Solution to Homework Assignment No. 3

1. (a) We have

$$\begin{aligned} s(t) &= A_c \cos\left[2\pi f_c t + k_p m(t)\right] \\ &= A_c \cos\left[k_p m(t)\right] \cos\left(2\pi f_c t\right) - A_c \sin\left[k_p m(t)\right] \sin\left(2\pi f_c t\right). \end{aligned}$$

Thus the in-phase and quadrature components of $s(t)$ are

$$s_I(t) = A_c \cos\left[k_p m(t)\right]$$

and

$$s_Q(t) = A_c \sin\left[k_p m(t)\right]$$

respectively. The envelope and the phase of $s(t)$ are A_c and $k_p m(t)$, respectively.

(b) We have

$$\begin{aligned} s(t) =& A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right] \\ =& A_c \cos\left[2\pi k_f \int_0^t m(\tau)d\tau\right] \cos\left(2\pi f_c t\right) \\ & - A_c \sin\left[2\pi k_f \int_0^t m(\tau)d\tau\right] \sin\left(2\pi f_c t\right). \end{aligned}$$

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respectively. The envelope and the phase of $s(t)$ are A_c and $2\pi k_f \int_0^t m(\tau)d\tau$, respectively.

2. (a) We have the PM signal given by

$$\begin{aligned} s(t) =& A_c \cos\left[2\pi f_c t + k_p A_m \cos\left(2\pi f_m t\right)\right] \\ =& A_c \cos\left[k_p A_m \cos\left(2\pi f_m t\right)\right] \cos(2\pi f_c t) \\ & - A_c \sin\left[k_p A_m \cos(2\pi f_m t)\right] \sin(2\pi f_c t) \\ \approx& A_c \cos\left(2\pi f_c t\right) - A_c k_p A_m \cos\left(2\pi f_m t\right) \sin\left(2\pi f_c t\right) \quad \text{(since } \beta_p = k_p A_m \text{ is small)} \\ =& A_c \cos\left(2\pi f_c t\right) - \frac{\beta_p A_c}{2}\left[\sin\left(2\pi(f_c - f_m)t\right) + \sin\left(2\pi(f_c + f_m)t\right)\right] \\ =& s'(t). \end{aligned}$$

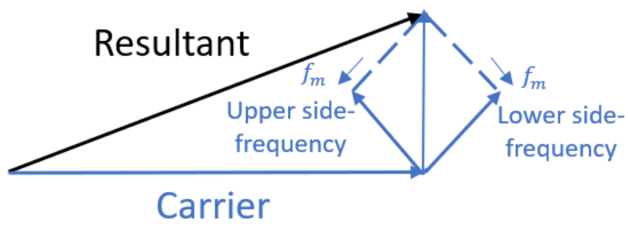

Figure 1: Pasor diagram for the PM signal in Problem 2.(b).

Thus the Fourier transform of $s'(t)$ is

$$\begin{aligned} S'(f) =& \frac{Ac}{2}\left[\delta\left(f-f_c\right)+\delta\left(f+f_c\right)\right] \\ &-\frac{\beta_p A_c}{4j}\left[\delta\left(f-f_c-f_m\right)-\delta\left(f+f_c+f_m\right)+\delta\left(f-f_c+f_m\right)-\delta\left(f+f_c-f_m\right)\right]. \end{aligned}$$

(b) The phasor diagram for this PM signal is shown in Figure 1. Comparing it with the narrow-band FM signal, we could see that they both keep the same amplitude and lead the carrier. However, the phase deviation of the PM signal increases as f_m becomes smaller, while in narrow-band FM the phase deviation increases as f_m becomes larger.

3. Since $\beta = 1$, wide-band FM is considered. Let $S(f)$ denote the Fourier transform of the FM signal $s(t)$. From class, we have

$$\begin{aligned} S(f) &= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\big[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\big] \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(1)\big[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\big]. \end{aligned}$$

Since the band-pass filter has mid-band frequency f_c and bandwidth $5f_m$, we only need to consider the frequency components of $S(f)$ in the range $f_c-(5/2)f_m$ to $f_c+(5/2)f_m$ and $-f_c-(5/2)f_m$ to $-f_c+(5/2)f_m$. Therefore, the amplitude spectrum of the filter output is

$$|S'(f)| = \frac{1}{2}\sum_{n=-2}^{2} |J_n(1)|\big[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\big].$$

4. (a) By Carson's rule, the transmission bandwidth of the FM signal is approximately given by

$$B_T \approx 2\Delta f\left(1+\frac{1}{\beta}\right)$$

where Δf is the frequency deviation and β is the modulation index. The frequency deviation is

$$\Delta f = k_f A_m = 10 \cdot 10^3 \cdot 10 = 100 \text{ kHz}$$

and the corresponding modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{10^5}{20 \cdot 10^3} = 5.$$

Therefore, we have

$$B_T \approx 2\Delta f \left(1 + \frac{1}{\beta}\right) = 2 \cdot 10^5 \cdot \left(1 + \frac{1}{5}\right) = 240 \text{ kHz}.$$

(b) The transmission bandwidth of the FM signal determined by considering only those side frequencies whose amplitudes exceed 1 percent of the unmodulated carrier amplitude is given by

$$B_T \approx 2n_{\max} f_m$$

where $2n_{\max}$ is number of significant side frequencies. From Table 4.1, for $\beta = 5$, $2n_{\max} = 16$. Therefore, we have

$$B_T \approx 2n_{\max} f_m = 16 \cdot 20 \cdot 10^3 = 320 \text{ kHz}.$$

(c) If the amplitude of the modulating wave is doubled, then $\Delta f = 200$ kHz and $\beta = 10$. Hence by Carson's rule, the transmission bandwidth is approximately

$$B_T \approx 2\Delta f \left(1 + \frac{1}{\beta}\right) = 2 \cdot 200 \cdot 10^3 \cdot \left(1 + \frac{1}{10}\right) = 440 \text{ kHz}.$$

From Table 4.1, for $\beta = 10$, $2n_{\max} = 28$. Therefore, the 1-percent bandwidth is

$$B_T \approx 2n_{\max} f_m = 28 \cdot 20 \cdot 10^3 = 560 \text{ kHz}.$$

5. The overall frequency multiplication ratio after two multipliers is

$$n = 3 \cdot 4 = 12.$$

Assume that the instantaneous frequency of the signal at the first multiplier input is

$$f_i(t) = f_c + \Delta f \cos(2\pi f_m t).$$

Then the instantaneous frequency of the signal at the second multiplier output is

$$f_i'(t) = n f_c + n\Delta f \cos(2\pi f_m t).$$

So the resulting frequency deviation is

$$n\Delta f = 12 \cdot 15 \text{ kHz} = 180 \text{ kHz},$$

and the modulation index is

$$\frac{n\Delta f}{f_m} = \frac{180 \text{ kHz}}{10 \text{ kHz}} = 18.$$

From class, the Fourier transform of the FM signal $s(t)$ is given by

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)\big[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\big].$$

Since the multipliers do not change f_m, the frequency separation at the second multiplier output is still $f_m = 10$ kHz.

6. The envelope detector iutput is

$$\begin{aligned} v(t) &= s(t) - s(t-T) \\ &= A_c \cos\left[2\pi f_c t + \phi(t)\right] - A_c \cos\left[2\pi f_c (t-T) + \phi(t-T)\right] \\ &= -2A_c \sin\left[\frac{2\pi f_c(2t-T) + \phi(t) + \phi(t-T)}{2}\right] \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \end{aligned}$$

where $\phi(t) = \beta \sin(2\pi f_m t)$. The phase difference is given by

$$\begin{aligned} \phi(t) - \phi(t-T) &= \beta \sin(2\pi f_m t) - \beta \sin\left[2\pi f_m (t-T)\right] \\ &= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t)\cos(2\pi f_m T) + \cos(2\pi f_m t)\sin(2\pi f_m T)] \\ &\approx \beta\left[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)\right] \\ &= 2\pi \Delta f T \cos(2\pi f_m t) \end{aligned}$$

where $\Delta f = \beta f_m$. Then, noting that $2\pi f_c T = \pi/2$, we have

$$\begin{aligned} \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] &= \sin\left[\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)\right] \\ &= \sin\left[\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)\right] \\ &= \frac{\sqrt{2}}{2} \cos\left[\pi \Delta f T \cos(2\pi f_m t)\right] + \frac{\sqrt{2}}{2} \sin\left[\pi \Delta f T \cos(2\pi f_m t)\right] \\ &\approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \pi \Delta f T \cos(2\pi f_m t) \end{aligned}$$

where we have used the fact that $\pi \Delta f T \ll 1$. We hence have

$$v(t) = -\sqrt{2} A_c \left[1 + \pi \Delta f T \cos(2\pi f_m t)\right] \sin\left[\pi f_c (2t - T) + \frac{\phi(t) + \phi(t-T)}{2}\right].$$

Therefore, the envelope-detector output is

$$a(t) = \sqrt{2} A_c [1 + \pi \Delta f T \cos(2\pi f_m t)].$$

7. From class, the FM signal is given by

$$\begin{aligned} s(t) &= A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right] \\ &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[2\pi (f_c + n f_m) t\right] \end{aligned}$$

where $\beta = k_f A_m / f_m$. Since the Hilbert transform of $\cos\left[2\pi (f_c + n f_m) t\right]$ is $\sin\left[2\pi (f_c + n f_m) t\right]$, by the linearity property of the Hilbert transform, we have

$$\begin{aligned} \hat{s}(t) &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \sin\left[2\pi (f_c + n f_m) t\right] \\ &= \operatorname{Im}\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c + n f_m)t}\right]. \end{aligned}$$

Using the fact that

$$\begin{aligned} s(t) &= \operatorname{Re}\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c + n f_m)t}\right] \\ &= A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right] \end{aligned}$$

we have

$$\hat{s}(t) = A_c \sin\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right].$$

8. First, $\hat{\phi}(t) = -\beta \cos(2\pi f_m t)$. Then,

$$s(t) = e^{\beta \cos(2\pi f_m t)} \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right].$$

The complex envelope is hence given by

$$\begin{aligned} \tilde{s}(t) &= e^{\beta \cos(2\pi f_m t) + j\beta \sin(2\pi f_m t)} \\ &= e^{\beta e^{j2\pi f_m t}} \\ &= \sum_{n=0}^{\infty} \frac{\left(\beta e^{j2\pi f_m t}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} e^{j2\pi n f_m t}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} s(t) &= \operatorname{Re}\left[\tilde{s}(t) e^{j2\pi f_c t}\right] \\ &= \operatorname{Re}\left[\sum_{n=0}^{\infty} \frac{\beta^n}{n!} e^{j2\pi(f_c + n f_m)t}\right] \\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos\left[2\pi (f_c + n f_m) t\right]. \end{aligned}$$

Note that $s(t)$ only contains discrete tones at f_c, $f_c + f_m$, $f_c + 2f_m$, $\ldots$, and $-f_c$, $-f_c - f_m$, $-f_c - 2f_m$, $\ldots$, leaving no frequency components in the interval $(-f_c, f_c)$.