Solution to Midterm Examination

1. (a) Ignoring the discontinuity, we have

$$\begin{aligned} g(t) = F^{-1}[G(f)] &= \int_0^\infty e^{-2f} \cdot e^{j2\pi f t} df \\ &= \frac{1}{-2+j2\pi t} e^{(-2f+j2\pi f t)} \Big|_0^\infty \\ &= \frac{1}{2-j2\pi t}. \end{aligned}$$

(b) From (a), we have

$$\begin{aligned} g(t) &= \frac{1}{2-j2\pi t} \\ &= \frac{1}{2+2\pi^2 t^2} + j\frac{\pi t}{2+2\pi^2 t^2} \end{aligned}$$

which is complex. To show that the real and imaginary parts of $g(t)$ constitute a Hilbert transform pair, it is sufficient to show that $g(t)$ is the pre-envelope of some real signal $h(t)$, i.e.,

$$g(t) = h_+(t) = h(t) + j\hat{h}(t)$$

where $\hat{h}(t)$ is the Hilbert transform of $h(t)$. Since the Fourier transform of $h_+(t)$ is given by

$$\begin{aligned} H_+(f) &= \begin{cases} 2H(f), & f > 0 \\ H(0), & f = 0 \\ 0, & f < 0 \end{cases} \\ &= G(f) \end{aligned}$$

we can have

$$H(f) = \begin{cases} (1/2)\exp(-2|f|), & |f| > 0 \\ \pi, & f = 0 \end{cases}$$

which is the Fourier transform of $h(t)$. Therefore,

$$g(t) = h_+(t) = h(t) + j\hat{h}(t)$$

where its real part

$$h(t) = \frac{1}{2+2\pi^2 t^2}$$

and the imaginary part

$$\hat{h}(t) = \frac{\pi t}{2+2\pi^2 t^2}$$

constitute a Hilbert transform pair.

2. (a) Since $x(t) = \text{Re}[\text{rect}(t/T)e^{j2\pi f_c t}]$, the complex envelope of $x(t)$ is

$$\tilde{x}(t) = \text{rect}(t/T).$$

(b) Since $h(t) = \text{Re}[u(t)e^{j2\pi f_c t}]$, the complex envelope of $h(t)$ is

$$\tilde{h}(t) = u(t).$$

(c) We have

$$\begin{aligned}
\tilde{y}(t) &= \frac{1}{2}\tilde{x}(t) \star \tilde{h}(t) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} u(\tau)\text{rect}\left(\frac{t-\tau}{T}\right) d\tau \\
&= \frac{1}{2}\int_{0}^{\infty} \text{rect}\left(\frac{t-\tau}{T}\right) d\tau \\
&= \begin{cases} 0, & t < -T/2 \\ \frac{1}{2}\left(t + \frac{T}{2}\right), & -T/2 \le t < T/2 \\ \frac{T}{2}, & t \ge T/2. \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= \text{Re}[\tilde{y}(t)e^{j2\pi f_c t}] \\
&= \begin{cases} 0, & t < -T/2 \\ \frac{1}{2}\left(t + \frac{T}{2}\right)\cos(2\pi f_c t), & -T/2 \le t < T/2 \\ \frac{T}{2}\cos(2\pi f_c t), & t \ge T/2. \end{cases}
\end{aligned}$$

The sketch of $y(t)$ is shown in Fig. 1.

3. (a) Since

$$\max_t |m(t)| = \max_t |3\cos(20\pi t) + 7\cos(60\pi t)| = 10$$

with 50% percentage modulation we have $k_a = 1/20$. Then

$$\begin{aligned}
s(t) &= 100\left[1 + \frac{1}{20}m(t)\right]\cos(200\pi t) \\
&= 100\left[1 + \frac{3}{20}\cos(20\pi t) + \frac{7}{20}\cos(60\pi t)\right]\cos(200\pi t) \\
&= 100\cos(200\pi t) + \frac{15}{2}\cos(180\pi t) + \frac{15}{2}\cos(220\pi t) + \frac{35}{2}\cos(140\pi t) \\
&\quad + \frac{35}{2}\cos(260\pi t).
\end{aligned}$$

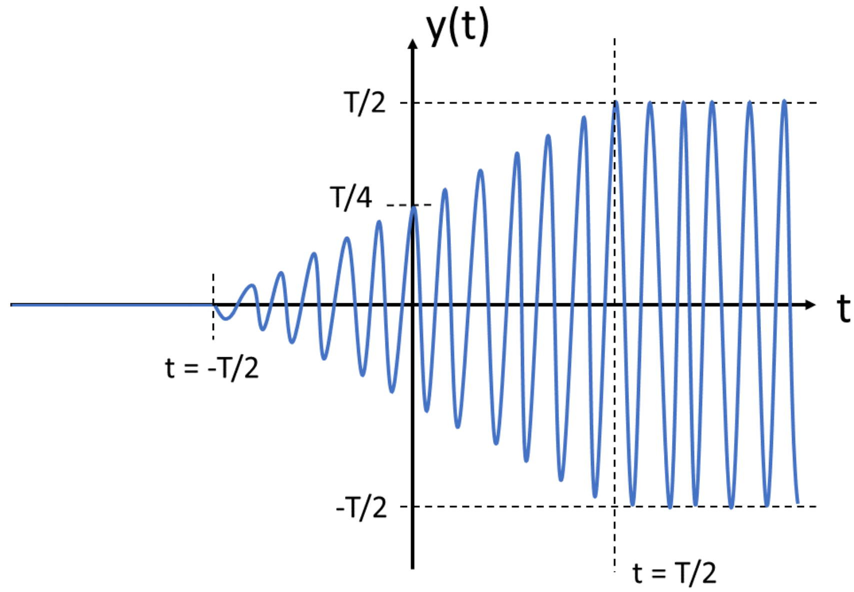

Figure 1: Sketch of $y(t)$ in Problem 2.(c).

(b) Correspondingly, we have

$$\begin{aligned} S(f) = {} & 50\left[\delta(f-100)+\delta(f+100)\right]+\frac{15}{4}\left[\delta(f-90)+\delta(f+90)\right] \\ & +\frac{15}{4}\left[\delta(f-110)+\delta(f+110)\right]+\frac{35}{4}\left[\delta(f-70)+\delta(f+70)\right] \\ & +\frac{35}{4}\left[\delta(f-130)+\delta(f+130)\right]. \end{aligned}$$

(c) The total power of $s(t)$ is given by

$$\frac{1}{2}\left[100^2+2\left(\frac{15}{2}\right)^2+2\left(\frac{35}{2}\right)^2\right]=\frac{10725}{2}$$

in which the power in the sidebands is 725/2. Therefore, the ratio of the power in the sidebands to the total power is

$$\frac{725/2}{10725/2}=\frac{29}{429}=0.0676.$$

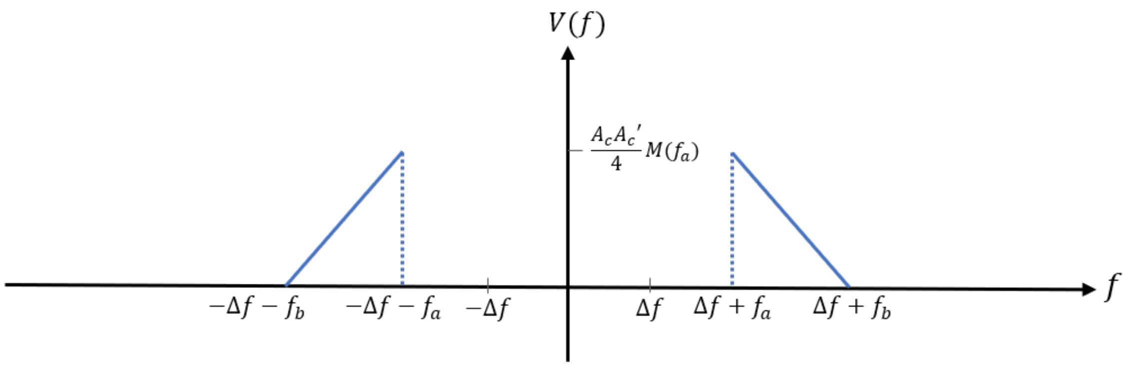

Figure 2: Fourier transform $V(f)$ of $v(t)$ in Problem 4.(b).

4. (a) First, the output of the product modulator for coherent demodulation is

$$\begin{aligned} s(t) \cdot c'(t) &= \left[\frac{A_c}{2} m(t) \cos(2\pi f_c t) + \frac{A_c}{2} \hat{m}(t) \sin(2\pi f_c t) \right] \cdot A_c' \cos[2\pi (f_c + \Delta f) t] \\ &= \frac{A_c A_c'}{2} m(t) \cos(2\pi f_c t) \cos[2\pi (f_c + \Delta f) t] \\ &\quad + \frac{A_c A_c'}{2} \hat{m}(t) \sin(2\pi f_c t) \cos[2\pi (f_c + \Delta f) t] \\ &= \frac{A_c A_c'}{4} m(t) \cos(2\pi \Delta f t) + \frac{A_c A_c'}{4} m(t) \cos[2\pi (2f_c + \Delta f) t] \\ &\quad - \frac{A_c A_c'}{4} \hat{m}(t) \sin(2\pi \Delta f t) + \frac{A_c A_c'}{4} \hat{m}(t) \sin[2\pi (2f_c + \Delta f) t]. \end{aligned}$$

Then after the low-pass filter, we have

$$v(t) = \frac{A_c A_c'}{4} m(t) \cos(2\pi \Delta f t) - \frac{A_c A_c'}{4} \hat{m}(t) \sin(2\pi \Delta f t).$$

(b) Note that

$$v(t) = \frac{A_c A_c'}{4} m(t) \cos(2\pi \Delta f t) - \frac{A_c A_c'}{4} \hat{m}(t) \sin(2\pi \Delta f t)$$

which can be seen as in the form of an upper-sideband SSB signal with carrier frequency Δf. The Fourier transform $V(f)$ of $v(t)$ is hence given in Fig. 2.

5. (a) From class, we know that DSB-SC modulation requires a transmission bandwidth of $2W$, where W is the message bandwidth. If we want to use DSB-SC to modulate these five message signals, the required bandwidth is then given by

$$\begin{aligned} B_T &= 2 \cdot 5 + 2 \cdot 10 + 2 \cdot 15 + 2 \cdot 10 + 2 \cdot 10 \\ &= 100 \text{ kHz}. \end{aligned}$$

However, the available bandwidth of the allocated band is $860 - 800 = 60$ kHz, which is less than 100 kHz. As a result, we can not use DSB-SC for transmission.

(b) Since VSB is used, the transmission band of each message signal $m_i(t)$, $i = 1, 2, 3, 4, 5$, consists of a 20% lower sideband and a full upper sideband.

	Transmission bandwidth (kHz)
$m_1(t)$	$(0.2+1)\cdot 5 = 6$
$m_2(t)$	$(0.2+1)\cdot 10 = 12$
$m_3(t)$	$(0.2+1)\cdot 15 = 18$
$m_4(t)$	$(0.2+1)\cdot 10 = 12$
$m_5(t)$	$(0.2+1)\cdot 10 = 12$

The carrier frequency f_c and the occupied frequency band for each message signal are then given in the following table.

	f_c (kHz)	Occupied frequency band (kHz)
$m_1(t)$	801	$800 \sim 806$
$m_2(t)$	808	$806 \sim 818$
$m_3(t)$	821	$818 \sim 836$
$m_4(t)$	838	$836 \sim 848$
$m_5(t)$	850	$848 \sim 860$

6. (a) By Carson's rule, the transmission bandwidth of the FM signal is approximately given by

$$B_T \approx 2\Delta f \left(1 + \frac{1}{\beta}\right)$$

where Δf is the frequency deviation and β is the modulation index. The frequency deviation is

$$\Delta f = k_f A_m = 4 \cdot 5 = 20 \text{ kHz}$$

and the corresponding modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{20}{10} = 2.$$

Therefore, we have

$$B_T \approx 2\Delta f \left(1 + \frac{1}{\beta}\right) = 2 \cdot 20 \cdot \left(1 + \frac{1}{2}\right) = 60 \text{ kHz}.$$

(b) The transmission bandwidth of the FM signal determined by considering only those side frequencies whose amplitudes exceed 1 percent of the unmodulated carrier amplitude is given by

$$B_T \approx 2 n_{\max} f_m$$

where

$$n_{\max} = \max\left\{n : |J_n(\beta)| > 1\%\right\}.$$

From A.4 Table of Bessel functions, for $\beta = 2$, $n_{\max} = 4$. Therefore, we have

$$B_T \approx 2 n_{\max} f_m = 2 \cdot 4 \cdot 10 = 80 \text{ kHz}.$$

(c) The required multiplication ratio is 80 kHz/20 kHz = 4, and hence we choose $n_1 = n_2 = 2$. Also since the carrier frequency of $s_2(t)$ is 92 MHz, we have

$$2 \cdot |2 \cdot f_{c_1} \pm f_l| = 92 \text{ MHz}$$

where f_{c_1} is 25 MHz and 1 MHz $\leq f_l \leq$ 10 MHz. Therefore, we have $f_l = 4$ MHz as

$$2 \cdot (2 \cdot 25 - 4) = 92 \text{ MHz}.$$

(d) For $s_2(t)$, $\Delta f = 80$ kHz and $f_m = 10$ kHz; then $\beta = 8$. Hence by Carson's rule, the transmission bandwidth is approximately

$$B_T \approx 2\Delta f \left(1 + \frac{1}{\beta}\right) = 2 \cdot 80 \cdot \left(1 + \frac{1}{8}\right) = 180 \text{ kHz}.$$

From A.4 Table of Bessel functions, for $\beta = 8$, $n_{\max} = 11$. Therefore, the 1-percent bandwidth is

$$B_T \approx 2 n_{\max} f_m = 2 \cdot 11 \cdot 10 = 220 \text{ kHz}.$$

(e) Since the multipliers do not change f_m, the frequency separation of the adjacent frequencies of $s_2(t)$ is still $f_m = 10$ kHz.

7. Since $\sin[\phi_e(t)] \approx \phi_e(t)$, we obtain

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \phi_e(t) = \frac{d\phi_1(t)}{dt}.$$

By taking the Fourier transform, we have

$$j2\pi f \Phi_e(f) + 2\pi K_0 \Phi_e(f) = j2\pi f \Phi_1(f)$$

which gives

$$\Phi_e(f) = \frac{jf}{K_0 + jf} \Phi_1(f).$$

Hence

$$V(f) = \frac{K_0}{k_v} \Phi_e(f) = \frac{K_0}{k_v} \frac{1}{1 + \frac{K_0}{jf}} \Phi_1(f).$$

Due to the assumption $K_0 \gg |f|$, we obtain

$$V(f) \approx \frac{K_0}{k_v} \frac{1}{\frac{K_0}{jf}} \Phi_1(f) = \frac{jf}{k_v} \Phi_1(f) = \frac{j2\pi f}{2\pi k_v} \Phi_1(f).$$

Therefore,

$$v(t) \approx \frac{1}{2\pi k_v} \frac{d\phi_1(t)}{dt} = \frac{2\pi k_f m(t)}{2\pi k_v} = \frac{k_f}{k_v} m(t)$$

which shows that FM demodulation is achieved.