Solution to Homework Assignment No. 2

1. (a) We have

$$\begin{aligned} s(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\ &= 20\cos(2000\pi t) + 5[\cos(2\pi(1000-100)t) + \cos(2\pi(1000+100)t)] \\ &= 20[1 + 0.5\cos(200\pi t)]\cos(2000\pi t) \\ &= A_c\,[1 + k_a m(t)]\cos(2\pi f_c t) \end{aligned}$$

where $A_c = 20$, $k_a m(t) = 0.5\cos(200\pi t)$, and $f_c = 1000$. Hence the percentage modulation is

$$\max_t |k_a m(t)| \times 100\% = \max_t |0.5\cos(200\pi t)| \times 100\% = 50\%.$$

(b) We have

$$\frac{\text{total sideband power}}{\text{total power}} = \frac{(5^2/2) + (5^2/2)}{(20^2/2) + (5^2/2) + (5^2/2)} = 1/9.$$

2. (a) We have

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where

$$v_1(t) = A_c\cos(2\pi f_c t) + m(t).$$

Then

$$\begin{aligned} v_2(t) &= a_1\left[A_c\cos(2\pi f_c t) + m(t)\right] + a_2\left[A_c\cos(2\pi f_c t) + m(t)\right]^2 \\ &= a_1 A_c\cos(2\pi f_c t) + a_1 m(t) + a_2 A_c^2\cos^2(2\pi f_c t) + 2a_2 A_c\cos(2\pi f_c t)\, m(t) + a_2 m^2(t) \\ &= [a_1 + 2a_2 m(t)]\, A_c\cos(2\pi f_c t) + \frac{1}{2}a_2 A_c^2\,[1 + \cos(4\pi f_c t)] + a_1 m(t) + a_2 m^2(t) \\ &= [a_1 + 2a_2 m(t)]\, A_c\cos(2\pi f_c t) + \frac{1}{2}a_2 A_c^2\cos(4\pi f_c t) + \frac{1}{2}a_2 A_c^2 + a_1 m(t) + a_2 m^2(t). \end{aligned}$$

(b) Suppose $m(t)$ is bandlimited to $-W \le f \le W$. Then $m^2(t)$ is bandlimited to $-2W \le f \le 2W$. To remove the unwanted terms in $v_2(t)$, we can design the tuned circuit as a band-pass filter with midband frequency f_c and bandwidth $2W$, where the requirement $f_c - W > 2W$, i.e., $f_c > 3W$, should be satisfied.

(c) Let the output of the tuned circuit be $v_2'(t)$. Then, by (a) and (b), $v_2'(t)$ is given by

$$\begin{aligned} v_2'(t) &= [a_1 + 2a_2 m(t)]\, A_c \cos(2\pi f_c t) \\ &= A_c a_1 \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos(2\pi f_c t). \end{aligned}$$

The amplitude sensitivity of this AM signal is hence

$$k_a = \frac{2a_2}{a_1}.$$

3. (a) Since

$$m(t) = 2\cos(2000\pi t) + \cos(6000\pi t)$$

we have the Fourier transform of $m(t)$ given by

$$M(f) = \delta(f-1000) + \delta(f+1000) + \frac{1}{2}\delta(f-3000) + \frac{1}{2}\delta(f+3000).$$

Now

$$s(t) = 100 m(t) \cos(2 \cdot 10^6 \pi t)$$

and hence the Fourier transform of $s(t)$ is

$$\begin{aligned} S(f) &= 50M\left(f - 10^6\right) + 50M\left(f + 10^6\right) \\ &= 50\delta\left(f - 10^6 - 1000\right) + 50\delta\left(f - 10^6 + 1000\right) \\ &\quad + 25\delta\left(f - 10^6 - 3000\right) + 25\delta\left(f - 10^6 + 3000\right) \\ &\quad + 50\delta\left(f + 10^6 - 1000\right) + 50\delta\left(f + 10^6 + 1000\right) \\ &\quad + 25\delta\left(f + 10^6 - 3000\right) + 25\delta\left(f + 10^6 + 3000\right). \end{aligned}$$

(b) The average power of $s(t)$ is given by

$$\frac{100^2}{2} + \frac{50^2}{2} + \frac{100^2}{2} + \frac{50^2}{2} = 100^2 + 50^2 = 12500.$$

4. The output of the AM modulator at the upper path is

$$s_1(t) = A_c\left[1 + k_a m(t)\right] \cos(2\pi f_c t)$$

where k_a is the amplitude sensitivity. Similarly, the output of the AM modulator at the lower path is

$$s_2(t) = A_c\left[1 - k_a m(t)\right] \cos(2\pi f_c t)$$

since the two modulators have the same amplitude sensitivity. Then

$$s(t) = s_1(t) - s_2(t) \quad = 2A_c k_a m(t) \cos(2\pi f_c t)$$

which represents a DSB-SC modulated signal.

5. The multiplexed signal is given by

$$s(t) = A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \sin(2\pi f_c t).$$

If the local oscillator in the reciever has a phase error ϕ, then the two carrier waves for demultiplexing are $2\cos(2\pi f_c t + \phi)$ and $2\sin(2\pi f_c t + \phi)$. The output of the top product modulator in the receiver is hence

$$\begin{aligned} 2s(t)\cos(2\pi f_c t + \phi) = A_c m_1(t) &[\cos\phi + \cos(4\pi f_c t + \phi)] \\ - A_c m_2(t) &[\sin\phi - \sin(4\pi f_c t + \phi)]. \end{aligned}$$

After the top low-pass filter, the output is

$$A_c m_1(t) \cos\phi - A_c m_2(t) \sin\phi.$$

Similarly, the output of the bottom product modulator in the receiver is

$$\begin{aligned} 2s(t)\sin(2\pi f_c t + \phi) = A_c m_1(t) &[\sin\phi + \sin(4\pi f_c t + \phi)] \\ + A_c m_2(t) &[\cos\phi - \cos(4\pi f_c t + \phi)] \end{aligned}$$

and then the low-pass filter output is

$$A_c m_1(t) \sin\phi + A_c m_2(t) \cos\phi.$$

It can be seen that there are cross-talks between the two demodulated signals at the receiver outputs.

6. (a) Since

$$m(t) = 2\cos(2\pi f_m t) + \cos(4\pi f_m t)$$

by the result of Problem 5.(a) of Homework No. 1

$$\hat{m}(t) = 2\sin(2\pi f_m t) + \sin(4\pi f_m t).$$

Then

$$\begin{aligned} s(t) &= 2m(t)\cos(2\pi f_c t) - 2\hat{m}(t)\sin(2\pi f_c t) \\ &= 2[2\cos(2\pi f_m t) + \cos(4\pi f_m t)]\cos(2\pi f_c t) \\ &\quad - 2[2\sin(2\pi f_m t) + \sin(4\pi f_m t)]\sin(2\pi f_c t) \\ &= 4[\cos(2\pi f_m t)\cos(2\pi f_c t) - \sin(2\pi f_m t)\sin(2\pi f_c t)] \\ &\quad + 2[\cos(4\pi f_m t)\cos(2\pi f_c t) - \sin(4\pi f_m t)\sin(2\pi f_c t)] \\ &= 4\cos[2\pi(f_c + f_m)t] + 2\cos[2\pi(f_c + 2f_m)t]. \end{aligned}$$

(b) From (a) we have

$$\begin{aligned} S(f) = 2&[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)] \\ &+ [\delta(f - f_c - 2f_m) + \delta(f + f_c + 2f_m)]. \end{aligned}$$

Since $S(f)$ contains components at $f_c + f_m$ and $f_c + 2f_m$ (and $-f_c - f_m$ and $-f_c - 2f_m$), $s(t)$ is an upper-sideband SSB signal.

7. (a) The VSB signal is given by

$$\begin{aligned} s(t) &= \frac{1}{2}aA_mA_c\cos\left[2\pi\left(f_c+f_m\right)t\right] + \frac{1}{2}A_mA_c(1-a)\cos\left[2\pi\left(f_c-f_m\right)t\right] \\ &= \frac{1}{2}aA_mA_c\left[\cos\left(2\pi f_ct\right)\cos\left(2\pi f_mt\right) - \sin\left(2\pi f_ct\right)\sin\left(2\pi f_mt\right)\right] \\ &\quad + \frac{1}{2}A_mA_c(1-a)\left[\cos\left(2\pi f_ct\right)\cos\left(2\pi f_mt\right) + \sin\left(2\pi f_ct\right)\sin\left(2\pi f_mt\right)\right] \\ &= \frac{A_mA_c}{2}\cos\left(2\pi f_mt\right)\cos\left(2\pi f_ct\right) + \frac{A_mA_c}{2}(1-2a)\sin\left(2\pi f_mt\right)\sin\left(2\pi f_ct\right) \\ &= s_I(t)\cos(2\pi f_ct) - s_Q(t)\sin(2\pi f_ct) \end{aligned}$$

where

$$\begin{aligned} s_I(t) &= \frac{A_mA_c}{2}\cos\left(2\pi f_mt\right) \\ s_Q(t) &= -\frac{A_mA_c}{2}(1-2a)\sin\left(2\pi f_mt\right). \end{aligned}$$

(b) From (a), we have

$$\begin{aligned} s(t) + A_c\cos(2\pi f_ct) = A_c\left[1+\frac{A_m}{2}\cos\left(2\pi f_mt\right)\right]\cos(2\pi f_ct) \\ + \frac{A_mA_c}{2}(1-2a)\sin\left(2\pi f_mt\right)\sin\left(2\pi f_ct\right). \end{aligned}$$

The output of the envelope detector is then given by

$$\begin{aligned} y\left(t\right) &= \sqrt{{A_c}^2\left[1+\frac{A_m}{2}\cos\left(2\pi f_mt\right)\right]^2 + \left[\frac{A_mA_c}{2}(1-2a)\sin\left(2\pi f_mt\right)\right]^2} \\ &= A_c\left[1+\frac{A_m}{2}\cos\left(2\pi f_mt\right)\right]\sqrt{1+\left[\frac{(A_m/2)(1-2a)\sin\left(2\pi f_mt\right)}{1+(A_m/2)\cos\left(2\pi f_mt\right)}\right]^2} \\ &= A_c[1+(1/2)m(t)]\cdot d(t) \end{aligned}$$

where the distortion

$$d(t) = \sqrt{1+\left[\frac{(A_m/2)(1-2a)\sin\left(2\pi f_mt\right)}{1+(A_m/2)\cos\left(2\pi f_mt\right)}\right]^2}.$$

(c) Since $0 \le a < 1$, the distortion $d(t)$ reaches the worst condition when $a = 0$.

8. Let f_c be the carrier frequency of the input AM wave. We have

$$0.535 \le f_c \le 1.605.$$

After mixing, the signal is translated to a frequency band centered at a fixed IF frequency 0.455 MHz, and hence

$$f_c - f_l = 0.455$$

which implies

$$f_l = f_c - 0.455.$$

When $f_c = 0.535$ MHz, we have $f_l = 0.08$ MHz; when $f_c = 1.605$ MHz, $f_l = 1.15$ MHz. Therefore, the required range of tuning of the local oscillator is 0.08 MHz to 1.15 MHz.