Noise

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Introduction

- The sources of noise may be external to the system (e.g., atmospheric noise, galactic noise, man-made noise), or internal to the system (e.g., noise arising from spontaneous fluctuations of current or voltage in electrical circuits).
- Two most common examples of spontaneous fluctuations in electrical circuits are shot noise and thermal noise.

Shot Noise

- Shot noise originates from the discrete nature of current flow in electronic devices such as diodes and transistors.
- Let

$$X(t) = \sum_{k=-\infty}^{\infty} h(t - \tau_k)$$

  where $h(t - \tau_k)$ is the current pulse generated at time τ_k.
- The random process governing the arrival times, i.e., the arrival process, is a Poisson process with rate λ:
  - The number of arrivals $N(t_1, t_2)$ in the interval (t_1, t_2) of length $t_0 = t_2 - t_1$ is a Poisson random variable with parameter λt_0:

$$P(N(t_1, t_2) = k) = \frac{(\lambda t_0)^k}{k!} e^{-\lambda t_0}, \quad \text{for } k = 0, 1, \ldots$$

  - If the intervals (t_1, t_2) and (t_3, t_4) are nonoverlapping, then the random variables $N(t_1, t_2)$ and $N(t_3, t_4)$ are independent.

Campbell's Theorem

- The mean of $X(t)$ is

$$\mu_X = \mathrm{E}[X(t)] = \lambda \int_{-\infty}^{\infty} h(t)\,dt.$$

- The autocovariance function of $X(t)$ is

$$C_X(\tau) = \mathrm{E}[(X(t+\tau) - \mu_X)(X(t) - \mu_X)] = \lambda \int_{-\infty}^{\infty} h(t)h(t+\tau)\,dt.$$

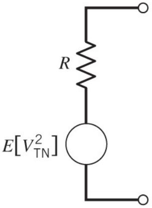
Thermal Noise

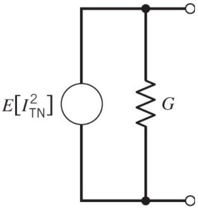
- Thermal noise is the electrical noise arising from the random motion of electrons in a conductor.
- The mean-square value of the thermal noise voltage V_{TN} across a resistor of resistance R in a bandwidth of $\triangle f$ is given by

$$\mathrm{E}[V_{\mathrm{TN}}^2] = 4kTR\triangle f$$

where k is the Boltzmann's constant equal to 1.38×10^{-23} and T is the absolute temperature.

- For the model of a noisy resistor, consider the Thevenin equivalent circuit or the Norton equivalent circuit:





- We have
$$\mathrm{E}[I^2_{\mathrm{TN}}] = \frac{1}{R^2}\mathrm{E}[V^2_{\mathrm{TN}}] = 4kTG\triangle f$$
where $G = 1/R$ is the conductance.
- The central limit theorem indicates that thermal noise is Gaussian distributed with zero mean.

- The maximum-power transfer theorem states that the maximum possible power is transferred from a source of internal resistance R to a load of resistance R_l is when $R_l = R$.
- Hence the available power of a noisy resistor is equal to

$$
\begin{aligned}
\mathrm{E}\left[\left(\frac{V_{\mathrm{TN}}}{2R}\right)^2 R\right] &= \frac{1}{4R}\mathrm{E}[V_{\mathrm{TN}}^2] \\
&= \frac{4kTR\triangle f}{4R} \\
&= kT\triangle f.
\end{aligned}
$$

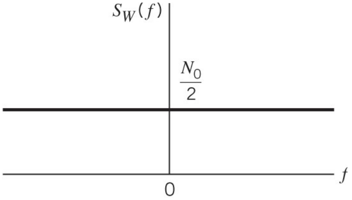
White Noise

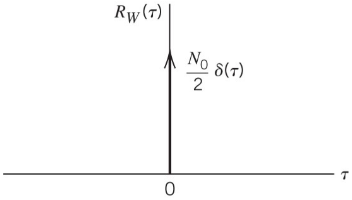
- A white noise $W(t)$ is wide-sense stationary and has power spectral density

$$S_W(f) = \frac{N_0}{2}, \quad \text{for all } f.$$

- Then its autocorrelation function is

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau).$$





- Assume $\mathrm{E}[W(t)] = 0$. Then for $\tau \neq 0$, $R_W(\tau) = 0$, i.e., $W(t_1)$ and $W(t_2)$, for $t_1 \neq t_2$, are uncorrelated.

- If $W(t)$ is white Gaussian with zero mean, then $W(t_1)$ and $W(t_2)$, for $t_1 \neq t_2$, are statistically independent Gaussian random variables.

Additive White Gaussian Noise

- Consider a channel with input $X(t)$ and output $Y(t)$. Suppose

$$Y(t) = X(t) + W(t).$$

  If $W(t)$ is white Gaussian, then $W(t)$ is called the additive white Gaussian noise (AWGN).
- The considered channel is called the additive white Gaussian noise channel.
- It may be expressed as

$$N_0 = kT_e$$

  where k is the Boltzmann's constant and T_e is called the equivalent noise temperature.
- Recall that the available power of the thermal noise produced by a noisy resistor is $kT\triangle f$ where T is the absolute temperature and $\triangle f$ is the bandwidth.

Ideal Low-Pass Filtered Noise

- Suppose a white Gaussian noise $W(t)$ of zero mean and power spectral density $N_0/2$ is applied to an ideal low-pass filter with transfer function

$$H(f) = \begin{cases} 1, & -B < f < B \\ 0, & \text{elsewhere.} \end{cases}$$

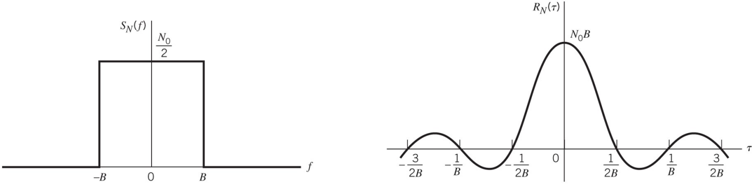
- The power spectral density of the output $N(t)$ is

$$\begin{aligned} S_N(f) &= S_W(f)|H(f)|^2 \\ &= \begin{cases} N_0/2, & -B < f < B \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

- Then the autocorrelation function of the output is

$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f \tau}\, df = N_0 B\, \mathrm{sinc}(2B\tau)$$

where $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$.



- Note that $R_N(\tau) = 0$ at $\tau = k/2B$, for $k = \pm 1, \pm 2, \ldots$.
- Since $W(t)$ is Gaussian, $N(t)$ is also Gaussian.
- The output noise samples $N(k/2B)$, for $k = 0, \pm 1, \pm 2, \ldots$, are statistically independent Gaussian random variables, each with zero mean and variance

$$\mathrm{E}[N^2(t)] = \int_{-\infty}^{\infty} S_N(f)\, df = N_0 B.$$

Noise Equivalent Bandwidth

- Suppose a white noise $W(t)$ with power spectral density $N_0/2$ is applied to a low-pass filter with transfer function $H(f)$.
- The average output noise power is

$$\begin{aligned} N_{\text{out}} &= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df \\ &= N_0\int_{0}^{\infty} |H(f)|^2\,df. \end{aligned}$$

- If $W(t)$ is through an ideal low-pass filter of zero frequency response $H(0)$ and bandwidth B, then

$$N_{\text{out}} = N_0 B H^2(0).$$

- The noise equivalent bandwidth is given by

$$B = \frac{\int_0^\infty |H(f)|^2\, df}{H^2(0)}.$$