EE 3640 Communication Systems I — Chi-chao Chao
Spring 2023

Homework Assignment No. 4
Due 1:20pm, May 15, 2023

Reading: Haykin & Moher, Chapter 5.

Problems for Solution:

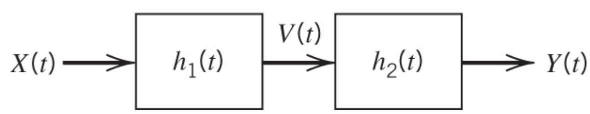
1. Let X and Y be independent Gaussian random variables, each with zero mean and unit variance. Define the random process

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t).$$

   (a) Let Z_1 and Z_2 be the random variables obtained by observing $Z(t)$ at times t_1 and t_2, respectively. Find the joint probability density function of Z_1 and Z_2. Assume that $\sin[2\pi(t_1 - t_2)] \neq 0$.

   (b) Is the process $Z(t)$ wide-sense stationary? Why?

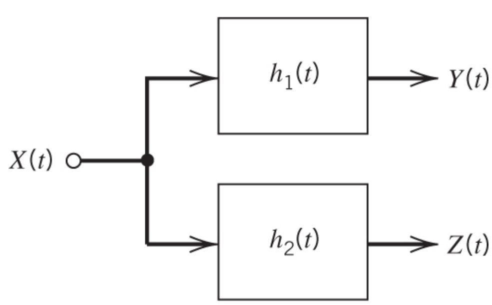
2. Consider two linear filters connected in cascade as shown below. Let $X(t)$ be a wide-sense stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.



   (a) Find the autocorrelation function of $Y(t)$.

   (b) Find the cross-correlation function $R_{VY}(\tau)$ of $V(t)$ and $Y(t)$.

3. A stationary Gaussian process $X(t)$ with mean μ_X and variance σ_X^2 is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$.



(a) Find the covariance of the random variables $Y(t_1)$ and $Z(t_2)$, for some times t_1 and t_2. Express your answer in terms of $h_1(t)$, $h_2(t)$, and the autocovariance function of $X(t)$ given by

$$C_X(\tau) = \mathrm{E}\left[(X(t+\tau) - \mu_X)(X(t) - \mu_X)\right].$$

(b) Let $H_1(f)$ and $H_2(f)$ be the corresponding transfer functions of $h_1(t)$ and $h_2(t)$, respectively. Show that $Y(t_1)$ and $Z(t_2)$ are statistically independent if $H_1(f)$ and $H_2(f)$ are non-overlapping, i.e., $H_1(f) = 0$ for all f such that $H_2(f) \neq 0$, and $H_2(f) = 0$ for all f such that $H_1(f) \neq 0$.

4. A white noise $w(t)$ of power spectral density $N_0/2$ is applied to a *Butterworth* low-pass filter of order n, whose amplitude response is given by

$$|H(f)| = \frac{1}{\left[1 + (f/f_0)^{2n}\right]^{1/2}}.$$

(a) Determine the noise equivalent bandwidth for this low-pass filter.

(b) What is the limiting value of the noise equivalent bandwidth as n approaches infinity?

(*Hint:* Note the definite integral

$$\int_0^\infty \frac{x^m}{x^n + a^n}\,dx = \frac{\pi a^{m-n+1}}{n \sin\left(\frac{m+1}{n}\pi\right)}, \qquad \text{for } 0 < m+1 < n.)$$

5. Let $X(t)$ be a wide-sense stationary process with zero mean, autocorrelation function $R_X(\tau)$, and power spectral density $S_X(f)$. We are required to find a linear filter with impulse response $h(t)$, such that the filter output is $X(t)$ when the input is white noise of zero mean and power spectral density $N_0/2$.

(a) Determine the condition that the impulse response $h(t)$ must satisfy in order to achieve this requirement.

(b) What is the corresponding condition on the transfer function $H(f)$ of the filter?

6. Consider a narrow-band noise $n(t)$ with autocorrelation function $R_N(\tau)$ and power spectral density $S_N(f)$. We have

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$

(a) Give a full derivation that the power spectral densities of the in-phase and quadrature components are given by

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \leq f \leq B \\ 0, & \text{elsewhere} \end{cases}$$

where it is assumed that $S_N(f)$ occupies the frequency interval $f_c - B \leq |f| \leq f_c + B$, and $f_c > B$.

(b) Give a full derivation that the cross-spectral densities of the in-phase and quadrature components are given by

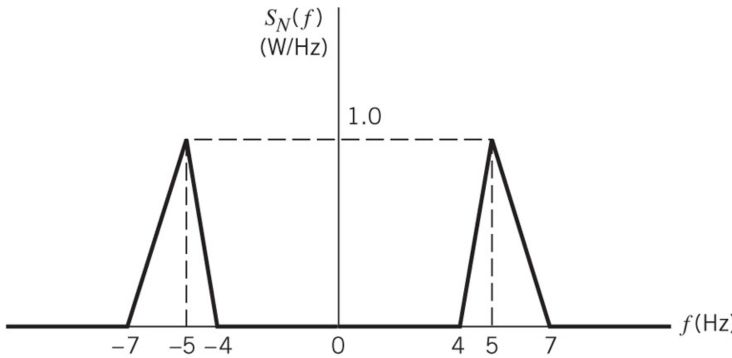
$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \begin{cases} j\left[S_N(f+f_c) - S_N(f-f_c)\right], & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

(*Hint:* You may use the results that

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_N(\tau)\sin(2\pi f_c \tau)$$

$$R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) = R_N(\tau)\sin(2\pi f_c \tau) - \hat{R}_N(\tau)\cos(2\pi f_c \tau).)$$

7. The power spectral density of a narrow-band noise $n(t)$ is shown below. The carrier frequency is 5 Hz.



(a) Plot the power spectral densities of the in-phase and quadrature components of $n(t)$.

(b) Plot their cross-spectral densities.

8. Consider a Gaussian noise $n(t)$ with zero mean and power spectral density $S_N(f)$ given by

$$S_N(f) = \begin{cases} N_0/2, & f_c - B \le f \le f_c + B \\ N_0/2, & -f_c - B \le f \le -f_c + B \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Let $r(t)$ be the envelope of $n(t)$ and R be the random variable obtained by observing $r(t)$ at time t_1. Find the probability density function of R.

(b) Find the mean and variance of R.

Homework Collaboration Policy: I allow and encourage discussion or collaboration on the homework. However, you are expected to write up your own solution and understand what you turn in. Late homework is subject to a penalty of 5% to 40% of your total points.