Solution to Homework Assignment No. 5

1. The transfer function of the low-pass RC filter is given by

$$H(f) = \frac{1}{1 + j2\pi fRC}.$$

The power spectral density of the output signal $y(t)$ is hence

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) \\ &= \frac{A_c^2[\delta(f - f_c) + \delta(f + f_c)]}{4\,[1 + (2\pi fRC)^2]} + \frac{N_0}{2\,[1 + (2\pi fRC)^2]} \\ &= S_{M'}(f) + S_{N'}(f). \end{aligned}$$

The average powers of the signal and noise components of $y(t)$ are

$$\int_{-\infty}^{\infty} S_{M'}(f)\,df = \frac{A_c^2}{2\,(1 + 4\pi^2 f_c^2 R^2 C^2)}$$

and

$$\int_{-\infty}^{\infty} S_{N'}(f)\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{df}{1 + (2\pi RC)^2 f^2} = \frac{N_0}{4RC}$$

respectively. Therefore, the output signal-to-noise ratio is given by

$$(\mathrm{SNR})_\mathrm{O} = \frac{2RCA_c^2}{(1 + 4\pi^2 f_c^2 R^2 C^2)\,N_0}.$$

2. If the DSB-SC signal is transmitted, we know that the output signal of the coherent detector in the receiver is

$$y(t) = \frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t).$$

Since the average power of the modulated signal is ${A_c}^2 P/2 = 10$ watts, where P is the average power of $m(t)$, the output signal-to-noise ratio of the receiver is then given by

$$\begin{aligned} (\mathrm{SNR})_\mathrm{O} &= \frac{(1/4)\cdot {A_c}^2 P}{(1/4)\cdot \int_{-W}^{W} S_{N_I}(f)\,df} = \frac{(1/2)\cdot 10}{(1/4)\cdot 8\cdot 10^3 \cdot 10^{-6}} = \frac{5}{2\cdot 10^{-3}} \\ &= 2500 \approx 34\,\mathrm{dB} \end{aligned}$$

where $W = 4$ kHz is the message bandwidth.

3. (a) For this AM system, the carrier power is 80 kilowatts, i.e.,

$$\frac{A_c^2}{2} = 80 \text{ kilowatts}$$

and the total sideband power is $2 \cdot 10 = 20$ kilowatts, i.e.,

$$\frac{A_c^2}{2} k_a^2 P = 20 \text{ kilowatts}$$

where P is the average power of the message signal. We hence have $k_a^2 P = 1/4$. Also the power spectral density of the white noise is $N_0/2 = 10^{-3}$ watt per Hertz and the message bandwidth W is 4 kHz. Therefore, when the carrier-to-noise ratio is high, the output signal-to-noise ratio of this AM system is

$$(\text{SNR})_{\text{O, AM}} = \frac{A_c^2 k_a^2 P}{2 N_0 W} = \frac{20 \cdot 10^3}{2 \cdot 10^{-3} \cdot 4 \cdot 10^3} = 2500 \approx 34 \text{ dB}.$$

(b) From class, we know that

$$(\text{SNR})_{\text{O, AM}} = \frac{k_a^2 P}{1 + k_a^2 P} (\text{SNR})_{\text{C, AM}}$$
$$(\text{SNR})_{\text{O, DSB-SC}} = (\text{SNR})_{\text{C, DSB-SC}} \cdot$$

Then given the same channel signal-to-noise ratio, we have

$$\frac{(\text{SNR})_{\text{O, AM}}}{(\text{SNR})_{\text{O, DSB-SC}}} = \frac{k_a^2 P}{1 + k_a^2 P} = \frac{1/4}{1 + (1/4)} = 0.2 \approx -7 \text{ dB}.$$

Therefore, this AM system is 7 dB inferior to a DSB-SC system.

4. The input signal of the envelop detector is

$$\begin{aligned} x(t) &= A_c \cos(2\pi f_c t) + n(t) \\ &= A_c \cos(2\pi f_c t) + n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \\ &= [A_c + n_I(t)] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t). \end{aligned}$$

The output signal of the envelop detector is then given by

$$y(t) = \sqrt{[A_c + n_I(t)]^2 + [n_Q(t)]^2}.$$

When the carrier-to-noise ratio is high, we may approximate $y(t)$ as

$$y(t) \approx A_c + n_I(t).$$

Therefore, the output signal-to-noise ratio is given by

$$(\text{SNR})_{\text{O}} = \frac{A_c^2}{\text{E}\left[n_I^2(t)\right]} = \frac{A_c^2}{2 N_0 W}.$$

5. (a) The input of the phase detector in the reciever is

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c \cos[2\pi f_c t + k_p m(t)] + r(t)\cos[2\pi f_c t + \psi(t)] \end{aligned}$$

where $n(t)$ is a narrowband noise. Let $\phi(t)$ denote $k_p m(t)$, the phase of the modulated signal. Given that the phase detector is ideal, its output is then given by

$$\theta(t) = \phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\psi(t) - \phi(t)]}{A_c + r(t)\cos[\psi(t) - \phi(t)]}\right\}.$$

When the carrier-to-noise ratio is high, we have

$$\theta(t) \approx \phi(t) + \frac{r(t)\sin[\psi(t) - \phi(t)]}{A_c}.$$

Similar to the FM system, it can be justified that the following approximation is good:

$$\theta(t) \approx \phi(t) + \frac{r(t)\sin[\psi(t)]}{A_c} = k_p m(t) + \frac{n_Q(t)}{A_c}$$

in which the noise power is $2N_0W/A_c^2$ and the signal power is $k_p^2 P$. Therefore, the output signal-to-noise ratio is

$$(\mathrm{SNR})_{\mathrm{O,\ PM}} = \frac{k_p^2 P}{2N_0W/A_c^2} = \frac{A_c^2 k_p^2 P}{2N_0W}.$$

(b) The channel signal-to-noise ratio of this PM system is the same as that of the FM system, which is given by

$$\frac{A_c^2}{2N_0W}.$$

Therefore, the figure of merit of this PM system is

$$\left.\frac{(\mathrm{SNR})_{\mathrm{O}}}{(\mathrm{SNR})_{\mathrm{C}}}\right|_{\mathrm{PM}} = \frac{A_c^2 k_p^2 P}{2N_0W}\frac{2N_0W}{A_c^2} = k_p^2 P.$$

6. (a) The average power of $m(t)$ is

$$\begin{aligned} P &= \int_{-\infty}^{+\infty} S_M(f)\,df \\ &= \int_{-W}^{W} \frac{S_0}{1 + (f/f_0)^2}\,df \\ &= S_0 f_0 \tan^{-1}\left(\frac{f}{f_0}\right)\Bigg|_{-W}^{W} \\ &= 2S_0 f_0 \tan^{-1}\left(\frac{W}{f_0}\right) \end{aligned}$$

and the average power of the emphasized message signal is

$$\begin{aligned}
P' &= \int_{-\infty}^{+\infty} S_M(f)\,|H_{\mathrm{pe}}(f)|^2\,df \\
&= \int_{-W}^{W} \frac{S_0}{1+(f/f_0)^2}\cdot k^2\left[1+\left(\frac{f}{f_0}\right)^2\right]df \\
&= 2WS_0k^2.
\end{aligned}$$

To keep the average power unchanged, i.e., $P' = P$, we have

$$k = \left[\frac{f_0}{W}\tan^{-1}\left(\frac{W}{f_0}\right)\right]^{1/2}.$$

(b) The improvement factor I is given by

$$\begin{aligned}
I &= \frac{\mathrm{E}\left[n_0^2(t)\right]}{\mathrm{E}\left[n_{\mathrm{de}}^2(t)\right]} = \frac{2W^3}{3\int_{-W}^{W} f^2\,|H_{\mathrm{de}}(f)|^2\,df} \\
&= \frac{2W^3}{3\int_{-W}^{W} f^2 \frac{1}{k^2\left[1+(f/f_0)^2\right]}\,df} \\
&= \frac{2W^3k^2}{3\int_{-W}^{W} \frac{f^2}{1+(f/f_0)^2}\,df} \\
&= \frac{W^3k^2}{3\left[f_0^2W - f_0^3\tan^{-1}(W/f_0)\right]} \\
&= \frac{(W/f_0)^2\tan^{-1}(W/f_0)}{3\left[(W/f_0) - \tan^{-1}(W/f_0)\right]}.
\end{aligned}$$