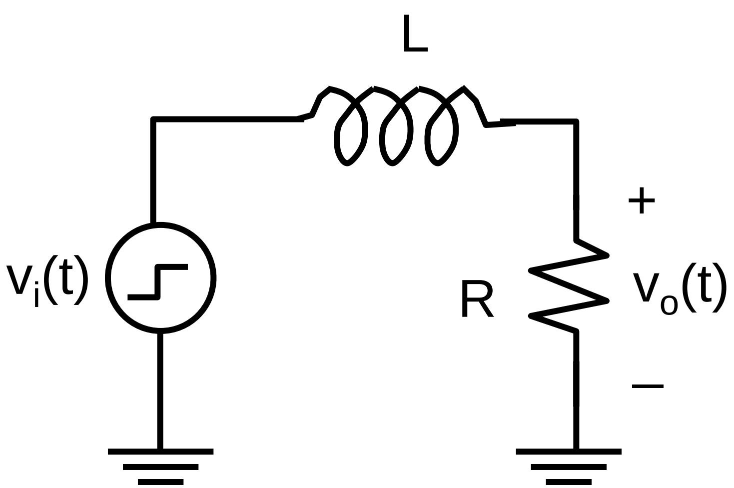




Example

- RLC impedances in Laplace Transform:

$$R, \frac{1}{sC} \text{ and } sL$$



- 經由阻抗分壓:

$$\frac{v_o(s)}{v_i(s)} = \frac{R}{Ls+R} \quad (1\text{st order})$$

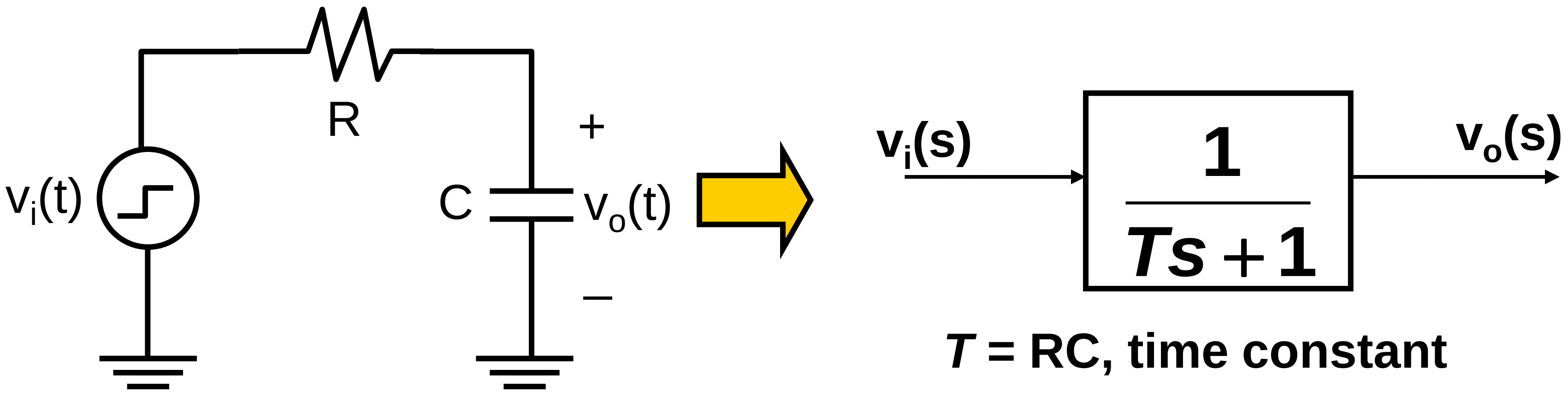
  - The original differential eqn. therefore is:

$$sL \cdot v_o(s) + R \cdot v_o(s) = R \cdot v_i(s)$$

$$\text{I.L.T.} \Rightarrow L\dot{v}_o(t) + Rv_o(t) = Rv_i(t)$$

Step Response of 1st-Order RC Circuit

- Given a unit-step input $v_i(t)$, assume $v_o(0) = 0$:



T = RC, time constant

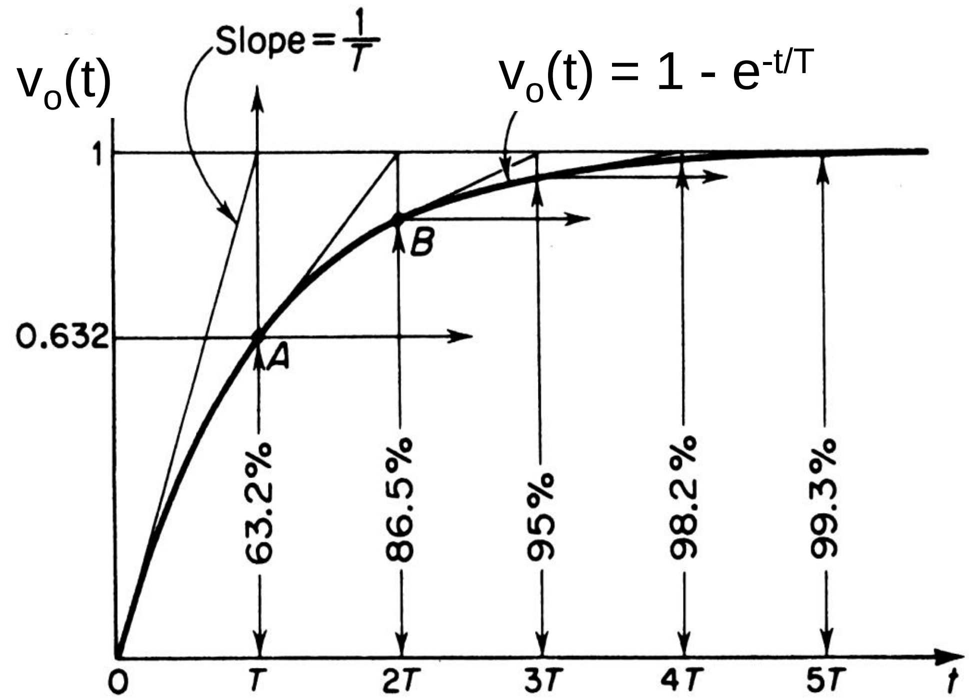
$$\frac{v_o(s)}{v_i(s)} = \frac{1}{RCs+1}$$

$$\Rightarrow v_o(s) = \frac{1}{RCs+1}\cdot\frac{1}{s} = \frac{1}{s} - \frac{RC}{RCs+1} = \frac{1}{s} - \frac{1}{s+(1/RC)}$$

$$Solve \quad v_o(t) = \left(1 - e^{-\frac{t}{RC}}\right)\cdot u(t)$$

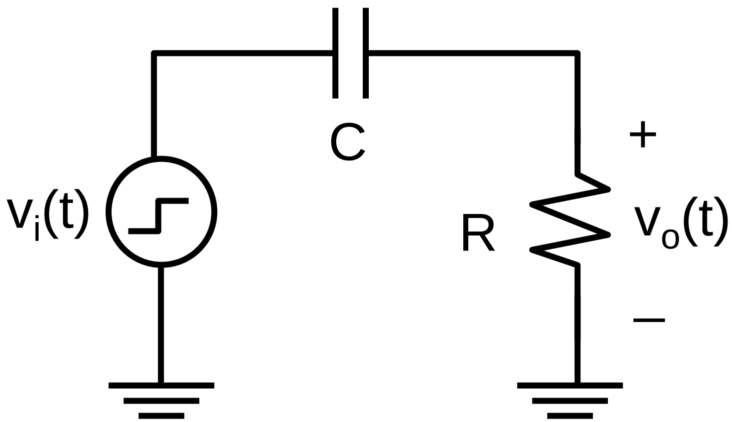
Cont'd

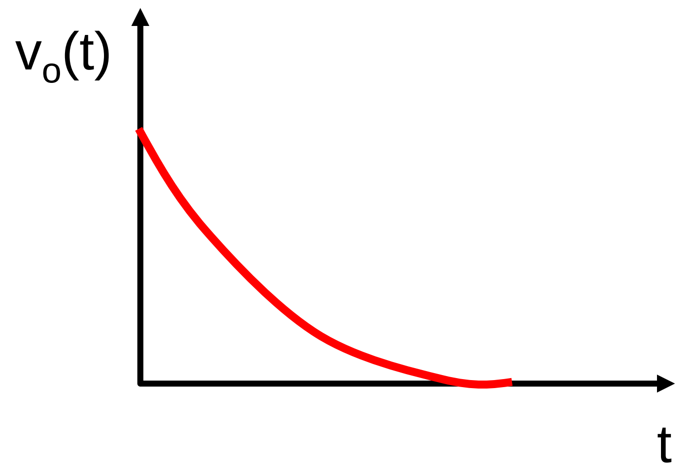
- For $t = T$ ($T = RC$), the response reaches 63% of the steady state
- The error is around 0.7% after $t = 5T$



Step Response: R and C Interchanged

- Given a step input $v_i(t)$, assume $v_o(0) = 0$:





$$\frac{v_o(s)}{v_i(s)} = \frac{RCs}{RCs+1}$$

$$\Rightarrow v_o(s) = \frac{RCs}{RCs+1} \cdot \frac{1}{s} = \frac{RC}{RCs+1} = \frac{1}{s+(1/RC)}$$

Solve $v_o(t) = e^{-\frac{t}{RC}}$

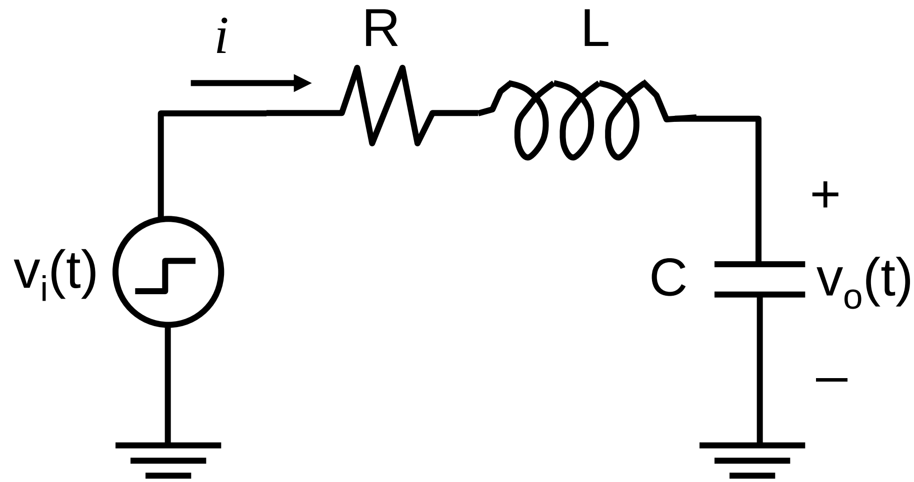
2nd-Order RLC Circuit

- By KVL:

$$v_i = iR + L\frac{di}{dt} + v_o$$

- Substituting i = C·dv$_o$/dt gives:

$$LC\frac{d^2v_o}{dt^2} + RC\frac{dv_o}{dt} + v_o = v_i$$



- By Laplace transform (assume v$_o$(0) = **v$_o$**'(0)= 0):

$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Introducing Damping Ratio and Natural Frequency

- 為便於分析二階系統的step response，the transfer function is re-written as:

$$\frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

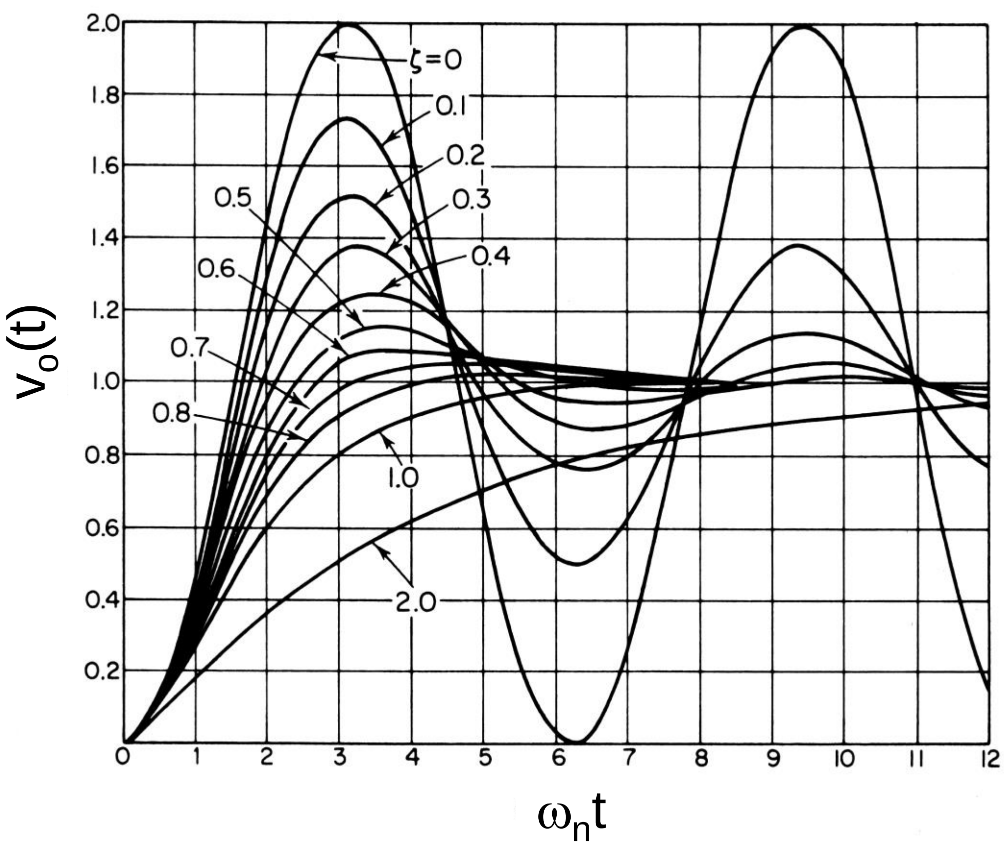
$$where \quad \omega_n = \frac{1}{\sqrt{LC}} \; (natural \;\; frequency) \quad and \quad \xi = \frac{R}{L}\cdot\frac{1}{2\omega_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

- ξ: damping ratio, dimensionless（無單位）；攸關於系統中能量的損失
  - ξ < 1, underdamped, producing two complex roots
  - ξ = 1, critically damped, producing two repeated real roots
  - ξ > 1, overdamped, producing two distinct real roots
- 二階系統的step response在damping ratio小於1時會有振盪行為
  - The roots of the denominator for ζ < 1 are:

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Step Responses of the 2^{nd}-Order Circuit

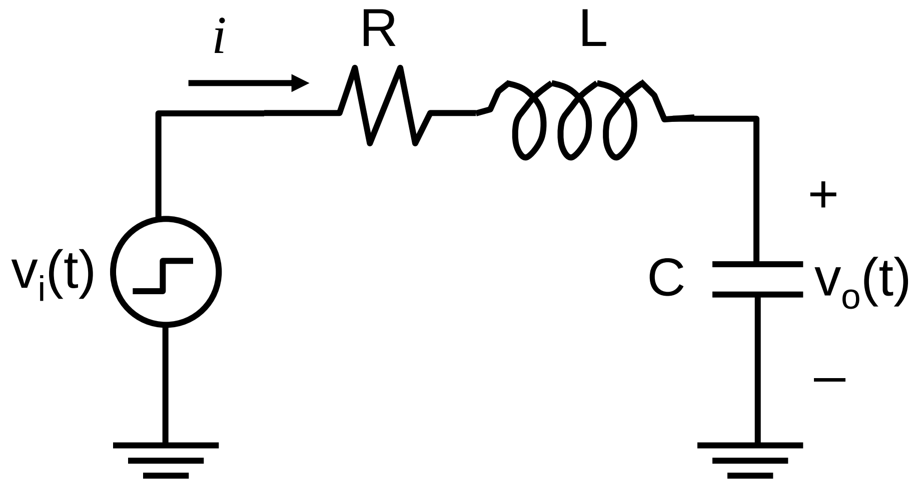
- The output grows exponentially without oscillation for $\xi \geq 1$
- 注意：$\xi < 1$ 則有振盪，$\xi = 0$ 則振盪不會衰減(due to no energy loss)



Example: Underdamped Step Response

- Given $v_i(t)$ is a unit step, and

$$\frac{v_o(s)}{v_i(s)}=\frac{100}{s^2+10s+100}$$



- First: ω_n = 10, and ξ = 0.5
- To solve:

$$v_o(s)=\frac{1}{s}\frac{100}{s^2+10s+100}$$

$$=\frac{1}{s}-\frac{s+10}{s^2+10s+100}$$

$$=\frac{1}{s}-\frac{s+5}{(s+5)^2+\left(5\sqrt{3}\right)^2}-\frac{5}{(s+5)^2+\left(5\sqrt{3}\right)^2}$$

$$=\frac{1}{s}-\frac{s+5}{(s+5)^2+\left(5\sqrt{3}\right)^2}-\frac{1}{\sqrt{3}}\frac{5\sqrt{3}}{(s+5)^2+\left(5\sqrt{3}\right)^2}$$

$$\therefore v_o(t)=\left(1-e^{-5t}\cos 5\sqrt{3}\,t-\frac{1}{\sqrt{3}}e^{-5t}\sin 5\sqrt{3}\,t\right)\cdot u(t)$$

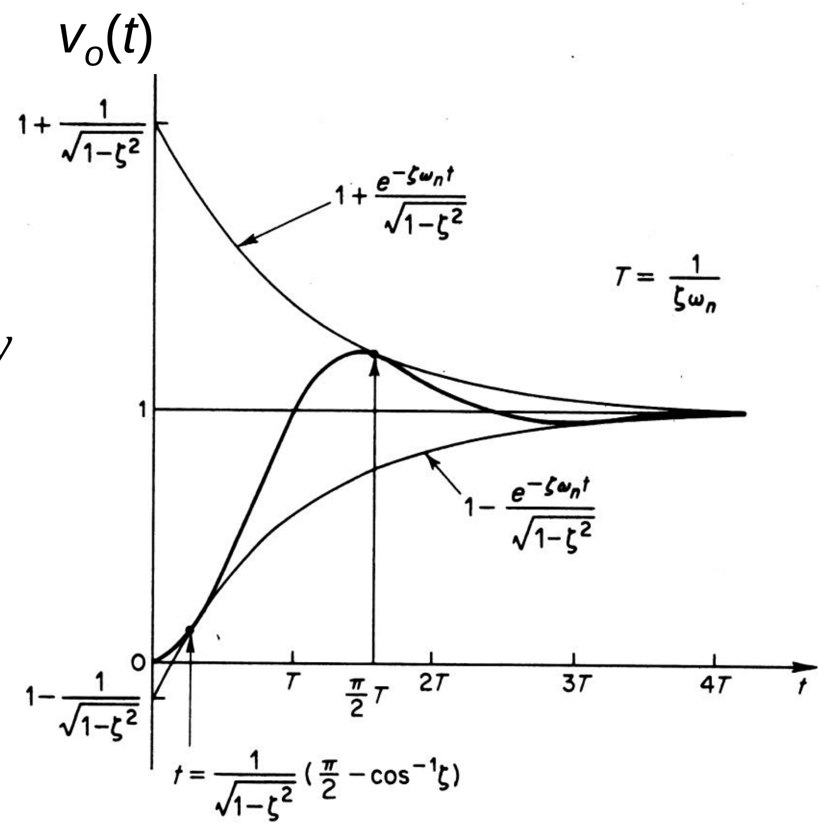
Underdamped Step Response

- Two complex poles: $s_{1,2} = -\xi\omega_n \pm j\underbrace{\omega_n\sqrt{1-\xi^2}}_{\omega_d}$
- 可以歸納成一個公式（不建議一定要背，但有些關鍵的物理意義要瞭解）：

$$v_o(t) = 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

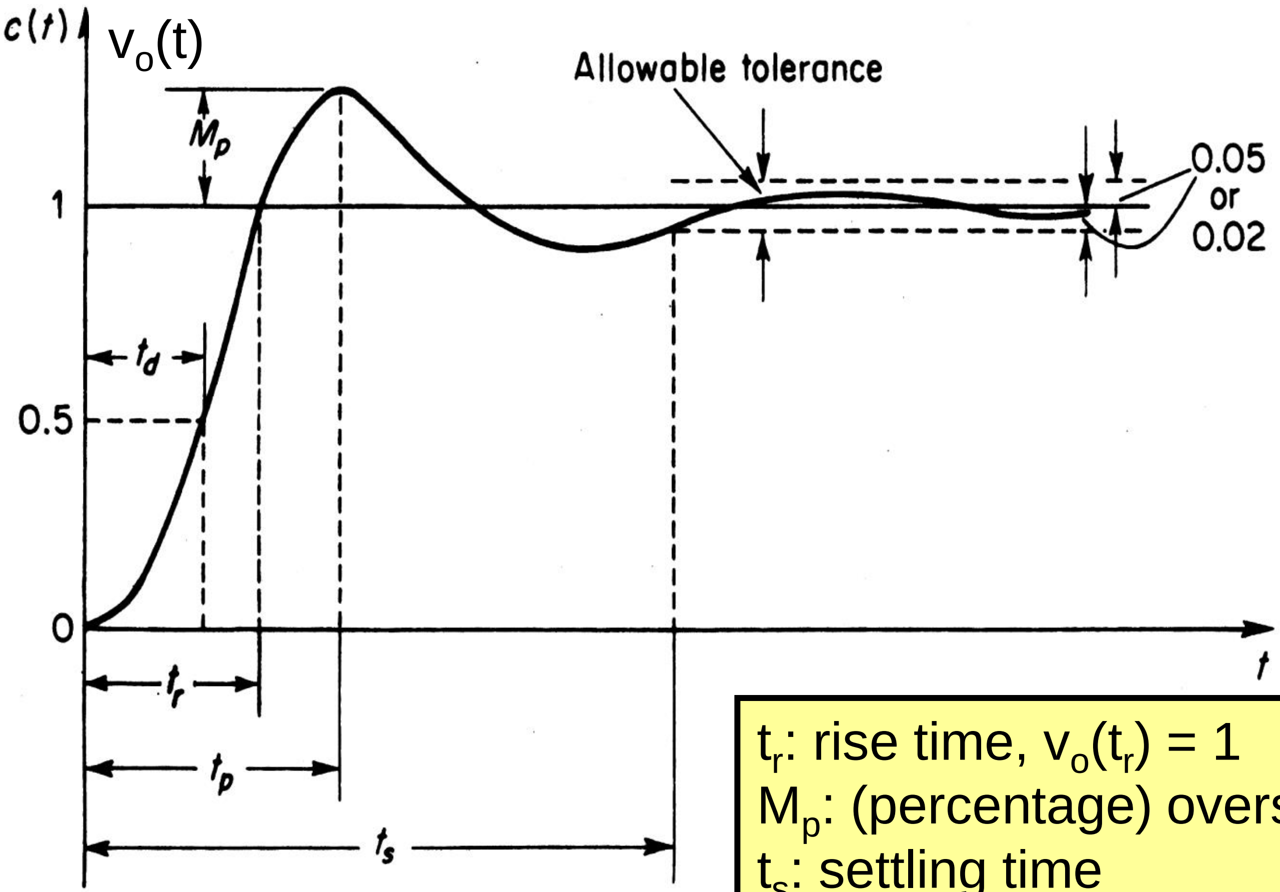
$$\omega_d = \omega_n\sqrt{1-\xi^2} \quad \textit{damped natural frequency}$$

Time constant: $\tau = \frac{1}{\xi\omega_n}$



Unit-Step Response Specifications（規格）for Underdamped Case

- 定義規格，便於之後設計我們所需的$v_o(t)$



t_r: rise time, $v_o(t_r) = 1$
M_p: (percentage) overshoot,最大超越量
t_s: settling time

Rise Time t_r

- Rise time t_r occurs at output $v_o(t) = 1$:

$$v_o(t_r) = 1 = 1 - e^{-\xi\omega_n t_r} \underbrace{\left(\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_d t_r \right)}_{=0}$$

$$\therefore t_r = \frac{\tan^{-1}\left(\frac{-\sqrt{1-\xi^2}}{\xi} \right)}{\omega_d}$$

Note: unit = radian, between $\pi/2$ and π

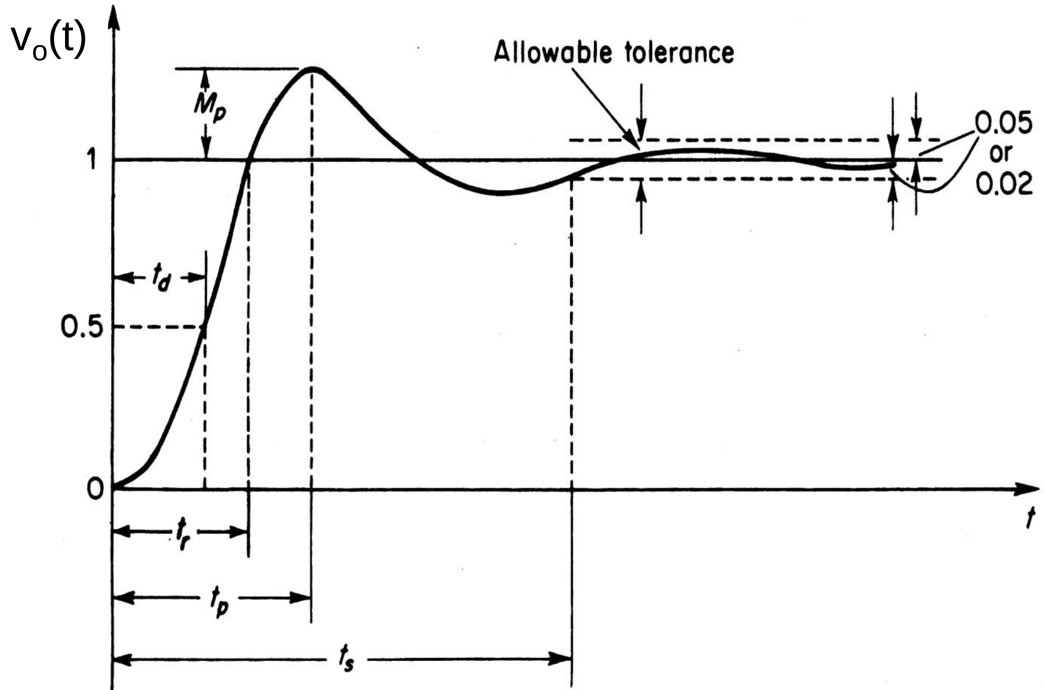
- 舉例： $\tan^{-1}(-1) = \frac{3}{4}\pi$

Peak Time t_p and Maximum Overshoot M_p

- t_p occurs at the first peak of $v_o(t)$:

$$\left.\frac{dv_o(t)}{dt}\right|_{t=t_p} = 0$$

$$\therefore t_p = \frac{\pi}{\omega_d}$$



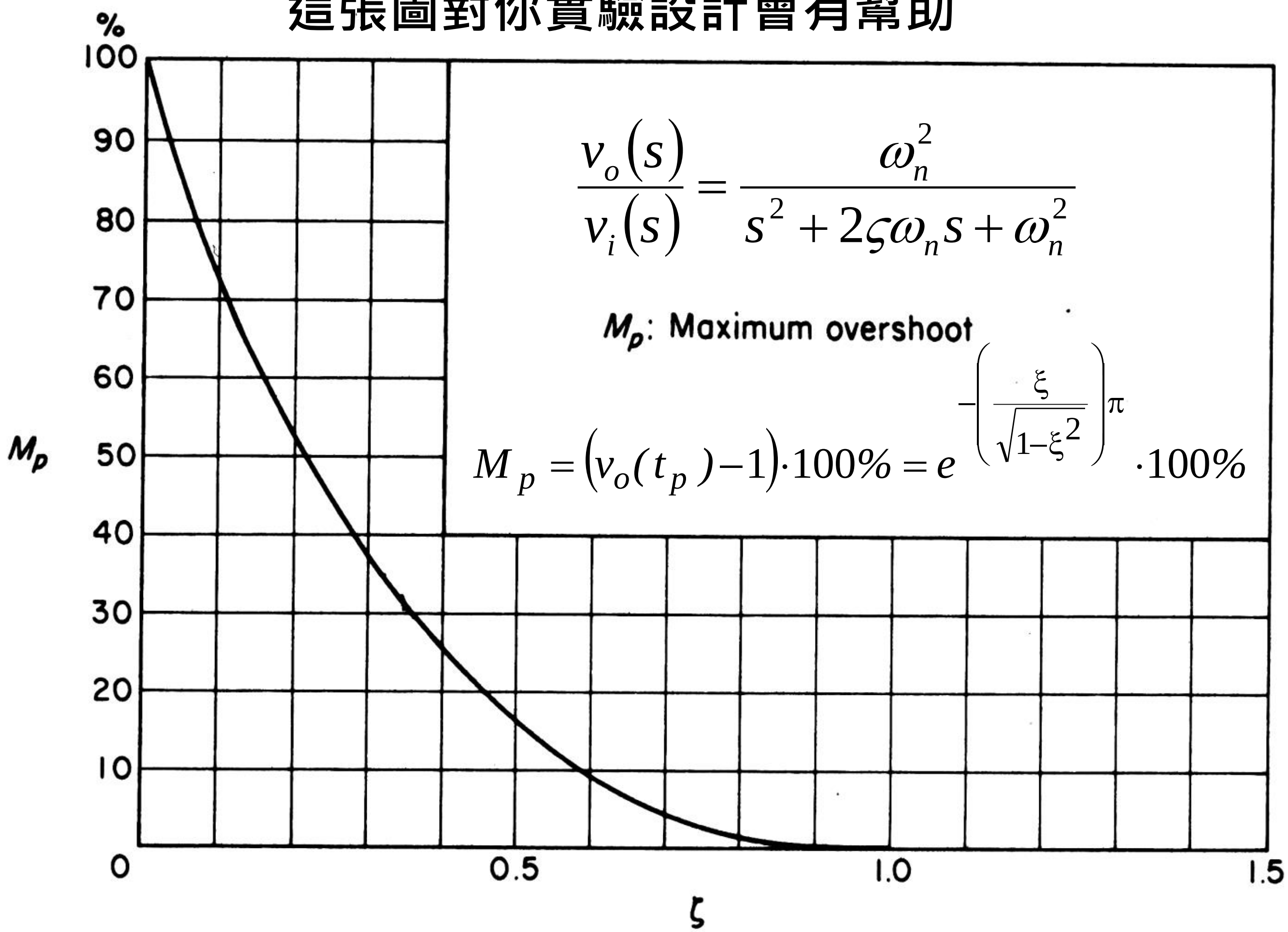
- So:

$$M_p = v_o(t_p) - 1 = e^{-\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\pi}$$

也請記得overshoot公式

Overshoot M_p vs. Damping Ratio ξ

這張圖對你實驗設計會有幫助



$$\frac{v_o(s)}{v_i(s)} = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$M_p = \left(v_o(t_p) - 1\right)\cdot 100\% = e^{-\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\pi} \cdot 100\%$$

Some Skills with the Oscilloscope: Averaging

- Purpose: to reduce noise and smooth the measured waveforms （波形變得較漂亮清楚，對於你紀錄波形有幫助）

Some Skills with the Oscilloscope: Cursor

- Purpose: 藉由二游標來量測波形中任兩點相對的時間差及電壓差(對於你紀錄波形有幫助)