EE 3640 Communication Systems I
Spring 2023

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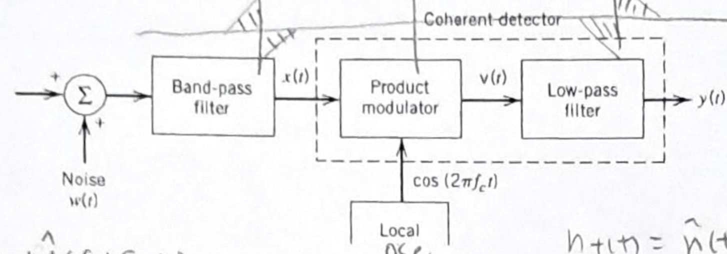
Final Examination
7:00pm to 10:00pm, June 16, 2023

Problems for Solution:

1. (10%) The random process $Y(t)$ is defined as
$$Y(t) = X(t) + X(t-T)$$
where $X(t)$ is a wide-sense stationary random process with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(f)$.
   (a) (5%) Show that the autocorrelation function of $Y(t)$ is given by
$$R_Y(\tau) = 2R_X(\tau) + R_X(\tau+T) + R_X(\tau-T).$$
   (b) (5%) Show that the power spectral density of $Y(t)$ is given by
$$S_Y(f) = 4S_X(f)\cos^2(\pi f T).$$

2. (25%) An ideal finite-time integrator is characterized by the input-output relationship:
$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\alpha)\,d\alpha$$
where $X(t)$ is the input and $Y(t)$ is the output. Suppose $X(t)$ is a stationary Gaussian process with mean μ_X and autocorrelation function $R_X(\tau) = (N_0/2)\delta(\tau)$.
   (a) (5%) Find the impulse response of this integrator.
   (b) (5%) Find the mean of $Y(t)$.
   (c) (5%) Find the power spectral density of $Y(t)$.
   (d) (5%) Find the autocorrelation function of $Y(t)$.
   (e) (5%) Suppose $\bar{Y}$ is the value of $Y(t)$ sampled at time $t = T$. Find the probability density function of $\bar{Y}$.

3. (25%) Consider the SSB modulated signal
$$s(t) = \frac{1}{2}m(t)\cos(2\pi f_c t) - \frac{1}{2}\hat{m}(t)\sin(2\pi f_c t)$$
where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. The message signal $m(t)$ has power spectral density $S_M(f)$ and average power P given by
$$P = \int_{-W}^{W} S_M(f)\,df$$
where W is the message bandwidth. The model of the receiver using coherent detection (with perfect synchronism) is shown below, where the noise $w(t)$ is white of zero mean and power spectral density $N_0/2$, and the band-pass filter is assumed to be ideal of which of the bandwidth is equal to the transmission bandwidth of the SSB modulated signal.



(a) (5%) Find the average power of $s(t)$ and the channel signal-to-noise ratio.

(b) (5%) Let $x(t) = s(t) + n(t)$ and $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$. Plot the power spectral density of $n_I(t)$.

(c) (5%) Give an expression for the demodulator output $y(t)$.

(d) (5%) Find the output signal-to-noise ratio and the figure of merit for this system.

(e) (5%) Could the threshold effect happen in this receiver? Why?

4. (a) (5%) Find the Nyquist rate for the signal $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$.

(b) (5%) A narrow-band signal has a bandwidth of 10 kHz centered on a carrier frequency of 100 kHz. It is proposed to represent this signal in discrete-time form by sampling its in-phase and quadrature components individually. What is the minimum sampling rate that can be used for this representation?

5. (10%) Consider the data stream $\{a_n\}_{n=1}^{10} = 1111011110$.

(a) (6%) Sketch the waveform for $\{a_n\}_{n=1}^{10}$ with each of the following line codes: unipolar nonreturn-to-zero, unipolar return-to-zero, bipolar return-to-zero.

(b) (4%) If differential encoding is used:

$$b_n = b_{n-1} \oplus a_n, \quad \text{for } n \geq 1$$

where $\oplus$ is the XOR operation and $b_0 = 1$, sketch the waveform for $\{b_n\}_{n=1}^{10}$ with each of the following line codes: polar nonreturn-to-zero, Manchester code.

6. (20%) An analog signal is sampled, quantized, and encoded into a binary PCM wave. The number of (quantization) representation levels used is 256. The PCM wave is transmitted over a baseband channel of bandwidth 24 kHz using a binary (two-level) PAM system with raised-cosine spectrum of roll-off factor $\alpha = 0.5$.

(a) (5%) Find the bit rate at which information is transmitted through the channel.

(b) (10%) Find the rate at which the analog signal is sampled. What is the maximum bandwidth of the analog signal for this scheme to work properly?

(c) (5%) If the PCM wave is transmitted using a *quaternary* (four-level) PAM system with raised-cosine spectrum of roll-off factor $\alpha = 0.5$ under the same channel bandwidth, find the resulting bit rate.

A.1 Properties of the Fourier transform

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\, df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau) d\tau \rightleftharpoons G_1(f) G_2(f)$
13. Rayleigh's energy theorem	$\int_{-\infty}^{\infty} \lvert g(t) \rvert^2 dt = \int_{-\infty}^{\infty} \lvert G(f) \rvert^2 df$

A.2 Fourier-transform pairs

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+j2\pi f}$
$\exp(-a\|t\|), \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1-\frac{\|t\|}{T}, & \|t\|<T \\ 0, & \|t\|\geq T \end{cases}$	$T\,\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty}\delta(t-iT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

Notes: $u(t)$ = unit step function
$\delta(t)$ = Dirac delta function
$\text{rect}(t)$ = rectangular function
$\text{sgn}(t)$ = signum function
$\text{sinc}(t)$ = sinc function

A.3 Trigonometric identities

$$\exp(\pm j\theta) = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$$

$$\sin\theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$