Solution to Final Examination

1. (a) The autocorrelation function of $Y(t)$ is

$$\begin{aligned}R_Y(\tau) =&\mathrm{E}\left[\left(X(t+\tau)+X(t+\tau-T)\right)\left(X(t)+X(t-T)\right)\right]\\ =&\mathrm{E}\left[X(t+\tau)X(t)\right]+\mathrm{E}\left[X(t+\tau)X(t-T)\right]\\ &+\mathrm{E}\left[X(t+\tau-T)X(t)\right]+\mathrm{E}\left[X(t+\tau-T)X(t-T)\right]\\ =&2R_X(\tau)+R_X(\tau+T)+R_X(\tau-T).\end{aligned}$$

(b) The power spectral density of $Y(t)$ is

$$\begin{aligned}S_Y(f) &= 2S_X(f)+e^{j2\pi fT}S_X(f)+e^{-j2\pi fT}S_X(f)\\ &= \left[2+2\cos(2\pi fT)\right]S_X(f)\\ &= 4S_X(f)\cos^2(\pi fT).\end{aligned}$$

2. (a) Since

$$Y(t)=\frac{1}{T}\int_{t-T}^{t}X(\alpha)\,d\alpha=\int_{\infty}^{\infty}h(t-\alpha)\,X(\alpha)\,d\alpha$$

we have

$$h(t-\alpha)=\begin{cases}1/T, & t-T<\alpha<t\\ 0, & \text{elsewhere.}\end{cases}$$

Hence the impulse response $h(t)$ is given by

$$h(t)=\frac{1}{T}\,\mathrm{rect}\left(\frac{t-(T/2)}{T}\right)=\begin{cases}1/T, & 0<t<T\\ 0, & \text{elsewhere.}\end{cases}$$

(b) The mean of $Y(t)$ is

$$\mathrm{E}\left[Y(t)\right]=\frac{1}{T}\int_{t-T}^{t}\mathrm{E}\left[X(\alpha)\right]d\alpha=\frac{1}{T}\int_{t-T}^{t}\mu_X d\alpha=\mu_X.$$

(c) The Fourier transform of $h(t)$ is

$$H(f)=\mathrm{sinc}\,(fT)\exp\left(-j\pi fT\right)$$

and the power spectral density of $X(t)$ is

$$S_X(f)=\frac{N_0}{2}.$$

Therefore, the power spectral density of $Y(t)$ is given by

$$S_Y(f)=\left|H(f)\right|^2 S_X(f)=\frac{N_0}{2}\,\mathrm{sinc}^2\left(fT\right).$$

(d) The autocorrelation function of $Y(t)$ is given by

$$\begin{aligned}
R_Y(\tau) &= R_X(\tau) \star h(\tau) \star h(-\tau) \\
&= \frac{N_0}{2} h(\tau) \star h(-\tau) \\
&= \frac{N_0}{2} \int_{-\infty}^{\infty} h(\alpha) h(\tau+\alpha) d\alpha \\
&= \begin{cases} \frac{N_0}{2T}\left(1-\frac{|\tau|}{T}\right), & |\tau| < T \\ 0, & \text{elsewhere} \end{cases} \\
&= \frac{N_0}{2T} \Lambda\left(\frac{\tau}{T}\right).
\end{aligned}$$

(e) Since $X(t)$ is a Gaussian process, so is $Y(t)$. Hence, $\bar{Y}$ follows a Gaussian distribution of mean μ_X and variance

$$\mathrm{Var}\left[\bar{Y}\right] = \mathrm{E}\left[\bar{Y}^2\right] - \mu_X^2 = R_Y(0) - \mu_X^2 = \frac{N_0}{2T} - \mu_X^2.$$

Therefore, the probability density function of $\bar{Y}$ is given by

$$f_{\bar{Y}}(\bar{y}) = \frac{1}{\sqrt{\pi\left(N_0/T - 2\mu_X^2\right)}} \exp\left(-\frac{\left(\bar{y}-\mu_X\right)^2}{N_0/T - 2\mu_X^2}\right).$$

3. (a) From class, the average power of the DSB-SC signal $m(t)\cos(2\pi f_c t)$ is $P/2$. The SSB signal $s(t)$ can be considered as the result of filtering out the lower sideband of the DSB-SC signal, and hence the average power of the SSB signal $s(t)$ is $(1/2)\cdot(P/2) = P/4$. The channel signal-to-noise ratio is therefore given by

$$(\mathrm{SNR})_\mathrm{C} = \frac{P/4}{(N_0/2)\cdot 2W} = \frac{P}{4N_0 W}.$$

(b) From class, we know that for $-W \le f \le W$,

$$S_{N_I}(f) = S_N(f-f_c) + S_N(f+f_c).$$

The power spectral density of $n_I(t)$ is hence plotted in Fig. 1

(c) First, the output of the product modulator is

$$\begin{aligned}
v(t) &= x(t)\cos(2\pi f_c t) \\
&= \left[\left(\frac{1}{2}m(t) + n_I(t)\right)\cos(2\pi f_c t) - \left(\frac{1}{2}\hat{m}(t) + n_Q(t)\right)\sin(2\pi f_c t)\right]\cos(2\pi f_c t) \\
&= \frac{1}{2}\left[\left(\frac{1}{2}m(t) + n_I(t)\right) + \left(\frac{1}{2}m(t) + n_I(t)\right)\cos(4\pi f_c t)\right] \\
&\quad - \frac{1}{2}\left[\left(\frac{1}{2}\hat{m}(t) + n_Q(t)\right)\sin(4\pi f_c t)\right].
\end{aligned}$$

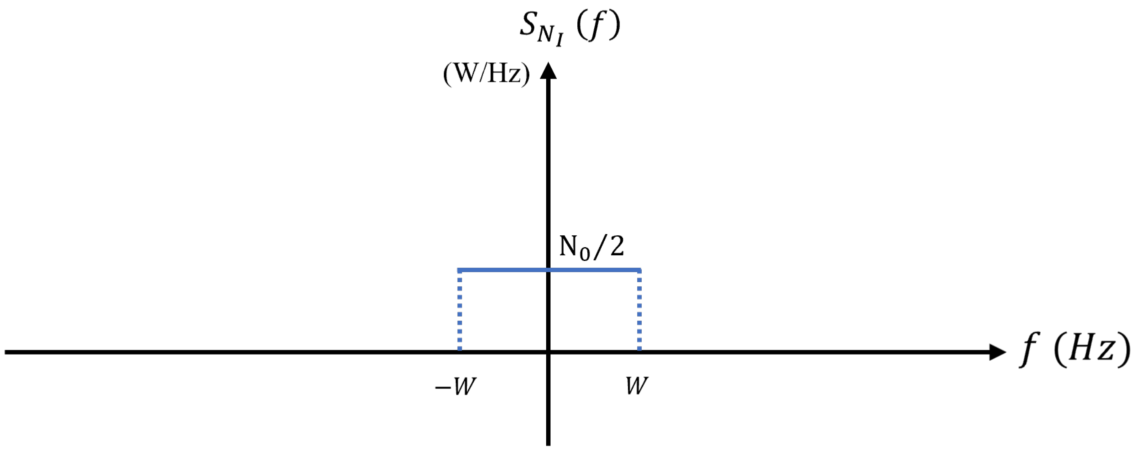

Figure 1: $S_{N_I(f)}$ in Problem 3.(b).

After the low-pass filter, the demodulator output is hence given by

$$y(t) = \frac{1}{4}m(t) + \frac{1}{2}n_I(t).$$

(d) The output signal-to-noise ratio and the figure of merit are therefore given by, respectively,

$$(\mathrm{SNR})_\mathrm{O} = \frac{(1/4)^2 P}{(1/2)^2 \cdot (N_0/2) \cdot 2W} = \frac{P}{4N_0 W}$$

and

$$\left.\frac{(\mathrm{SNR})_\mathrm{O}}{(\mathrm{SNR})_\mathrm{C}}\right|_{\mathrm{SSB}} = \frac{\frac{P}{4N_0W}}{\frac{P}{4N_0W}} = 1.$$

(e) Since the coherent detector is a linear device and the threshold effect only exists in nonlinear devices, it will not happen in this receiver.

4. (a) By taking the Fourier transform of $g(t)$, we obtain

$$\begin{aligned} G(f) &= \frac{1}{200}\mathrm{rect}\left(\frac{f}{200}\right) + \frac{1}{200}\mathrm{rect}\left(\frac{f}{200}\right) \star \frac{1}{200}\mathrm{rect}\left(\frac{f}{200}\right) \\ &= \frac{1}{200}\mathrm{rect}\left(\frac{f}{200}\right) + \frac{1}{200}\Lambda\left(\frac{f}{200}\right) \end{aligned}$$

which implies that the message bandwidth is 200 Hz. Thus, the Nyquist rate is $2 \cdot 200 = 400$ Hz.

(b) Note that the bandwidth of 10 kHz means that the in-phase and quadrature components are band-limited from -5 kHz to 5 kHz. Thus, the minimum sampling rate is $2 \cdot 5 = 10$ kHz.

5. (a) The waveforms with unipolar nonreturn-to-zero signaling, unipolar return-to-zero signaling, and bipolar return-to-zero signaling for $\{a_n\}_{n=1}^{10}$ are shown in Figs. 2, 3, and 4, respectively.

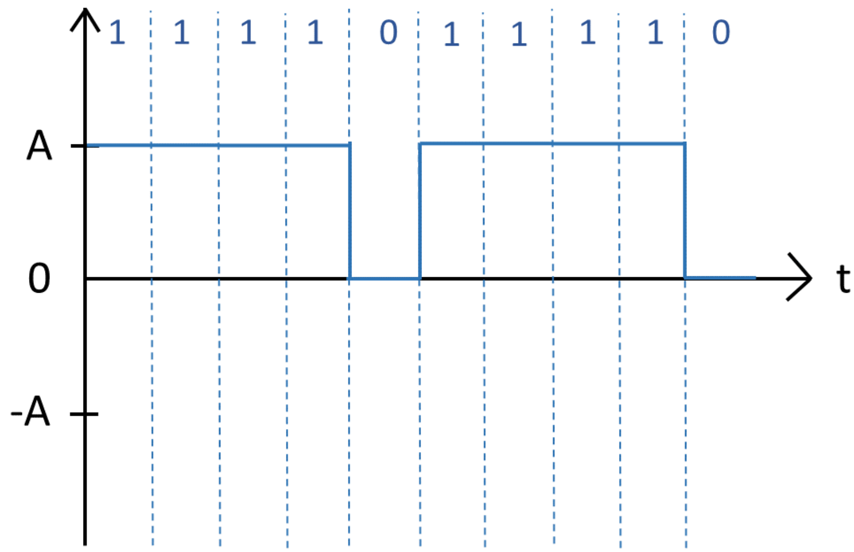

Figure 2: Unipolar nonreturn-to-zero signaling in Problem 5.(a).

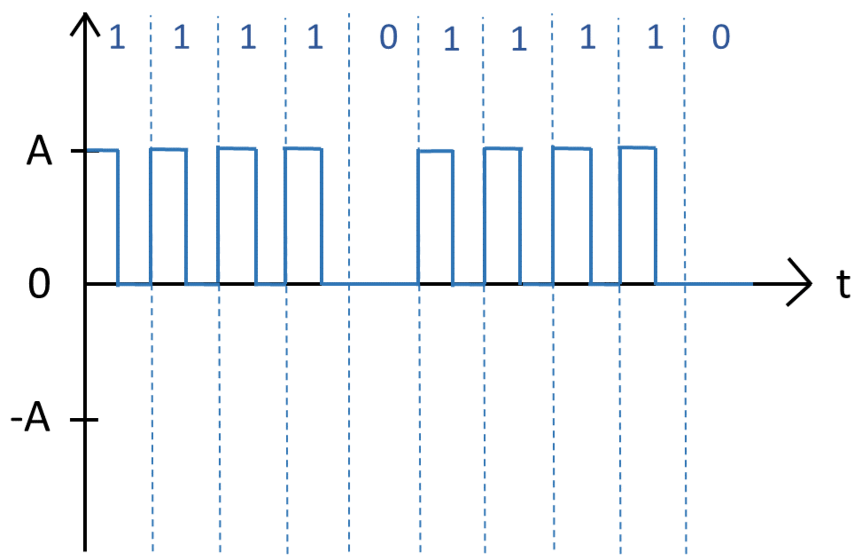

Figure 3: Unipolar return-to-zero signaling in Problem 5.(a).

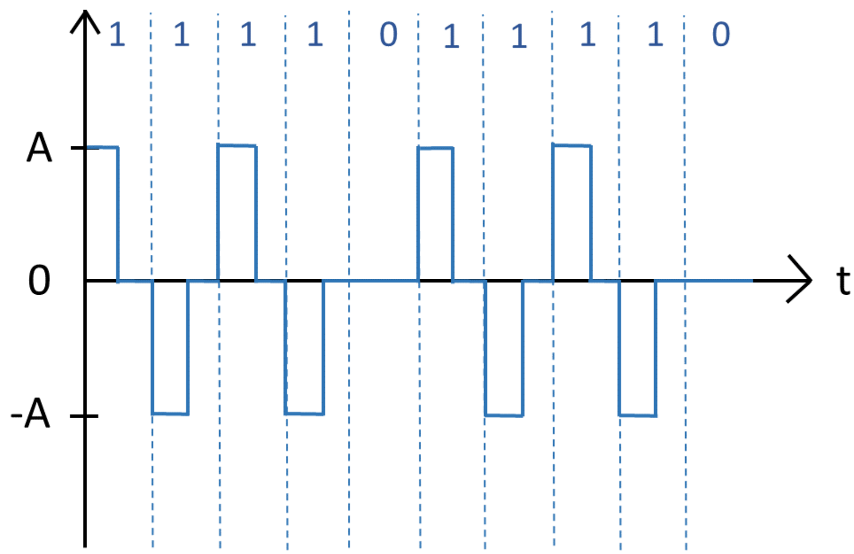

Figure 4: Bipolar return-to-zero signaling in Problem 5.(a).

(b) First note that $\{b_n\}_{n=1}^{10} = 0101101011$. The waveforms with polar nonreturn-to-zero signaling and Manchester code signaling for $\{b_n\}_{n=1}^{10}$ are shown in Figs. 5 and 6, respectively.

6. (a) Let T be the symbol duration of the PAM system. We have the transmission bandwidth given by

$$B_T = \frac{1}{2T}(1+\alpha)$$

for a raised-cosine spectrum with roll-off factor α. Since binary PAM is used, the bit duration $T_b = T$. Hence the bit rate is

$$R_b = \frac{1}{T_b} = \frac{2B_T}{1+\alpha} = \frac{2 \cdot 24}{3/2} = 32 \text{ kb/s}.$$

(b) For 256 representation levels, $\log_2 256 = 8$ bits are required to transmit a sample. From (a) the bit rate 32 kb/s, so the sampling rate is given by

$$f_s = \frac{32 \text{ kb/s}}{8 \text{ bits/sample}} = 4 \text{ kHz}.$$

In order to avoid aliasing, the maximum bandwidth of the analog signal is $f_s/2 = 2$ kHz.

(c) For a quaternary PAM system, we have $T = (\log_2 4)T_b = 2T_b$. Given the same transmission bandwidth and same roll-off factor of the raised-cosine spectrum, the symbol duration T is the same as that in (a). Hence the bit rate is twice faster than that in (a), which is $2 \cdot 32 = 64$ kb/s.

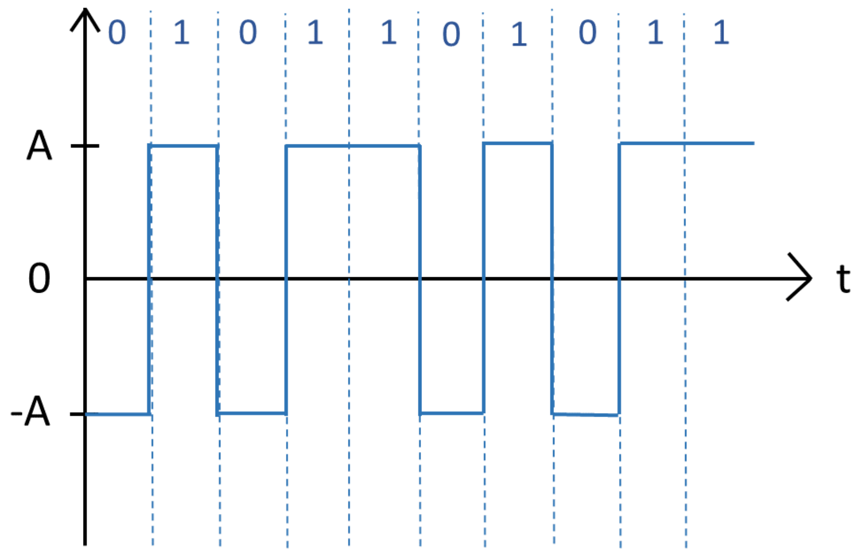

Figure 5: Polar nonreturn-to-zero signaling in Problem 5.(b).

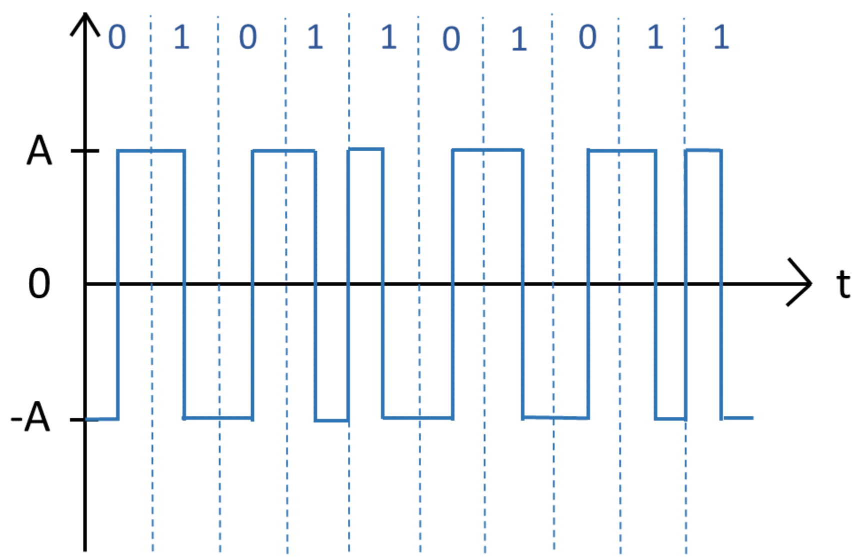

Figure 6: Manchester code signaling in Problem 5.(b).