EECS1010 Logic Design
Lecture 3 Gate-Level Minimization

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Outline

- Digital systems and information
- Boolean algebra and logic gates
- Gate-level minimization
- Combinational logic
- Sequential circuits
- Registers and counters
- Memory

Chapter Outline

- The map method
- Technology mapping
- Hazards

The Map Method

Logic Simplification

- For a logic function, its truth-table representation is unique, while its algebraic expression is not unique.
- Complexity of digital circuit (gate count) is proportional to the complexity of algebraic expression (literal count).
- The simplest algebraic expression is one that has minimum number of terms with the smallest possible number of literals in each term.

Merging Minterms

- The minterms in a function can be merged to form a simpler product term.
- Example:
  - m_1 and m_3:
    $d'c'b'a + d'c'ba = d'c'a(b'+b) = \bar{d}\bar{c}a$
  - m_2 and m_3:
    $d'c'ba' + d'c'ba = \bar{d}\bar{c}b$

$L = 8$ $\qquad L = 3$

$G = 8 + 2 = 10$ $\qquad G = 3$

$G_N = 10 + 3 = 13$ $\qquad G_N = 3 + 2 = 5$

Implicant

$w=0$
$x=0$
$y=1$

- Implicant of a function: a product term that is true implies the function is true.
- Example 1: for the prime detector, F = Σm(1, 2, 3, 5, 7, 11, 13).

SOM

$m_1 = \bar{w}\bar{x}\bar{y}z$

implicants: $m_1, m_2, m_3, m_5, m_7, m_{11}, m_{13}, \bar{w}\bar{x}y, \ldots$

$m_2 + m_3 = \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}yz = \bar{w}\bar{x}y(\bar{z}+z) = \bar{w}\bar{x}y \cdot 1 = \bar{w}\bar{x}y$

w	x	y	z	F	
0	0	0	0	0	m_0
0	0	0	1	1	m_1
0	0	1	0	1	m_2
0	0	1	1	1	m_3
0	1	0	0	0	
⋮				⋮	

- Example 2: F(x,y,z) = x'yz + xy'z + xyz' + xyz.

$= \Sigma m(\,?\,)$

$F(x,y,z)$

$\bar{x}yz + xyz$
$= yz$

implicants: $\bar{x}yz, x\bar{y}z, xy\bar{z}, xyz, xy, yz, xz,$

Total: 7 implicants.

Prime Implicant and Essential Prime Implicant

- Prime implicant (PI)
  - The implicant that cannot be merged into a larger one.
- Essential prime implicant (EPI)
  - The one and only one prime implicant that contains particular minterm of a function.

- Example: $F(x,y,z)=m_3+m_5+m_6+m_7$

implicants: m_3, m_5, $m6$, m_7, m_3+m_7, m_5+m_7, m_6+m_7
(7)

PIs: m_3+m_7 (yz),
3 PIs. m_5+m_7 (xz) ✓
m_6+m_7 (xy) ✓

EPIs: yz, xz, xy

Karnaugh Map (K-Map)

- A K-map is a collection of squares.
  - Each square represents a minterm | maxterm
  - Each K-map is a graphical representation of a Boolean function.
  - Adjacent squares differ in the value of one variable.
  - Alternative algebraic expressions for the same function are derived by recognizing patterns of squares.
- Each K-map defines a unique Boolean function.
- K-map is a visual diagram of all possible ways a function may be expressed.
  - Provide visual aid to identify PIs and EPIs.
  - Provide a means for finding optimum or near optimum of a Boolean function by combining squares.

Two-Variable K-Map

x, y

$x \backslash y$	0	1
0	m_0 $x=0$ $y=0$	m_1 $x=0$ $y=1$
1	m_2 $x=1$ $y=0$	m_3 $x=1$ $y=1$

x	y	
0	0	m_0
0	1	m_1
1	0	m_2
1	1	m_3

Three-Variable K-Map

$x \backslash yz$	00	01	11	10
0	$x=0$ m_0	$x=0$ $y=0$ $z=1$ m_1	$x=0$ m_3	$x=0$ m_2
1	$x=1$ m_4	$x=1$ $y=0$ $z=1$ m_5	$x=1$ m_7	$x=1$ m_6

$xy \backslash z$	0	1
00	m_0	m_1
01	m_2	m_3
11	m_6	m_7
10	m_4	m_5

$m_0 + m_2 = \bar{x}\bar{z}$

$m_2 + m_3 = \bar{x}y$

$m_2 + m_6 = \bar{x}y\bar{z} + xy\bar{z} = y\bar{z}(\bar{x}+x) = y\bar{z} \cdot 1 = \underline{\underline{y\bar{z}}}$

$x=1,\ y=1,\ z=0$, minterm $= xy\bar{z} = m_6$

Maxterm $= \bar{x} + \bar{y} + z = M_6$

Four-Variable K-Map

$wx \backslash yz$	00	01	11	10
00	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
01	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
11	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
10	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

Five-Variable K-Map

A, B, C, D, E

- For five variable problems, we use two adjacent K-maps. It becomes harder to visualize adjacent minterms for selecting. You can extend the problem to six variables by using four K-maps.

A = 0

BC \ DE	00	01	11	10
00	m_0 0	1	3	2
01	4	5	7	6
11	12	13	m_{15} 15	14
10	8	9	11	10

D: DE = 11, 10; E: DE = 01, 11; B: BC = 11, 10; C: BC = 01, 11

A = 1

BC \ DE	00	01	11	10
00	m_{16} 16	17	19	18
01	20	21	23	22
11	28	29	m_{31} 31	30
10	24	25	27	26

D: DE = 11, 10; E: DE = 01, 11; B: BC = 11, 10; C: BC = 01, 11

Covering a Function

- Procedure to select an inexpensive set of implicants
  - Start with an empty cover.
  - Add all essential prime implicants.
  - For the remaining uncovered minterms, add the largest implicant that covers the minterms.
- The procedure results in a good cover, but no guarantee the lowest cost cover.

K-Map Example 1

F(x,y,z)=Σm(2,3,6,7) $= (m_3+m_7)+(m_2+m_6) = yz + y\bar{z} = y$

x \ yz	00	01	11	10
0	0	0	1	1
1	0	0	1	1

y=1

implicants: $m_2, m_3, m_6, m_7, y\bar{z}, yz, y,$

(9)

$m_2+m_3\ (\bar{x}y),\ m_6+m_7\ (xy)$

PIs: y

EPIs: y

G(x,y,z)=Σm(3,4,6,7)

K-Map Example 1

$F(x,y,z)=\Sigma m(2,3,6,7)$ $= (m_3 + m_7) + (m_2 + m_6) = yz + y\bar{z} = y$

$x \backslash yz$	00	01	11	10
0	0	0	1	1
1	0	0	1	1

$y=1$

implicants: $m_2, m_3, m_6, m_7, y\bar{z}, yz, y$, (9)

m_2+m_3 $(\bar{x}y)$, m_6+m_7 (xy)

PIs: y

EPIs: y

Practice

$G(x,y,z)=\Sigma m(3,4,6,7)$

$F = m_3 + m_4 + m_6 + m_7$

K-Map Example 2

F(w,x,y,z)=Σm(0,2,4,5,6,7,8,10,11,15)

wx \ yz	00	01	11	10
00	1 m_0	0 m_1	0 m_3	1 m_2
01	1 m_4	1 m_5	1 m_7	1 m_6
11	0 m_{12}	0 m_{13}	1 m_{15}	0 m_{14}
10	1 m_8	0 m_9	1 m_{11}	1 m_{10}

$F = \bar{w}x(\bar{y}\bar{z} + \bar{y}z + yz + y\bar{z}) + \bar{x}\bar{z}$

$+ wyz$

$= \bar{w}x + \bar{x}\bar{z} + wyz$

$L = 7$

$G = 7 + 3 = 10$

$GN = 10 + 3 = 13$

implicants: $m_0, m_2, \ldots m_{11}, m_{15}, \bar{w}\bar{y}\bar{z}, \ldots$

Prime implicants: $\bar{w}x,\ \bar{x}\bar{z},\ wyz,\ \bar{w}\bar{z},\ xyz,\ w\bar{x}y$

EPIs: $\bar{w}x,\ \bar{x}\bar{z},$

K-Map Example 3

B=0
C=1
D=0

F(A,B,C,D)=B'CD'+A'B'C'+A'BCD'+AB'C'

- Practice: optimize F using Boolean algebra.

AB\CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

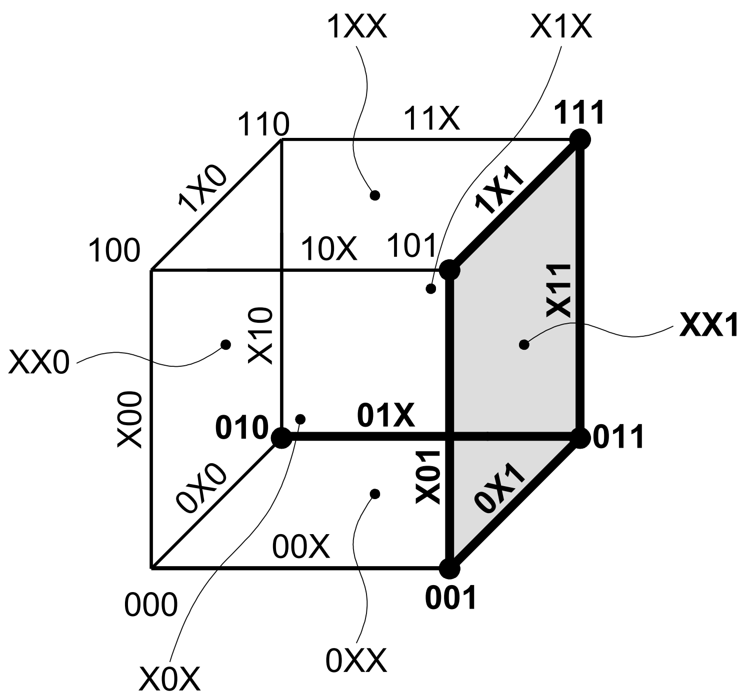
$$F = \overline{B}\,\overline{D} + \overline{B}\,\overline{C} + \overline{A}C\overline{D}$$

- Prime implicants: $\overline{B}\,\overline{D}$, $\overline{B}\,\overline{C}$, $\overline{A}C\overline{D}$
- EPIs: $\overline{B}\,\overline{D}$, $\overline{B}\,\overline{C}$, $\overline{A}C\overline{D}$

Cube Method

- Implicants can be visualized on a cube.
  - Vertex: Gray-coded minterm
  - Edge: product (implicant) consisting of 2 minterms
  - Shaded face: product (implicant) consisting of 4 minterms

$$f = \sum m(1,2,3,5,7)$$

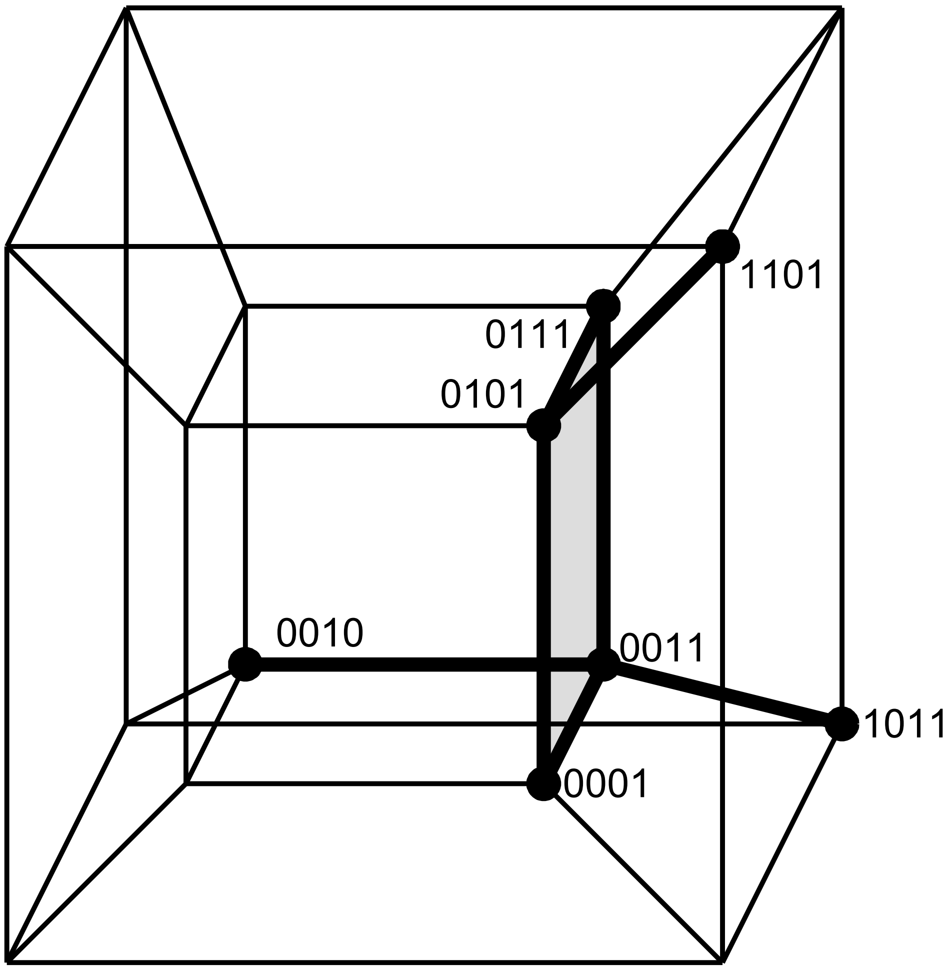


Practice

4D Hypercube for 4-Bit Input

$$f = \sum_{dcba} m(1,2,3,5,7,11,13)$$

- Shaded face:

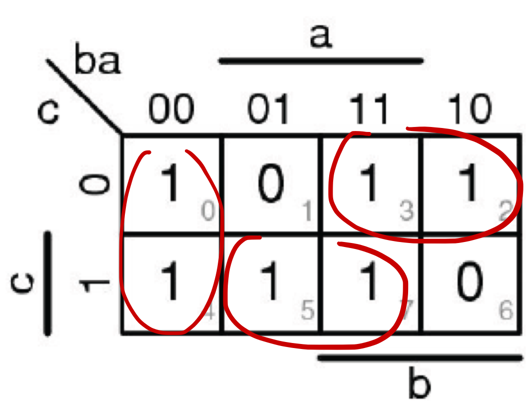


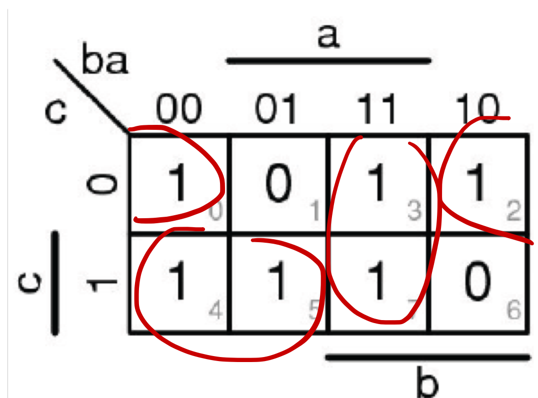
Example: PIs and EPIs

- $F = \Sigma m(0,1,4,5,7,10)$

Non-Unique Minimum Covering

- When there is no essential prime implicants, more than one solution exist.





$F = \bar{b}\bar{a} + ca + \bar{c}b$

$F = \bar{c}\bar{a} + c\bar{b} + ba$

- All product terms are not EPIs.

K-Map Summary

- Any 2^k adjacent squares, k = 0, 1, ..., n, in an n-variable map represent an area that gives a product term of n-k literals.

K	# of adjacent squares	# of literals in a term in an n-variable map			
		n=2	n=3	n=4	n=5
0	1	2	3	4	5
1	2	1	2	3	4
2	4	0	1	2	3
3	8		0	1	2
4	16			0	1
5	32				0

Example: From Cover to Gates

$$f(d,c,b,a) = d'a + d'c'b + cb'a + c'ba$$

Practice.

Don’t Care

- The output value need not be defined:
  - The input values for the minterm will never occur, or
  - The output value for the minterm is not used
  - Instead, the output value is defined as a “don't care”
- By placing “don’t cares” (“x”) in the map, the cost of the logic circuit may be lowered.
- Example: A logic function using BCD digits as its inputs.

wx \ yz	00	01	11	10
00				0
01	0		1	1
11	x	x	x	x
10			x	x

Example: Boolean Function with Don't Cares

- $F = \Sigma m(1,2,3,5,7) + d(10, 11, 12, 13, 14, 15)$.

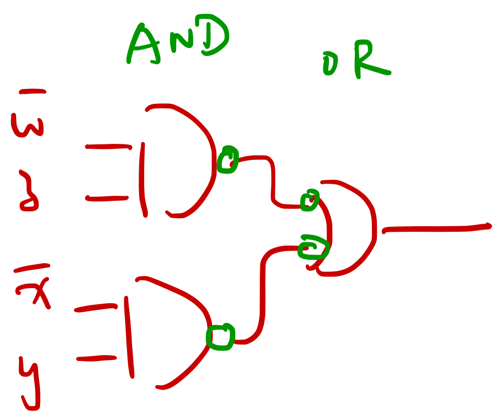
wx \ yz	00	01	11	10
00	0	1	1	1
01	0	1	1	0
11	x	x	x	x
10	0	0	x	x

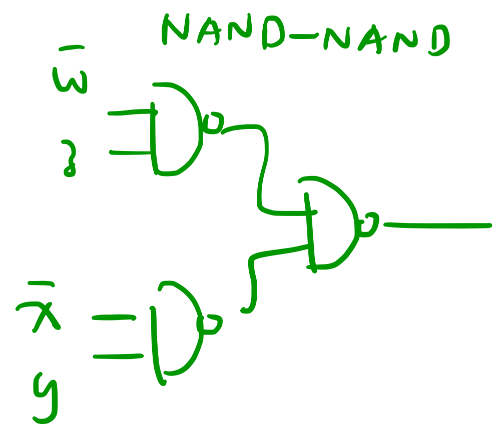
$$F = \underbrace{\bar{w}z + \bar{x}y}$$

SOP

NAND

NOR





Product of Sums Simplification

- Based on DeMorgan's Theorem

  $\overline{F}$: sop $\Rightarrow \overline{\overline{F}} = \overline{\text{sop}} =$

  - Choose 0's in the k-map: simplified F' in the form of sum of products
  - Apply DeMorgan's theorem
  - F': sum of products
  - F: product of sums

Example: Product of Sums Simplification

F(w,x,y,z)=Σm(0,1,2,5,8,9,10)

pos (cover 0's)

wx \ yz	00	01	11	10
00	1	1	0	1
01	0	1	0	0
11	0	0	0	0
10	1	1	0	1

$$\bar{F} = yz + x\bar{z} + wx$$

$$F = \bar{\bar{F}} = \left(\overline{yz + x\bar{z} + wx}\right)$$

$$= \left(\overline{yz}\right) \cdot \left(\overline{x\bar{z}}\right) \cdot \left(\overline{wx}\right)$$

$$= \left(\bar{y} + \bar{z}\right) \cdot \left(\bar{x} + z\right) \cdot \left(\bar{w} + \bar{x}\right) \quad \text{pos}$$

Simplification Example 1

- $F(w,x,y,z) = \Pi(13,15)$

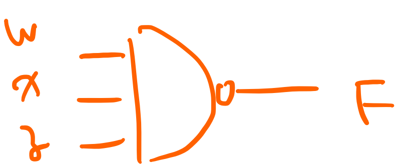
wx \ yz	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11	1	0	0	1
10	1	1	1	1

SOP

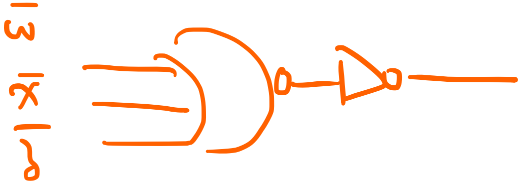
$F = \bar{w} + \bar{z} + \bar{x}$

$\bar{F} = wxz$

$\bar{\bar{F}} = \left(\overline{wxz}\right) = \bar{w} + \bar{x} + \bar{z}$



NAND



NOR