Solution to Homework Assignment No. 1

1. (a) Since

$$\cos(2\pi f_c t) = \frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right)$$

and

$$e^{j2\pi f_c t} g(t) \rightleftharpoons G(f - f_c)$$

we have

$$m(t)\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right].$$

(b) Similarly, since

$$\sin(2\pi f_c t) = \frac{1}{2j}\left(e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right)$$

and

$$e^{j2\pi f_c t} g(t) \rightleftharpoons G(f - f_c)$$

we have

$$m(t)\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}\left[M(f - f_c) - M(f + f_c)\right].$$

2. (a) First, we have

$$\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \\
&= \int_{0^+}^{\infty} e^{-(a+j2\pi f)t} dt - \int_{-\infty}^{0^-} e^{(a-j2\pi f)t} dt \\
&= \frac{1}{-(a+j2\pi f)} e^{-(a+j2\pi f)t}\Big|_{0^+}^{\infty} - \frac{1}{(a-j2\pi f)} e^{(a-j2\pi f)t}\Big|_{-\infty}^{0^-} \\
&= \frac{1}{(a+j2\pi f)} - \frac{1}{(a-j2\pi f)} \\
&= \frac{-j4\pi f}{a^2 + (2\pi f)^2}.
\end{aligned}$$

As $\mathrm{sgn}(t) = \lim_{a\to 0} g(t)$, we have the Fourier transform of $\mathrm{sgn}(t)$ given by

$$\lim_{a\to 0} G(f) = \lim_{a\to 0} \frac{-j4\pi f}{a^2 + 4\pi^2 f^2} = \frac{1}{j\pi f}.$$

(b) Since

$$u(t) = (1/2)[\mathrm{sgn}(t) + 1]$$

we have the Fourier transform of $u(t)$ given by

$$\frac{1}{2} F[\mathrm{sgn}(t)] + \frac{1}{2} F[1] = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f).$$

(c) We have

$$\begin{aligned}\int_{-\infty}^{t} g(\tau)d\tau &= \int_{0}^{\infty} g(t-\lambda)d\lambda \qquad (\text{let } \lambda = t-\tau)\\ &= \int_{-\infty}^{\infty} g(t-\lambda)u(\lambda)d\lambda \\ &= g(t)\star u(t).\end{aligned}$$

Then, from (b), the corresponding Fourier transform is

$$G(f)\cdot\left(\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)\right) = \frac{1}{j2\pi f}G(f)+\frac{1}{2}G(0)\delta(f).$$

3. (a) Let $g'(t) = e^{-t}u(t)$ and then $g(t) = g'(t)\cos(2\pi f_c t)$. Thus their Fourier transforms are related by

$$G(f) = \frac{1}{2}\left[G'(f-f_c)+G'(f+f_c)\right].$$

We have

$$\begin{aligned}G'(f) &= \int_{-\infty}^{\infty} e^{-t}u(t)\,e^{-j2\pi f t}dt\\ &= \int_{0}^{\infty} e^{-t}e^{-j2\pi f t}dt\\ &= \int_{0}^{\infty} e^{-(1+j2\pi f)t}dt\\ &= \frac{1}{1+j2\pi f}.\end{aligned}$$

Hence

$$\begin{aligned}G(f) &= \frac{1}{2}\left[\frac{1}{1+j2\pi(f-f_c)}+\frac{1}{1+j2\pi(f+f_c)}\right]\\ &= \frac{1+j2\pi(f+f_c)+1+j2\pi(f-f_c)}{2\left[(1+j2\pi f)^2-(j2\pi f_c)^2\right]}\\ &= \frac{1+j2\pi f}{1+4\pi^2 {f_c}^2-4\pi^2 f^2+j4\pi f}.\end{aligned}$$

(b) Ignoring the discontinuity, we have

$$G(f) = 3u(f).$$

Since from the result of Problem 2.(b),

$$u(t) \rightleftharpoons \frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$$

by the duality property the inverse Fourier transform of $G(f)$ is given by

$$\begin{aligned}g(t) &= \frac{3}{j2\pi(-t)}+\frac{3}{2}\delta(-t)\\ &= \frac{3}{2}\delta(t)+j\frac{3}{2\pi t}.\end{aligned}$$

4. (a) The Fourier series of $g_{T_0}(t)$ is given by

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

where $f_0 = 1/T_0$ and

$$\begin{aligned} c_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-j2\pi n f_0 t} dt \\ &= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi n f_0 t} dt \\ &= \frac{1}{\pi n}\cdot \sin\left(\frac{\pi n}{2}\right) \\ &= \begin{cases} 1/2, & n = 0 \\ 0, & n = 2m \\ \frac{(-1)^{m-1}}{(2m-1)\pi}, & n = 2m-1. \end{cases} \end{aligned}$$

It is clear that $c_{-n} = c_n$. We have

$$g_{T_0}(t) = \frac{1}{2} + \sum_{m=-\infty}^{\infty} \frac{(-1)^{m-1}}{(2m-1)\pi} e^{j2\pi(2m-1)f_0 t}.$$

(b) We know that $H(f)$ is an ideal band-pass filter with mid-band frequency f_0 and bandwidth f_0. The output is then given by

$$\begin{aligned} y(t) &= c_1 \cdot e^{j2\pi f_0 t} H(f_0) + c_{-1}\cdot e^{-j2\pi f_0 t} H(-f_0) \\ &= \frac{1}{\pi} e^{j2\pi f_0 t} + \frac{1}{\pi} e^{-j2\pi f_0 t} \\ &= \frac{2}{\pi}\cos(2\pi f_0 t). \end{aligned}$$

5. (a) Let

$$g(t) = A\cos(2\pi f_c t + \theta) = A\cos\theta\cos(2\pi f_c t) - A\sin\theta\sin(2\pi f_c t).$$

We have

$$G(f) = F[g(t)] = \frac{A}{2}\cos\theta\left[\delta(f-f_c) + \delta(f+f_c)\right] - \frac{A}{2j}\sin\theta\left[\delta(f-f_c) - \delta(f+f_c)\right].$$

For the Hilbert transform, we obtain

$$\begin{aligned} \hat{G}(f) &= G(f)\cdot -j\,\mathrm{sgn}(f) \\ &= \frac{A}{2}\sin\theta\left[\delta(f-f_c)+\delta(f+f_c)\right] + \frac{A}{2j}\cos\theta\left[\delta(f-f_c) - \delta(f+f_c)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{g}(t) &= F^{-1}\left[\hat{G}(f)\right] = A\left[\sin\theta\cos(2\pi f_c t) + \cos\theta\sin(2\pi f_c t)\right] \\ &= A\sin(2\pi f_c t + \theta). \end{aligned}$$

(b) Let $y(t) = m(t)\cos(2\pi f_c t)$. According to the result of Problem 1.(a), we have

$$Y(f) = F[y(t)] = \frac{1}{2}[M(f - f_c) + M(f + f_c)].$$

For the Hilbert transform, we obtain

$$\begin{aligned}\hat{Y}(f) &= Y(f) \cdot -j\,\mathrm{sgn}(f)\\ &= \frac{1}{2j}[M(f - f_c) - M(f + f_c)].\end{aligned}$$

Therefore,

$$\hat{y}(t) = F^{-1}\left[\hat{Y}(f)\right] = m(t)\sin(2\pi f_c t).$$

6. (a) We have

$$\begin{aligned}g_I(t) &= \mathrm{Re}[\tilde{g}(t)]\\ &= \mathrm{Re}[g_+(t)e^{-j2\pi f_c t}]\\ &= \mathrm{Re}[\big(g(t) + j\hat{g}(t)\big)e^{-j2\pi f_c t}]\\ &= g(t)\cos(2\pi f_c t) + \hat{g}(t)\sin(2\pi f_c t).\end{aligned}$$

Similarly,

$$\begin{aligned}g_Q(t) &= \mathrm{Im}[\tilde{g}(t)]\\ &= \mathrm{Im}[\big(g(t) + j\hat{g}(t)\big)e^{-j2\pi f_c t}]\\ &= -g(t)\sin(2\pi f_c t) + \hat{g}(t)\cos(2\pi f_c t).\end{aligned}$$

(b) From (a), we have

$$g_I(t) = \frac{1}{2}[g(t)e^{j2\pi f_c t} + g(t)e^{-j2\pi f_c t}] + \frac{1}{2j}[\hat{g}(t)e^{j2\pi f_c t} - \hat{g}(t)e^{-j2\pi f_c t}].$$

Then

$$G_I(f) = \frac{1}{2}[G(f - f_c) + G(f + f_c)] + \frac{1}{2j}[\hat{G}(f - f_c) - \hat{G}(f + f_c)].$$

Since $\hat{G}(f) = G(f)\big(-j\mathrm{sgn}(f)\big)$, we obtain

$$\hat{G}(f - f_c) = \begin{cases} jG(f - f_c)\ , & -W \le f \le W\\ -jG(f - f_c)\ , & \text{elsewhere}\end{cases}$$

and

$$\hat{G}(f + f_c) = \begin{cases} -jG(f + f_c)\ , & -W \le f \le W\\ jG(f + f_c)\ , & \text{elsewhere.}\end{cases}$$

Combining together, we have

$$G_I(f) = \begin{cases} G(f - f_c) + G(f + f_c)\ , & -W \le f \le W\\ 0\ , & \text{elsewhere.}\end{cases}$$

(c) From (a), we have

$$g_Q(t) = \frac{1}{2j}[g(t)e^{-j2\pi f_c t} - g(t)e^{j2\pi f_c t}] + \frac{1}{2}[\hat{g}(t)e^{-j2\pi f_c t} + \hat{g}(t)e^{j2\pi f_c t}].$$

Then

$$G_Q(f) = \frac{1}{2j}[G(f+f_c) - G(f-f_c)] + \frac{1}{2}[\hat{G}(f+f_c) + \hat{G}(f-f_c)].$$

Similarly, we have

$$G_Q(f) = \begin{cases} j[G(f-f_c) - G(f+f_c)] \ , \ -W \leq f \leq W \\ 0 \ , \text{ elsewhere.} \end{cases}$$

7. We have

$$\begin{aligned} h(t) &= x(T-t) \\ &= \begin{cases} A\cos\left(2\pi f_c(T-t)\right), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases} \\ &= \begin{cases} A\cos\left(2\pi f_c t\right), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases} \\ &= x(t) \end{aligned}$$

since f_c is a large integer multiple of $1/T$. We obtain

$$\tilde{x}(t) = \tilde{h}(t) = A\,\mathrm{rect}((t-(T/2))/T).$$

Hence the complex envelope of the output

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{2}\left(\tilde{x}(t) \star \tilde{h}(t)\right) \\ &= \frac{A^2T}{2}\Lambda\left(\frac{t-T}{T}\right) \end{aligned}$$

where

$$\Lambda(t) = \begin{cases} 1-|t|, & |t| < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Therefore, the output

$$y(t) = \frac{A^2T}{2}\Lambda\left(\frac{t-T}{T}\right)\cos\left(2\pi f_c t\right).$$

8. Since

$$x(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

we have the complex envelope of $x(t)$ given by

$$\tilde{x}(t) = A_c e^{j2\pi k_f \int_0^t m(\tau)d\tau}.$$

For the low-pass equivalent model,

$$\tilde{H}(f) = \begin{cases} j4\pi a\left(f + \frac{BT}{2}\right), & \frac{BT}{2} \le f \le \frac{BT}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Then the output

$$\tilde{Y}(f) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f) = j2\pi a\left(f + \frac{B_T}{2}\right)\tilde{X}(f)$$

for $-B_T/2 \le f \le B_T/2$. Hence

$$\tilde{y}(t) = a\frac{d\tilde{x}(t)}{dt} + j\pi a B_T \tilde{x}(t).$$

Since $\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$, we have

$$\tilde{y}(t) = j\pi B_T a A_c \left[1 + \frac{2k_f m(t)}{B_T}\right] e^{j2\pi k_f \int_0^t m(\tau)d\tau}.$$

Therefore,

$$y(t) = \pi B_T a A_c \left[1 + \frac{2k_f m(t)}{B_T}\right] \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau + \frac{\pi}{2}\right).$$