Narrow-Band Noise

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Band-Pass Processes

Definition

A real random process $n(t)$ is band-pass if its power spectral density $S_N(f)$ is zero outside the vicinity of a central (or mid-band) frequency f_c, i.e.,

$$S_N(f) = 0, \quad \text{for all } |f \pm f_c| > B.$$

- The band-pass process is called narrow-band if $2B \ll f_c$.
- Assume that $n(t)$ is a wide-sense stationary narrow-band noise with zero mean, autocorrelation function $R_N(\tau)$, and power spectral density $S_N(f)$.

- Let the pre-envelope of $n(t)$ be

$$n_+(t) = n(t) + j\hat{n}(t)$$

and the complex envelope be

$$\tilde{n}(t) = n_+(t)e^{-j2\pi f_c t} = n_I(t) + jn_Q(t).$$

- Then

$$\begin{aligned} n(t) &= \text{Re}\left[n_+(t)\right] = \text{Re}\left[\tilde{n}(t)e^{j2\pi f_c t}\right] \\ &= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t). \end{aligned}$$

- We also have

$$\begin{aligned} n_I(t) &= \text{Re}\left[\tilde{n}(t)\right] = \text{Re}\left[n_+(t)e^{-j2\pi f_c t}\right] \\ &= n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t) \\ n_Q(t) &= \text{Im}\left[\tilde{n}(t)\right] = \text{Im}\left[n_+(t)e^{-j2\pi f_c t}\right] \\ &= -n(t)\sin(2\pi f_c t) + \hat{n}(t)\cos(2\pi f_c t). \end{aligned}$$

Property 1

Both $n_I(t)$ and $n_Q(t)$ have zero mean.

Proof. Since

$$\mathrm{E}\left[\hat{n}(t)\right] = \mathrm{E}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n(\tau)}{t-\tau}\,d\tau\right] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{E}\left[n(\tau)\right]}{t-\tau}\,d\tau = 0$$

we have

$$\mathrm{E}\left[n_I(t)\right] = \mathrm{E}\left[n(t)\right]\cos(2\pi f_c t) + \mathrm{E}\left[\hat{n}(t)\right]\sin(2\pi f_c t) = 0.$$

Similarly,

$$\mathrm{E}\left[n_Q(t)\right] = -\mathrm{E}\left[n(t)\right]\sin(2\pi f_c t) + \mathrm{E}\left[\hat{n}(t)\right]\cos(2\pi f_c t) = 0.$$

Property 2

$$R_{N\hat{N}}(\tau) = -\hat{R}_N(\tau)$$

$$R_{\hat{N}N}(\tau) = \hat{R}_N(\tau)$$

$$R_{\hat{N}}(\tau) = R_N(\tau).$$

Proof. Since $\hat{n}(t)$ is the output of $n(t)$ passed through a linear time-invariant (LTI) system with impulse response $1/(\pi t)$, we have

$$\begin{aligned} R_{N\hat{N}}(\tau) &= R_N(\tau) \star \frac{1}{\pi(-\tau)} \\ &= -R_N(\tau) \star \frac{1}{\pi\tau} \\ &= -\hat{R}_N(\tau) \end{aligned}$$

where $\star$ denotes convolution and $\hat{R}_N(\tau)$ is the Hilbert transform of $R_N(\tau)$.

Also

$$R_{\hat{N}N}(\tau) = R_N(\tau) \star \frac{1}{\pi\tau} = \hat{R}_N(\tau).$$

Finally,

$$S_{\hat{N}}(f) = S_N(f) \left|-j\mathrm{sgn}(f)\right|^2 = S_N(f)$$

which yields

$$R_{\hat{N}}(\tau) = R_N(\tau).$$

■

Property 3

$$
\begin{aligned}
& \mathrm{E}\left[n_I(t+\tau)n_I(t)\right] = R_{N_I}(\tau) \\
= \quad & \mathrm{E}\left[n_Q(t+\tau)n_Q(t)\right] = R_{N_Q}(\tau) \\
= \quad & R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{E}\left[n_I(t+\tau)n_Q(t)\right] = R_{N_I N_Q}(\tau) \\
= \quad & -\mathrm{E}\left[n_Q(t+\tau)n_I(t)\right] = -R_{N_Q N_I}(\tau) \\
= \quad & R_N(\tau)\sin(2\pi f_c\tau) - \hat{R}_N(\tau)\cos(2\pi f_c\tau).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& \mathrm{E}\left[n_I(t+\tau)n_I(t)\right] \\
= \; & \mathrm{E}\left[(n(t+\tau)\cos(2\pi f_c(t+\tau)) + \hat{n}(t+\tau)\sin(2\pi f_c(t+\tau)))\right. \\
& \left.(n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t))\right] \\
= \; & \frac{1}{2}\left[R_N(\tau) + R_{\hat{N}}(\tau)\right]\cos(2\pi f_c \tau) \\
& + \frac{1}{2}\left[-R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\sin(2\pi f_c \tau) \\
& + \frac{1}{2}\left[R_N(\tau) - R_{\hat{N}}(\tau)\right]\cos(2\pi f_c(2t+\tau)) \\
& + \frac{1}{2}\left[R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\sin(2\pi f_c(2t+\tau)) \\
= \; & R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_N(\tau)\sin(2\pi f_c \tau) \quad \text{(by Property 2)} \\
= \; & R_{N_I}(\tau).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \mathsf{E}\left[n_Q(t+\tau)n_Q(t)\right] \\
= \; & \mathsf{E}\left[(-n(t+\tau)\sin(2\pi f_c(t+\tau)) + \hat{n}(t+\tau)\cos(2\pi f_c(t+\tau)))\right. \\
& \left.(-n(t)\sin(2\pi f_c t) + \hat{n}(t)\cos(2\pi f_c t))\right] \\
= \; & \frac{1}{2}\left[R_N(\tau) + R_{\hat{N}}(\tau)\right]\cos(2\pi f_c \tau) \\
& + \frac{1}{2}\left[-R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\sin(2\pi f_c \tau) \\
& + \frac{1}{2}\left[-R_N(\tau) + R_{\hat{N}}(\tau)\right]\cos(2\pi f_c (2t+\tau)) \\
& + \frac{1}{2}\left[-R_{N\hat{N}}(\tau) - R_{\hat{N}N}(\tau)\right]\sin(2\pi f_c (2t+\tau)) \\
= \; & R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_N(\tau)\sin(2\pi f_c \tau) \quad \text{(by Property 2)} \\
= \; & R_{N_I}(\tau) \\
= \; & R_{N_Q}(\tau).
\end{aligned}
$$

Also

$$
\begin{aligned}
& \mathrm{E}\left[n_I(t+\tau)n_Q(t)\right] \\
= \; & \mathrm{E}\left[(n(t+\tau)\cos(2\pi f_c(t+\tau)) + \hat{n}(t+\tau)\sin(2\pi f_c(t+\tau)))\right. \\
& \left.(-n(t)\sin(2\pi f_c t) + \hat{n}(t)\cos(2\pi f_c t))\right] \\
= \; & \frac{1}{2}\left[R_{N\hat{N}}(\tau) - R_{\hat{N}N}(\tau)\right]\cos(2\pi f_c \tau) \\
& + \frac{1}{2}\left[R_N(\tau) + R_{\hat{N}}(\tau)\right]\sin(2\pi f_c \tau) \\
& + \frac{1}{2}\left[R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\cos(2\pi f_c(2t+\tau)) \\
& + \frac{1}{2}\left[-R_N(\tau) + R_{\hat{N}}(\tau)\right]\sin(2\pi f_c(2t+\tau)) \\
= \; & R_N(\tau)\sin(2\pi f_c \tau) - \hat{R}_N(\tau)\cos(2\pi f_c \tau) \quad \text{(by Property 2)} \\
= \; & R_{N_I N_Q}(\tau).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \mathrm{E}\left[n_Q(t+\tau) n_I(t)\right] \\
= \; & \mathrm{E}\left[(-n(t+\tau)\sin(2\pi f_c(t+\tau)) + \hat{n}(t+\tau)\cos(2\pi f_c(t+\tau)))\right. \\
& \left.(n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t))\right] \\
= \; & \frac{1}{2}\left[-R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\cos(2\pi f_c \tau) \\
& + \frac{1}{2}\left[-R_N(\tau) - R_{\hat{N}}(\tau)\right]\sin(2\pi f_c \tau) \\
& + \frac{1}{2}\left[R_{N\hat{N}}(\tau) + R_{\hat{N}N}(\tau)\right]\cos(2\pi f_c(2t+\tau)) \\
& + \frac{1}{2}\left[-R_N(\tau) + R_{\hat{N}}(\tau)\right]\sin(2\pi f_c(2t+\tau)) \\
= \; & -R_N(\tau)\sin(2\pi f_c \tau) + \hat{R}_N(\tau)\cos(2\pi f_c \tau) \quad \text{(by Property 2)} \\
= \; & -R_{N_I N_Q}(\tau) \\
= \; & R_{N_Q N_I}(\tau).
\end{aligned}
$$

Property 4

Both $n_I(t)$ and $n_Q(t)$ are wide-sense stationary; also $n_I(t)$ and $n_Q(t)$ are jointly wide-sense stationary.

Proof. This property follows directly from Properties 1 and 3. ∎

Property 5

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

Proof. This property can be obtained by taking the Fourier transform of both sides of the following equation in Property 3:

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_N(\tau)\sin(2\pi f_c \tau).$$

∎

Property 6

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \begin{cases} j\left[S_N(f+f_c) - S_N(f-f_c)\right], & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

Proof. This property can be obtained by taking the Fourier transform of both sides of the following equation in Property 3:

$$R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) = R_N(\tau)\sin(2\pi f_c \tau) - \hat{R}_N(\tau)\cos(2\pi f_c \tau).$$

∎

Property 7

$$\mathrm{Var}\left[n_I(t)\right] = \mathrm{Var}\left[n_Q(t)\right] = \mathrm{Var}\left[n(t)\right].$$

Proof. Since $n_I(t)$, $n_Q(t)$, and $n(t)$ all have zero mean, it is equivalent to showing that

$$\mathrm{E}\left[n_I^2(t)\right] = \mathrm{E}\left[n_Q^2(t)\right] = \mathrm{E}\left[n^2(t)\right]$$

which follows from the fact that

$$\int_{-\infty}^{\infty} S_{N_I}(f)\,df = \int_{-\infty}^{\infty} S_{N_Q}(f)\,df = \int_{-\infty}^{\infty} S_N(f)\,df$$

by Property 5. ∎

Property 8

$$R_N(\tau) = \text{Re}\left[\frac{1}{2}R_{\tilde{N}}(\tau)e^{j2\pi f_c \tau}\right]$$

where

$$R_{\tilde{N}}(\tau) = \text{E}\left[\tilde{n}(t+\tau)\tilde{n}^*(t)\right].$$

Proof.

$$
\begin{aligned}
R_N(\tau) &= \mathrm{E}\left[n(t+\tau)n(t)\right] \\
&= \mathrm{E}\left[(n_I(t+\tau)\cos(2\pi f_c(t+\tau)) - n_Q(t+\tau)\sin(2\pi f_c(t+\tau)))\right. \\
&\quad \left.(n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t))\right] \\
&= \frac{1}{2}\left[R_{N_I}(\tau) + R_{N_Q}(\tau)\right]\cos(2\pi f_c \tau) \\
&\quad + \frac{1}{2}\left[R_{N_I N_Q}(\tau) - R_{N_Q N_I}(\tau)\right]\sin(2\pi f_c \tau) \\
&\quad + \frac{1}{2}\left[R_{N_I}(\tau) - R_{N_Q}(\tau)\right]\cos(2\pi f_c(2t+\tau)) \\
&\quad - \frac{1}{2}\left[R_{N_I N_Q}(\tau) + R_{N_Q N_I}(\tau)\right]\sin(2\pi f_c(2t+\tau)) \\
&= R_{N_I}(\tau)\cos(2\pi f_c \tau) + R_{N_I N_Q}(\tau)\sin(2\pi f_c \tau) \quad \text{(by Property 3).}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left[\tilde{n}(t+\tau)\tilde{n}^*(t)\right] &= \mathrm{E}\left[\left(n_I(t+\tau)+jn_Q(t+\tau)\right)\left(n_I(t)-jn_Q(t)\right)\right] \\
&= R_{N_I}(\tau)+R_{N_Q}(\tau)-jR_{N_IN_Q}(\tau)+jR_{N_QN_I}(\tau) \\
&= 2R_{N_I}(\tau)-j2R_{N_IN_Q}(\tau) \\
&= R_{\tilde{N}}(\tau).
\end{aligned}
$$

Therefore,

$$
R_N(\tau) = \mathrm{Re}\left[\frac{1}{2}R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau}\right].
$$

■

- Since also $\mathrm{E}\left[\tilde{n}(t)\right] = \mathrm{E}\left[n_I(t)\right] + j\mathrm{E}\left[n_Q(t)\right] = 0$, $\tilde{n}(t)$ is wide-sense stationary.

Property 9

$$S_N(f) = \frac{1}{4}\left[S_{\tilde{N}}(f - f_c) + S_{\tilde{N}}(-f - f_c)\right].$$

Proof.

$$\begin{aligned} R_N(\tau) &= \mathrm{Re}\left[\frac{1}{2}R_{\tilde{N}}(\tau)e^{j2\pi f_c \tau}\right] \\ &= \frac{1}{4}\left(R_{\tilde{N}}(\tau)e^{j2\pi f_c \tau} + R^*_{\tilde{N}}(\tau)e^{-j2\pi f_c \tau}\right). \end{aligned}$$

Since

$$\begin{aligned} F\left[R_{\tilde{N}}(\tau)\right] &= S_{\tilde{N}}(f) \\ F\left[R^*_{\tilde{N}}(\tau)\right] &= S^*_{\tilde{N}}(-f) \end{aligned}$$

we have

$$\begin{aligned} F\left[R_{\tilde{N}}(\tau)e^{j2\pi f_c \tau}\right] &= S_{\tilde{N}}(f - f_c) \\ F\left[R^*_{\tilde{N}}(\tau)e^{-j2\pi f_c \tau}\right] &= S^*_{\tilde{N}}(-f - f_c) \end{aligned}$$

where $F[\bullet]$ denotes the Fourier transform.

Also since

$$\begin{aligned} S_{\tilde{N}}^*(f) &= \int_{-\infty}^{\infty} R_{\tilde{N}}^*(\tau) e^{j2\pi f_c \tau}\, d\tau \\ &= \int_{-\infty}^{\infty} R_{\tilde{N}}(-\tau) e^{j2\pi f_c \tau}\, d\tau \\ &= \int_{-\infty}^{\infty} R_{\tilde{N}}(\tau') e^{-j2\pi f_c \tau'}\, d\tau' \\ &= S_{\tilde{N}}(f) \end{aligned}$$

$S_{\tilde{N}}(f)$ is real. Therefore,

$$S_N(f) = F\left[R_N(\tau)\right] = \frac{1}{4}\left[S_{\tilde{N}}(f - f_c) + S_{\tilde{N}}(-f - f_c)\right].$$

■

Property 10

If $n(t)$ is a Gaussian process, then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian processes.

Proof. Recall that $\hat{n}(t)$ can be considered as the output of $n(t)$ passed through an LTI system. If $n(t)$ is a Gaussian process, then $\hat{n}(t)$ is a Gaussian process, and $n(t)$ and $\hat{n}(t)$ are jointly Gaussian processes. Since

$$n_I(t) = n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t)$$

and

$$n_Q(t) = -n(t)\sin(2\pi f_c t) + \hat{n}(t)\cos(2\pi f_c t)$$

$n_I(t)$ and $n_Q(t)$ are jointly Gaussian processes. ∎

Property 11

If the power spectral density $S_N(f)$ is locally symmetric around f_c, then

$$R_{N_I N_Q}(\tau) = \mathrm{E}\left[n_I(t+\tau)n_Q(t)\right] = 0, \quad \text{for all } \tau$$

i.e., $n_I(t+\tau)$ and $n_Q(t)$ are uncorrelated.

Proof. If $S_N(f)$ is locally symmetric around f_c, then

$$S_N(f - f_c) = S_N(f + f_c), \quad \text{for } -B \leq f \leq B$$

and hence by Property 6

$$S_{N_I N_Q}(f) = S_{N_Q N_I}(f) = 0.$$

Therefore, $R_{N_I N_Q}(\tau) = R_{N_Q N_I}(\tau) = 0$, for all τ. ∎

Property 12

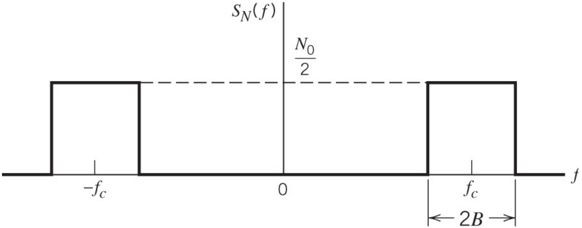
If $n(t)$ is a Gaussian process with power spectral density $S_N(f)$ locally symmetric around f_c, then $n_I(t_1)$ and $n_Q(t_2)$ are statistically independent, for all t_1, t_2.

Proof. This property follows from Properties 10 and 11 and also the fact that uncorrelated jointly Gaussian random variables are statistically independent. ■

Ideal Band-Pass Filtered Noise

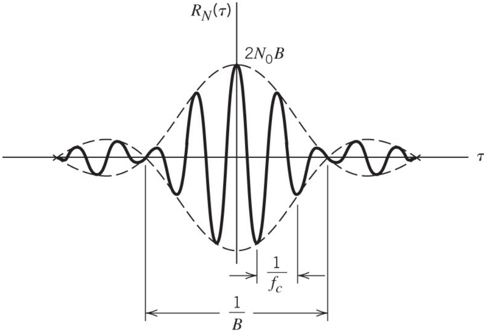
- Consider a white Gaussian noise of zero mean and power spectral density $N_0/2$ which is passed through an ideal band-pass filter of mid-band frequency f_c and bandwidth $2B$.
- The power spectral density of the filtered noise $n(t)$ is given by

$$S_N(f) = \begin{cases} N_0/2, & -f_c - B \leq f \leq -f_c + B \\ N_0/2, & f_c - B \leq f \leq f_c + B \\ 0, & \text{elsewhere.} \end{cases}$$



- Then the autocorrelation function of $n(t)$ is

$$\begin{aligned} R_N(\tau) &= \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} e^{j2\pi f\tau}\, df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} e^{j2\pi f\tau}\, df \\ &= N_0 B\, \mathrm{sinc}(2B\tau)\left(e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}\right) \\ &= 2N_0 B\, \mathrm{sinc}(2B\tau)\cos(2\pi f_c\tau). \end{aligned}$$



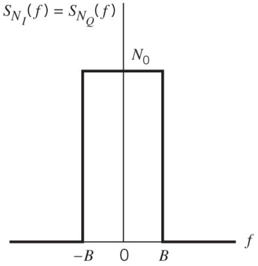
- Since $S_N(f)$ is locally symmetric around f_c, we have

$$S_{N_I N_Q}(f) = S_{N_Q N_I}(f) = 0 \quad \text{and} \quad R_{N_I N_Q}(\tau) = R_{N_Q N_I}(\tau) = 0.$$

- We also have

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} N_0, & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$



- Hence

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = 2N_0 B \operatorname{sinc}(2B\tau).$$

Envelope and Phase of Narrow-Band Noise

- Consider a narrow-band noise $n(t)$ which is a Gaussian process of zero mean and power spectral density $S_N(f)$.
- Assume that $S_N(f)$ is locally symmetric around f_c and $\mathrm{E}[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\, df = \sigma^2$.
- We have

$$
\begin{aligned}
n(t) &= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\
&= r(t)\cos\left[2\pi f_c t + \psi(t)\right]
\end{aligned}
$$

where

$$
r(t) = \left[n_I^2(t) + n_Q^2(t)\right]^{1/2}
$$

$$
\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right].
$$

- Let N_I and N_Q represent $n_I(t)$ and $n_Q(t)$ at some fixed time t, respectively. Also R and Ψ represent $r(t)$ and $\psi(t)$ at the same fixed time t, respectively.
- Then N_I and N_Q are independent Gaussian random variables with joint probability density function given by

$$f_{N_I,N_Q}(n_I, n_Q) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right).$$

- We have

$$\begin{aligned} n_I &= r\cos\psi \\ n_Q &= r\sin\psi. \end{aligned}$$

- The Jacobian of the transformation is then given by

$$J = \begin{vmatrix} \frac{\partial n_I}{\partial r} & \frac{\partial n_I}{\partial \psi} \\ \frac{\partial n_Q}{\partial r} & \frac{\partial n_Q}{\partial \psi} \end{vmatrix} = \begin{vmatrix} \cos\psi & -r\sin\psi \\ \sin\psi & r\cos\psi \end{vmatrix} = r\left(\cos^2\psi + \sin^2\psi\right) = r.$$

- Hence the joint probability density function of R and Ψ is

$$f_{R,\Psi}(r,\psi) = f_{N_I,N_Q}(r\cos\psi, r\sin\psi)|J| = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right).$$

- We then have

$$f_{R,\Psi}(r,\psi) = f_R(r)f_\Psi(\psi)$$

where R is Rayleigh distributed with probability density function given by

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), & r \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

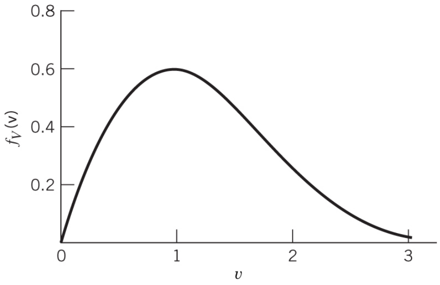
and Ψ is uniformly distributed with probability density function given by

$$f_\Psi(\psi) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \psi \leq 2\pi \\ 0, & \text{elsewhere.} \end{cases}$$

- Note that R and Ψ are independent.

- Let $v = r/\sigma$ and then $f_V(v) = \sigma f_R(r)$.
- The probability density function of the Rayleigh distribution in the normalized form is then given by

$$f_V(v) = \begin{cases} v \exp\left(-\frac{v^2}{2}\right), & v \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$



Sinusoidal Signal Plus Narrow-Band Noise

- Assume that the narrow-band noise $n(t)$ is a Gaussian process of zero mean and power spectral density $S_N(f)$.
- Also assume that $S_N(f)$ is locally symmetric around f_c and $\mathrm{E}[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\,df = \sigma^2$.
- Consider

$$\begin{aligned} x(t) &= A\cos(2\pi f_c t) + n(t) \\ &= A\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= n_I'(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

where

$$n_I'(t) = A + n_I(t).$$

- Let N_I' and N_Q represent $n_I'(t)$ and $n_Q(t)$ at some fixed time t, respectively.

- The joint probability density function of N_I' and N_Q is then given by

$$f_{N_I',N_Q}(n_I', n_Q) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(n_I' - A)^2 + n_Q^2}{2\sigma^2}\right].$$

- Consider

$$r(t) = \left\{[n_I'(t)]^2 + n_Q^2(t)\right\}^{1/2}$$
$$\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I'(t)}\right].$$

- Let R and Ψ represent $r(t)$ and $\psi(t)$ at the same fixed time t, respectively.
- We can then obtain

$$f_{R,\Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2 - 2Ar\cos\psi}{2\sigma^2}\right).$$

- Note that now R and Ψ are dependent.

- The marginal probability density function of R can be found by

$$\begin{aligned} f_R(r) &= \int_0^{2\pi} f_{R,\Psi}(r,\psi)\,d\psi \\ &= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)\int_0^{2\pi}\exp\left(\frac{Ar}{\sigma^2}\cos\psi\right)\,d\psi. \end{aligned}$$

- Note that the modified Bessel function of the first kind of zero order is given by

$$I_0(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\exp(\alpha\cos\psi)\,d\psi.$$

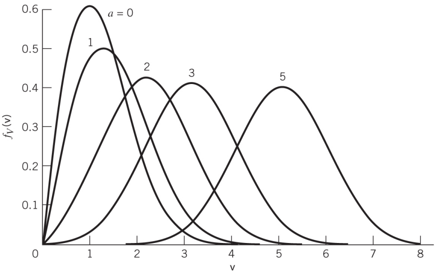
- We hence obtain

$$f_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right), \quad r \geq 0$$

which is called the Rician distribution.

- Let $v = r/\sigma$ and $a = A/\sigma$; then $f_V(v) = \sigma f_R(r)$.
- The probability density function of the Ricain distribution in the normalized form is then given by

$$f_V(v) = v \exp\left(-\frac{v^2 + a^2}{2}\right) I_0(av), \quad v \geq 0.$$



- When $a = 0$, the Rician distribution reduces to the Rayleigh distribution.