Solution to Homework Assignment No. 6

1. (a) The Fourier transform of $g(t)$ is

$$G(f) = \frac{1}{200}\,\mathrm{rect}\left(\frac{f}{200}\right)$$

and then the message bandwidth is 100 Hz. The Nyquist rate and the Nyquist interval for $g(t)$ are therefore 200 Hz and 5 ms, respectively.

(b) Similarly, the Fourier transform of $g(t)$ is

$$G(f) = \frac{1}{200}\Lambda\left(\frac{f}{200}\right) = \begin{cases} (1/200)(1 - |f/200|), & |f| < 200 \\ 0, & \text{elsewhere} \end{cases}$$

and then the message bandwidth is 200 Hz. The Nyquist rate and the Nyquist interval for $g(t)$ are therefore 400 Hz and 2.5 ms, respectively.

2. (a) Since a sampling rate of 8 kHz corresponds to a sampling interval of 125 μs, the time allotted to each pulse in this time-division multiplexed signal is $125/25 = 5$ μs. The spacing between successive pulses is therefore $5 - 1 = 4$ μs.

(b) The Nyquist rate for the voice signals is 6.8 kHz, and then the sampling interval $T_s = 147$ μs. The time allotted to each pulse in this time-division multiplexed signal is $147/25 = 5.88$ μs. The spacing between successive pulses is therefore $5.88 - 1 = 4.88$ μs.

3. First note that $\{b_n\}_{n=1}^{10} = 1110110010$.

(a) The polar nonreturn-to-zero signaling is shown in Fig. 1.

(b) The bipolar return-to-zero signaling is shown in Fig. 2.

(c) The Manchester code signaling is shown in Fig. 3.

4. (a) Since a 7-bit codeword is used, the bit duration T_b is given by

$$T_b = \frac{T_s}{7} = \frac{1}{7f_s}$$

where T_s is the sampling period and f_s is the sampling rate. We then have the bit rate

$$\frac{1}{T_b} = 7f_s = 50 \times 10^6 \text{ b/s}.$$

According to the sampling theorem, the minimum sampling rate is $2W$, where W is the bandwidth of the message signal. Therefore, the maximum bandwidth is given by

$$\frac{f_s}{2} = \frac{50 \times 10^6}{2 \cdot 7} \approx 3.57 \times 10^6 \text{ Hz}.$$

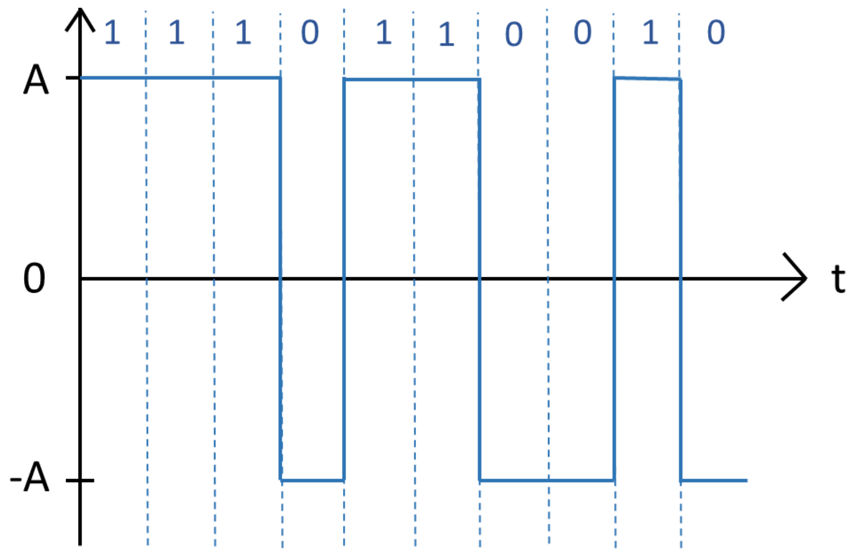

Figure 1: Polar nonreturn-to-zero signaling in Problem 3.(a).

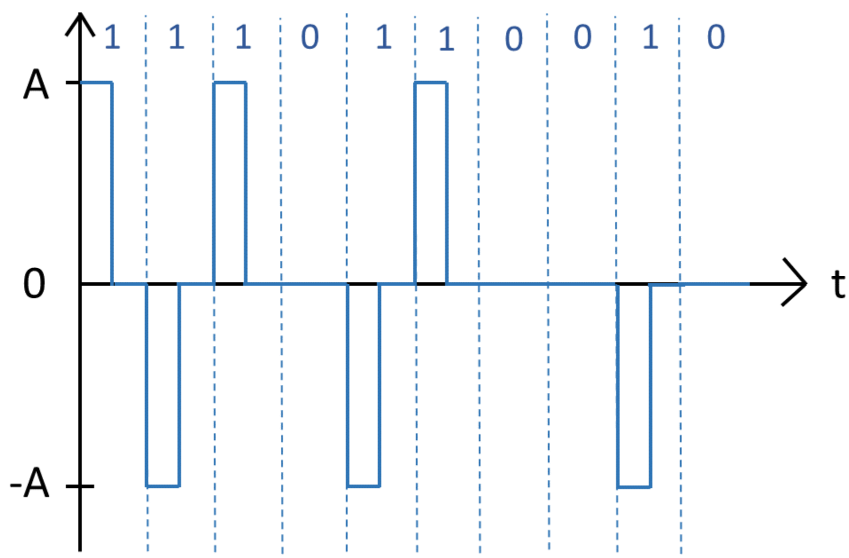

Figure 2: Bipolar return-to-zero signaling in Problem 3.(b).

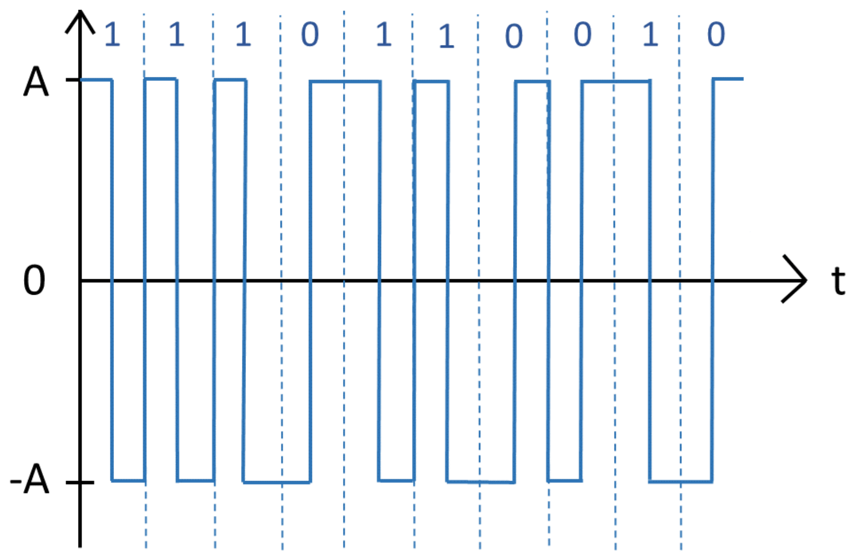

Figure 3: Manchester code signaling in Problem 3.(c).

(b) From the example in class, the output signal-to-noise ratio of a full-load sinusoidal modulating wave is $(3/2)2^{2R}$, where R is the number of bits per sample. Since $R = 7$, we have

$$(\mathrm{SNR})_\mathrm{O} = \frac{3}{2} \cdot 2^{14} = 24576 \quad \approx 43.9\,\mathrm{dB}.$$

5. (a) For $n \geq 1$, using the fact that two inversions cancel each other we have

$$\begin{aligned} p_{n+1} &= \Pr\{\text{a binary symbol is in error after the } (n+1)\text{-th repeater}\} \\ &= \Pr\{\text{the bit is in error after the } n\text{-th repeater}\} \\ &\quad \cdot \Pr\{\text{it is not inverted by the } (n+1)\text{-th repeater}\} \\ &\quad + \Pr\{\text{it is correct after the } n\text{-th repeater}\} \\ &\quad \cdot \Pr\{\text{it is inverted by the } (n+1)\text{-th repeater}\} \end{aligned}$$

where the inversion probability of a single repeater is p_1. Hence, we obtain

$$p_{n+1} = p_n\,(1 - p_1) + (1 - p_n)\,p_1.$$

(b) We first prove this by induction. For $n = 1$, we have

$$\frac{1}{2}\,[1 - (1 - 2p_1)] = p_1.$$

Assume that, for $k \geq 1$,

$$p_k = \frac{1}{2}\left[1 - (1 - 2p_1)^k\right].$$

Then from (a)

$$\begin{aligned} p_{k+1} &= p_k (1 - p_1) + (1 - p_k) p_1 \\ &= p_k (1 - 2p_1) + p_1 \\ &= \frac{1}{2} \left[1 - (1 - 2p_1)^k \right] (1 - 2p_1) + p_1 \\ &= \frac{1}{2} \left[1 - (1 - 2p_1)^{k+1} \right] - p_1 + p_1 \\ &= \frac{1}{2} \left[1 - (1 - 2p_1)^{k+1} \right]. \end{aligned}$$

Therefore, by induction, $p_n = (1/2) [1 - (1 - 2p_1)^n]$ for $n \geq 1$.
Alternatively, the equation in (a) can be considered as a first-order nonhomogeneous difference equation (or linear recursion):

$$p_{n+1} - (1 - 2p_1) p_n = p_1, \qquad \text{for } n \geq 1$$

with the initial condition $p_n = p_1$ when $n = 1$. By the method of solving difference equations, we have the general solution to this nonhomogeneous difference equation given by

$$p_n = a (1 - 2p_1)^n + \frac{1}{2}, \qquad \text{for } n \geq 1$$

where a is a constant. For $n = 1$,

$$p_n = a (1 - 2p_1)^1 + \frac{1}{2} = p_1$$

which gives $a = -1/2$. Therefore,

$$p_n = \frac{1}{2} [1 - (1 - 2p_1)^n], \qquad \text{for } n \geq 1.$$

(c) When p_1 is small, we have the following approximation:

$$(1 - 2p_1)^n \approx 1 - 2np_1.$$

The appoximate value of p_n is then given by

$$p_n \approx \frac{1}{2} [1 - (1 - 2np_1)] = np_1.$$

6. The modulating wave is

$$m(t) = A_m \cos (2\pi f_m t)$$

and then the slope of $m(t)$ is given by

$$\frac{dm(t)}{dt} = -2\pi f_m A_m \sin (2\pi f_m t).$$

In order to avoid slope overload, the requirement

$$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right|$$

should be satisfied. With

$$\max\left|\frac{dm(t)}{dt}\right| = 2\pi f_m A_m$$

we can obtain

$$\frac{\Delta}{T_s} \geq 2\pi f_m A_m$$

which implies

$$A_m \leq \frac{\Delta}{2\pi f_m T_s}.$$

Therefore, the maximum amplitude of this modulating signal required to avoid slope overload is $\Delta/(2\pi f_m T_s)$.