EE 3640 Communication Systems I — Chi-chao Chao
Spring 2023

Homework Assignment No. 1
Due 1:20pm, March 9, 2023

Reading: Haykin & Moher, Chapters 1 and 2 (except Sections 2.13 and 2.14).

Problems for Solution:

1. (a) Show that

$$m(t)\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}\left[M(f-f_c)+M(f+f_c)\right]$$

where $M(f)$ is the Fourier transform of $m(t)$.

(b) Show that

$$m(t)\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}\left[M(f-f_c)-M(f+f_c)\right].$$

2. (a) The signum function $\operatorname{sgn}(t)$ does not satisfy the Dirichlet's conditions, and therefore, strictly speaking, it may not have a Fourier transform. However, we may define a Fourier transform for $\operatorname{sgn}(t)$ by viewing it as the limiting form of

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases}$$

as the parameter $a \to 0$. First show that the Fourier transform of $g(t)$ is given by

$$G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2}.$$

Then find the Fourier transform for $\operatorname{sgn}(t)$ by $\lim_{a\to 0} G(f)$ and show that

$$\operatorname{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}.$$

(b) The *unit step function* $u(t)$ is defined by

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0. \end{cases}$$

Then we have $u(t) = (1/2)[\operatorname{sgn}(t) + 1]$. Find the Fourier transform of $u(t)$ by using the result of (a).

(c) First show that

$$\int_{-\infty}^{t} g(\tau)\, d\tau = g(t) \star u(t)$$

where $\star$ is the convolution operation. Then show that

$$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{1}{2} G(0)\delta(f)$$

where $G(f)$ is the Frourier transform of $g(t)$.

3. (a) Find the Fourier transform of

$$g(t) = \exp(-t)\cos(2\pi f_c t) u(t)$$

where $u(t)$ is the unit step function.

(b) Find the inverse Fourier transform of

$$G(f) = \begin{cases} 3, & f > 0 \\ 1/e, & f = 0 \\ 0, & f < 0. \end{cases}$$

4. (a) Find the Fourier series expansion for the periodic signal $g_{T_0}(t)$ given by

$$g_{T_0}(t) = \begin{cases} 1, & |t| \leq T_0/4 \\ 0, & T_0/4 < |t| \leq T_0/2 \end{cases}$$

$$g_{T_0}(t + T_0) = g_{T_0}(t).$$

(b) Suppose the signal $g_{T_0}(t)$ in (a) is fed to a filter with transfer function

$$H(f) = \begin{cases} \mathrm{rect}\left(\frac{f - f_0}{f_0}\right), & f > 0 \\ \mathrm{rect}\left(\frac{f + f_0}{f_0}\right), & f < 0 \end{cases}$$

where $f_0 = 1/T_0$. Find the corresponding output.

5. (a) Find the Hilbert transform of $A\cos(2\pi f_c t + \theta)$.

(b) Find the Hilbert transform of $m(t)\cos(2\pi f_c t)$, where $m(t)$ is a low-pass signal whose Fourier transform $M(f)$ vanishes for $|f| > W$ and $f_c > W$.

6. Consider a band-pass signal $g(t)$ whose spectrum is limited to the interval $f_c - W \leq |f| \leq f_c + W$, where $W < f_c$. We can represent $g(t)$ in the canonical form:

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t).$$

(a) Show that

$$g_I(t) = g(t)\cos(2\pi f_c t) + \hat{g}(t)\sin(2\pi f_c t)$$

and

$$g_Q(t) = -g(t)\sin(2\pi f_c t) + \hat{g}(t)\cos(2\pi f_c t)$$

where $\hat{g}(t)$ is the Hilbert transform of $g(t)$.

(b) Let $G_I(f)$, $G_Q(f)$, and $G(f)$ be the Fourier transforms of $g_I(t)$, $g_Q(t)$, and $g(t)$, respectively. Show that

$$G_I(f) = \begin{cases} G(f-f_c) + G(f+f_c), & -W \le f \le W \\ 0, & \text{elsewhere.} \end{cases}$$

(c) Show that

$$G_Q(f) = \begin{cases} j\left[G(f-f_c) - G(f+f_c)\right], & -W \le f \le W \\ 0, & \text{elsewhere.} \end{cases}$$

7. The rectangular radio frequency pulse

$$x(t) = \begin{cases} A\cos(2\pi f_c t), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

is applied to a filter with impulse response

$$h(t) = x(T-t).$$

Assume that the frequency f_c equals a large integer multiple of $1/T$. Determine the output of the filter. (*Hint:* Consider the equivalent low-pass model.)

8. Consider a signal

$$x(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right]$$

whose spectrum vanishes outside the frequency interval $f_c - B_T/2 \le |f| \le f_c + B_T/2$. Suppose $x(t)$ is fed to a linear time-invariant system with transfer function

$$H(f) = \begin{cases} j2\pi a\left(f - f_c + \frac{B_T}{2}\right), & f_c - \frac{B_T}{2} \le f \le f_c + \frac{B_T}{2} \\ j2\pi a\left(f + f_c - \frac{B_T}{2}\right), & -f_c - \frac{B_T}{2} \le f \le -f_c + \frac{B_T}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Please use the equivalent low-pass model to find the output $y(t)$. (*Hint:* Recall $\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$.)

Homework Collaboration Policy: I allow and encourage discussion or collaboration on the homework. However, you are expected to write up your own solution and understand what you turn in. Late homework is subject to a penalty of 5% to 40% of your total points.