Solution to Homework Assignment No. 4

1. (a) Since X and Y are independent Gaussian random variables and Z_1 and Z_2 are both linear combinations of X and Y, Z_1 and Z_2 are jointly Gaussian random variables. We have

$$\mathrm{E}[Z(t)] = \mathrm{E}[X]\cos(2\pi t) + \mathrm{E}[Y]\sin(2\pi t) = 0$$

and hence

$$\mathrm{E}[Z_1] = \mathrm{E}[Z_2] = 0.$$

Moreover,

$$\begin{aligned}
\mathrm{E}[Z(t+\tau)Z(t)] =& \mathrm{E}[(X\cos(2\pi(t+\tau)) + Y\sin(2\pi(t+\tau))) \\
& (X\cos(2\pi t) + Y\sin(2\pi t))] \\
=& \cos(2\pi(t+\tau))\cos(2\pi t)\mathrm{E}\left[X^2\right] + \sin(2\pi(t+\tau))\sin(2\pi t)\mathrm{E}\left[Y^2\right] \\
& + [\cos(2\pi(t+\tau))\sin(2\pi t) + \sin(2\pi(t+\tau))\cos(2\pi t)]\,\mathrm{E}[XY] \\
=& \cos(2\pi(t+\tau))\cos(2\pi t) + \sin(2\pi(t+\tau))\sin(2\pi t) \\
=& \cos(2\pi\tau)
\end{aligned}$$

and thus

$$\mathrm{E}\left[Z_1^2\right] = \mathrm{E}\left[Z_2^2\right] = 1$$
$$\mathrm{E}\left[Z_1 Z_2\right] = \cos\left(2\pi\left(t_1 - t_2\right)\right).$$

The correlation coefficient between Z_1 and Z_2 is then given by

$$\rho = \cos\left(2\pi\left(t_1 - t_2\right)\right).$$

Therefore, the joint probability density function of Z_1 and Z_2 is

$$\begin{aligned}
f_{Z_1,Z_2}(z_1, z_2) &= \frac{1}{2\pi\sqrt{1-\cos^2\left(2\pi\left(t_1-t_2\right)\right)}} \exp\left\{-\frac{z_1^2 - 2\cos\left(2\pi\left(t_1-t_2\right)\right)z_1 z_2 + z_2^2}{2\left[1-\cos^2\left(2\pi\left(t_1-t_2\right)\right)\right]}\right\} \\
&= \frac{1}{2\pi\left|\sin\left(2\pi\left(t_1-t_2\right)\right)\right|} \exp\left\{-\frac{z_1^2 - 2\cos\left(2\pi\left(t_1-t_2\right)\right)z_1 z_2 + z_2^2}{2\sin^2\left(2\pi\left(t_1-t_2\right)\right)}\right\}.
\end{aligned}$$

(b) From (a), we have the mean of $Z(t)$

$$\mathrm{E}[Z(t)] = 0$$

and the autocorrelation function of $Z(t)$

$$\mathrm{E}[Z(t+\tau)Z(t)] = \cos(2\pi\tau).$$

Therefore, $Z(t)$ is wide-sense stationary.

2. (a) From class, we know that the autocorrelation function of $V(t)$

$$R_V(\tau) = R_X(\tau) \star h_1(\tau) \star h_1(-\tau)$$

where $\star$ denotes the convolution. Then we get the autocorrelation function of $Y(t)$

$$\begin{aligned} R_Y(\tau) &= R_V(\tau) \star h_2(\tau) \star h_2(-\tau) \\ &= R_X(\tau) \star h_1(\tau) \star h_1(-\tau) \star h_2(\tau) \star h_2(-\tau). \end{aligned}$$

(b) Since

$$R_{VY}(\tau) = R_V(\tau) \star h_2(-\tau)$$

we obtain

$$R_{VY}(\tau) = R_X(\tau) \star h_1(\tau) \star h_1(-\tau) \star h_2(-\tau).$$

3. (a) The random variable $Y(t_1)$ is given by

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1 - u)\, h_1(u)\, du$$

and the mean of $Y(t_1)$ is

$$\begin{aligned} \mu_{Y_1} &= \mathrm{E}\left[Y(t_1)\right] \\ &= \mathrm{E}\left[\int_{-\infty}^{\infty} X(t_1 - u)\, h_1(u)\, du\right] = \int_{-\infty}^{\infty} \mathrm{E}[X(t_1 - u)] h_1(u)\, du \\ &= \int_{-\infty}^{\infty} \mu_X h_1(u)\, du = \mu_X \int_{-\infty}^{\infty} h_1(u)\, du. \end{aligned}$$

Similarly, the random variable $Z(t_2)$ is given by

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2 - u)\, h_2(u)\, du$$

and the mean of $Z(t_2)$ is

$$\mu_{Z_2} = \mathrm{E}\left[Z(t_2)\right] = \mu_X \int_{-\infty}^{\infty} h_2(u)\, du.$$

The covariance of $Y(t_1)$ and $Z(t_2)$ is

$$\begin{aligned} &\mathrm{Cov}\left[Y(t_1),\ Z(t_2)\right] \\ &= \mathrm{E}\left[\left(Y(t_1) - \mu_{Y_1}\right)\left(Z(t_2) - \mu_{Z_2}\right]\right) \\ &= \mathrm{E}\left[\int_{-\infty}^{\infty} \left(X(t_1 - \tau_1) - \mu_X\right) h_1(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} \left(X(t_2 - \tau_2) - \mu_X\right) h_2(\tau_2)\, d\tau_2\right] \\ &= \mathrm{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(X(t_1 - \tau_1) - \mu_X\right)\left(X(t_2 - \tau_2) - \mu_X\right) h_1(\tau_1)\, h_2(\tau_2)\ d\tau_1 d\tau_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\left(X(t_1 - \tau_1) - \mu_X\right)\left(X(t_2 - \tau_2) - \mu_X\right)\right] h_1(\tau_1)\, h_2(\tau_2)\ d\tau_1 d\tau_2. \end{aligned}$$

With the the autocovariance function of $X(t)$

$$C_X(\tau) = \mathrm{E}\left[\left(X(t+\tau) - \mu_X\right)\left(X(t) - \mu_X\right)\right]$$

we can further obtain

$$\begin{aligned}
&\operatorname{Cov}\left[Y\left(t_1\right),\ Z\left(t_2\right)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X\left(\tau - \tau_1 + \tau_2\right) h_1\left(\tau_1\right) h_2\left(\tau_2\right)\, d\tau_1 d\tau_2 \\
&= C_X\left(\tau\right) \star h_1\left(\tau\right) \star h_2\left(-\tau\right)
\end{aligned}$$

where $\tau = t_1 - t_2$.

(b) Since $Y(t)$ and $Z(t)$ are jointly Gaussian processes, the random variables $Y\left(t_1\right)$ and $Z\left(t_2\right)$ are statistically independent if they are uncorrelated. We know that $Y\left(t_1\right)$ and $Z\left(t_2\right)$ are uncorrelated if and only if their covariance is zero. If $H_1(f)$ and $H_2(f)$ are non-overlapping, we have $H_1(f)H_2(f) = 0$ and $H_1(f)H_2^*(f) = 0$. Then

$$F_{C_X}(f)H_1(f)H_2^*(f) = 0$$

where $F_{C_X}(f)$ is the Fourier transform of $C_X(\tau)$. By taking the inverse Fourier transform, from (a) we obtain

$$0 = C_X(\tau) \star h_1(\tau) \star h_2(-\tau) = \operatorname{Cov}\left[Y\left(t_1\right),\ Z\left(t_2\right)\right]$$

which implies that $Y\left(t_1\right)$ and $Z\left(t_2\right)$ are statistically independent.

4. (a) We have the noise equivalent bandwidth B given by

$$\begin{aligned}
B &= \frac{1}{H^2\left(0\right)} \int_0^{\infty} \left|H\left(f\right)\right|^2 df \\
&= \int_0^{\infty} \frac{df}{1 + \left(f/f_0\right)^{2n}} \\
&= f_0^{2n} \int_0^{\infty} \frac{df}{f^{2n} + f_0^{2n}} \\
&= f_0^{2n} \frac{\pi f_0^{-2n+1}}{2n \sin\left(\pi/2n\right)} \\
&= \frac{f_0 \pi}{2n \sin\left(\pi/2n\right)} \\
&= \frac{f_0}{\operatorname{sinc}\left(1/2n\right)}.
\end{aligned}$$

(b) As n approaches infinity, $\operatorname{sinc}\left(1/2n\right)$ approaches $\operatorname{sinc}\left(0\right)$, which is 1. Hence,

$$\lim_{n\to\infty} B = f_0.$$

5. (a) Let $R_W(\tau)$ denote the autocorrelation of the white noise, which is $(N_0/2)\delta(\tau)$; then

$$\begin{aligned} R_X(\tau) &= R_W(\tau) \star h(\tau) \star h(-\tau) \\ &= \frac{N_0}{2}\delta(\tau) \star h(\tau) \star h(-\tau) \\ &= \frac{N_0}{2} h(\tau) \star h(-\tau). \end{aligned}$$

So the condition that $h(t)$ must satisfy is

$$h(\tau) \star h(-\tau) = \frac{2}{N_0} R_X(\tau).$$

(b) By taking the Fourier transform on both sides of the equation $h(\tau) \star h(-\tau) = (2/N_0)R_X(\tau)$, we get

$$|H(f)|^2 = \frac{2}{N_0} S_X(f)$$

implying

$$|H(f)| = \sqrt{\frac{2S_X(f)}{N_0}}$$

which is the corresponding condition on $H(f)$.

6. (a) From the result

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_N \sin(2\pi f_c \tau)$$

we can get

$$S_{N_I}(f) = S_{N_Q}(f) = \frac{1}{2}\left[S_N(f - f_c) + S_N(f + f_c)\right] + \frac{1}{2j}\left[\hat{S}_N(f - f_c) - \hat{S}_N(f + f_c)\right]$$

by taking the Fourier transform. Since $\hat{S}_N(f) = S_N(f)(-j\,\mathrm{sgn}(f))$, we then obtain

$$\hat{S}_N(f - f_c) = \begin{cases} jS_N(f - f_c), & -B \le f \le B \\ -jS_N(f - f_c), & \text{elsewhere} \end{cases}$$

and

$$\hat{S}_N(f + f_c) = \begin{cases} -jS_N(f + f_c), & -B \le f \le B \\ jS_N(f + f_c), & \text{elsewhere.} \end{cases}$$

Combining the above results, we have

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

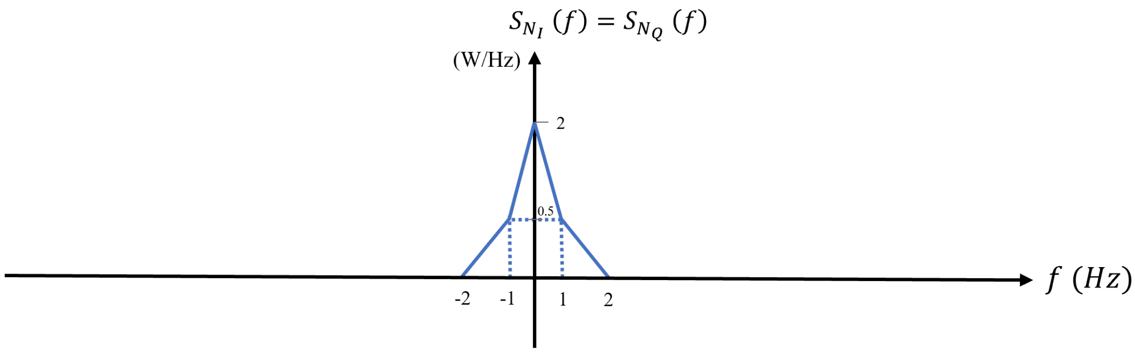

Figure 1: $S_{N_I}(f)$ and $S_{N_Q}(f)$ in Problem 7.(a).

(b) From the result

$$R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) = R_N(\tau)\sin(2\pi f_c \tau) - \hat{R}_N \cos(2\pi f_c \tau),$$

we can get

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \frac{1}{2j}\left[S_N(f - f_c) - S_N(f + f_c)\right] - \frac{1}{2}\left[\hat{S}_N(f - f_c) + \hat{S}_N(f + f_c)\right]$$

by taking the Fourier transform. By the results of $\hat{S}_N(f + f_c)$ and $\hat{S}_N(f + f_c)$ in (a), we then obtain

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \begin{cases} j\left[S_N(f + f_c) - S_N(f - f_c)\right], & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

7. (a) According to the result of Problem 6.(a), we have

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

The power spectral densities of the in-phase and quadrature components are then plotted in Fig. 1.

(b) According to the result of Problem 6.(b), we have

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \begin{cases} j\left[S_N(f + f_c) - S_N(f - f_c)\right], & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

The cross-spectral densities of the in-phase and quadrature components are plotted in Figs. 2 and 3.

8. (a) Since $n(t)$ is band-pass, we have

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$

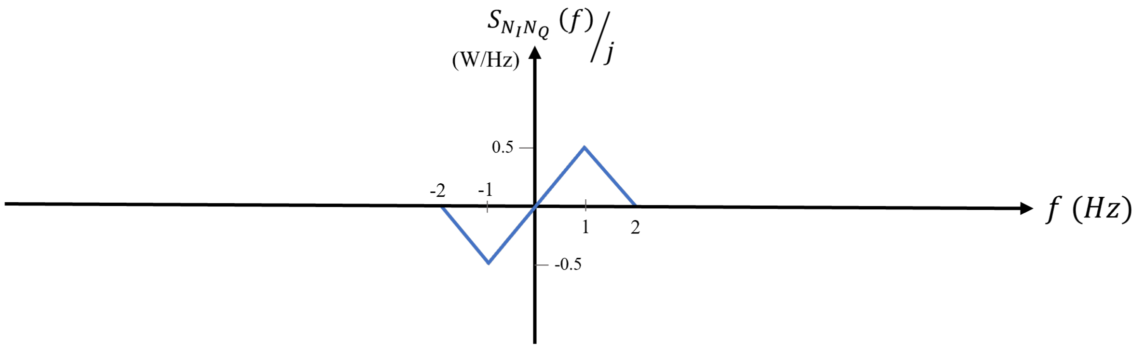

Figure 2: $S_{N_I N_Q}(f)$ in Problem 7.(b).

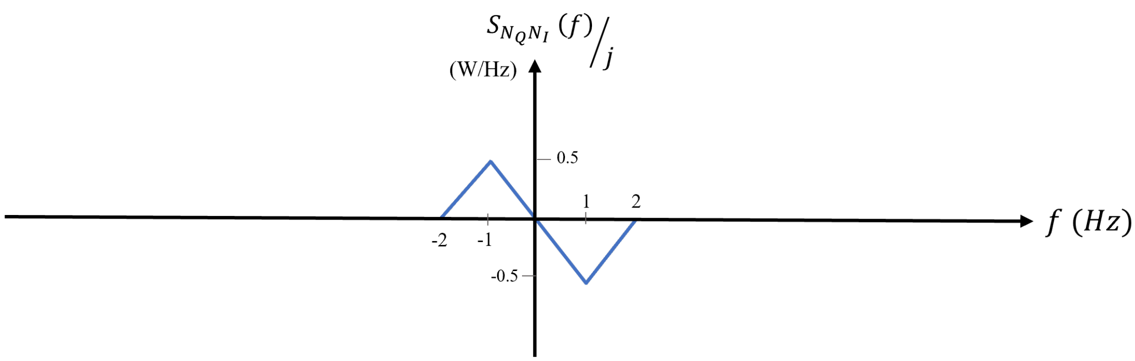

Figure 3: $S_{N_Q N_I}(f)$ in Problem 7.(b).

Then the envelope $r(t)$ is given by

$$\sqrt{n_I^2(t) + n_Q^2(t)}.$$

At time t_1, $n_I(t_1)$ and $n_Q(t_1)$ are jointly Gaussian random variables, both with zero mean and variance $2N_0B$, and are independent of each other since $S_N(f)$ is locally symmetric around $\pm f_c$. Thus, R follows a Rayleigh distribution with probability density function given by

$$f_R(r) = \frac{r}{2N_0B} \exp\left(-\frac{r^2}{4N_0B}\right).$$

(b) The mean of R is given by

$$\mathrm{E}[R] = \int_0^\infty r \frac{r}{2N_0B} \exp\left(-\frac{r^2}{4N_0B}\right) dr.$$

By integration by parts, we have

$$\begin{aligned}\mathrm{E}[R] &= -r \exp\left(-\frac{r^2}{4N_0B}\right)\Bigg|_0^\infty + \int_0^\infty \exp\left(-\frac{r^2}{4N_0B}\right) dr \\ &= \int_0^\infty \exp\left(-\frac{r^2}{4N_0B}\right) dr.\end{aligned}$$

Note that for a Gaussian random variable of zero mean and variance σ^2,

$$1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{x^2}{2\sigma^2}\right) dx.$$

Hence,

$$\mathrm{E}[R] = \int_0^\infty \exp\left(-\frac{r^2}{4N_0B}\right) dr = \frac{\sqrt{2N_0B}\sqrt{2\pi}}{2} = \sqrt{\pi N_0 B}.$$

The variance of R is given by

$$\begin{aligned}\mathrm{E}\left[R^2\right] - \mathrm{E}^2[R] &= \mathrm{E}\left[n_I^2(t_1) + n_Q^2(t_1)\right] - \pi N_0 B \\ &= 4N_0B - \pi N_0 B = (4-\pi) N_0 B.\end{aligned}$$