Homework #7

Due: Jan. 7

1. Assume a noise $n_1(t)$ is stationary with a power spectral density shown in Fig. 1. Another noise process $n_2(t)$ is related with $n_1(t)$ by:

$$n_2(t) = n_1(t)\cos(2\pi f_c t + \theta) - n_1(t)\sin(2\pi f_c t + \theta),$$

where f_c is a carrier frequency and θ is the value of a random variable Θ uniformly distributed within $(0, 2\pi)$.

(a) Please find and sketch the autocorrelation function of $n_1(t)$.

(b) Show that the cross correlation of $n_1(t)$ and $n_2(t)$ is 0. Assume that random variables N_1 and Θ are statistically independent.

(c) Demonstrate that $n_2(t) = \sqrt{2}n_1(t)\cos(2\pi f_c t + \pi/4 + \theta)$.

(d) Using (c) to find the autocorrelation function of $n_2(t)$ in terms of the autocorrelation of $n_1(t)$: $R_{N_1}(\tau)$, where τ is the time difference.

(e) Plot power spectral density of $n_2(t)$.

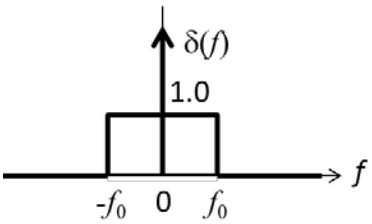


Fig. 1

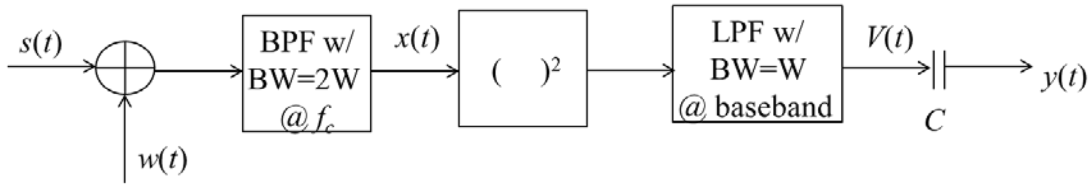
2. Consider a phase modulation (PM) signal with the modulated wave defined by $s(t) = A_c\cos[2\pi f_c t + k_p m(t)]$, where k_p is a phase sensitivity and $m(t)$ is the message. The filtered additive white Gaussian noise $n(t)$ at the input of the phase detector is $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$. Assume the carrier to noise ratio is large, please determine

(a) Output SNR

(b) FOM.

(c) Please compare this PM signal with FM one in terms of sinusoidal modulation.

3. A conventional AM signal is expressed as: $s(t) = A_c[1 + \mu m(t)]\cos(2\pi f_c t)$.
Assume we apply a "square-law" detector to detect this signal with the following receiving block diagram:



If $m(t) = \cos(2\pi f_m t)$.

(a) Find SNR_C.

(b) Find SNR_I.

(c) Please find $V(t)$.

(d) Please find $y(t)$.

(e) Assume $|\mu m(t)| << 1$, please find SNR_O.

(f) If $SNR_C >> 1$, Please find an approximation value of FOM.

(g) If $SNR_C << 1$, Please find an approximation value of FOM.

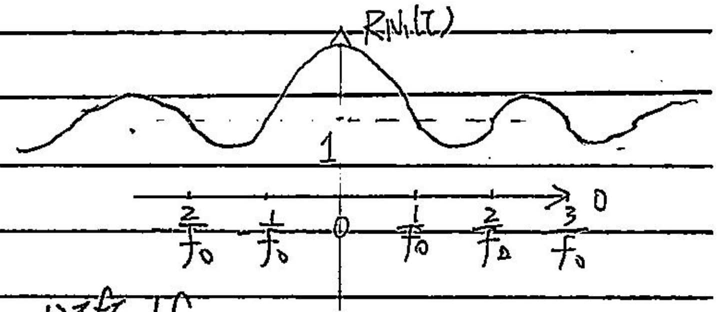
You may need the following Gaussian Integrals:

$$\int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}} dx = \sqrt{\pi}\,\frac{a^{2n+1}(1\times 3\times 5\cdots\times(2n-1))}{2^{n+1}}$$

$$\int_0^\infty x^{2n+1} e^{-\frac{x^2}{a^2}} dx = \frac{n!}{2} a^{2n+2}$$

Please note: Homework must be turned in by the beginning of class.
No late homework submission is allowable!

1. (a) $S_{N_1}(f) = \delta(f) + \begin{cases} 1 & |f| \le f_0 \\ 0 & \text{elsewhere} \end{cases}$



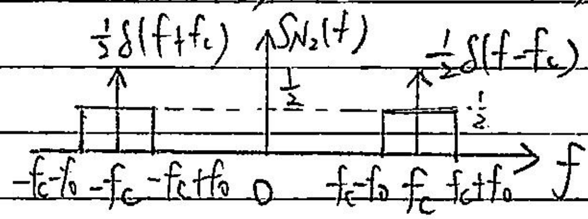
$$
\begin{aligned}
R_{N_1}(\tau) &= \mathcal{F}^{-1}\{S_{N_1}(f)\} \\
&= \int_{-\infty}^{\infty} S_{N_1}(f)\, e^{j2\pi f\tau}\, df \\
&= \int_{-\infty}^{\infty} \delta(f)\, e^{j2\pi f\tau}\, df + \int_{-f_0}^{f_0} 1\, e^{j2\pi f\tau}\, df \\
&= 1 + 2f_0 \operatorname{sinc}(2f_0\tau)
\end{aligned}
$$

(b)
$$
\begin{aligned}
R_{N_1N_2}(\tau) &= \mathbb{E}[N_1(t+\tau)\, N_2(t)] \\
&= \mathbb{E}[N_1(t+\tau)\cdot(N_1(t)\cos(2\pi f_c t+\Theta) - N_1(t)\sin(2\pi f_c t+\Theta))] \\
&= \mathbb{E}[N_1(t+\tau)N_1(t)]\cdot\mathbb{E}[\cos(2\pi f_c t+\Theta)] \\
&\quad - \mathbb{E}[N_1(t+\tau)N_1(t)]\cdot\mathbb{E}[\sin(2\pi f_c t+\Theta)] \\
&= R_{N_1}(\tau)\cdot 0 - R_{N_1}(\tau)\cdot 0 = 0
\end{aligned}
$$

(c)
$$
\begin{aligned}
n_2(t) &= n_1(t)\cos(2\pi f_c t+\theta) - n_1(t)\sin(2\pi f_c t+\theta) \\
&= n_1(t)[\cos(2\pi f_c t+\theta) - \sin(2\pi f_c t+\theta)] \\
&= \sqrt{2}\, n_1(t)\left[\tfrac{1}{\sqrt{2}}\cos(2\pi f_c t+\theta) - \tfrac{1}{\sqrt{2}}\sin(2\pi f_c t+\theta)\right] \\
&= \sqrt{2}\, n_1(t)\left[\cos\tfrac{\pi}{4}\cos(2\pi f_c t+\theta) - \sin\tfrac{\pi}{4}\sin(2\pi f_c t+\theta)\right] \\
&= \sqrt{2}\, n_1(t)\cos\left(2\pi f_c t + \tfrac{\pi}{4} + \theta\right).
\end{aligned}
$$

(d)
$$
\begin{aligned}
R_{N_2}(\tau) &= \mathbb{E}[N_2(t+\tau)\cdot N_2(t)] \\
&= 2\,\mathbb{E}[N_1(t+\tau)\cdot N_1(t)]\,\mathbb{E}\left[\cos\left(2\pi f_c(t+\tau)+\tfrac{\pi}{4}+\Theta\right)\cos\left(2\pi f_c t+\tfrac{\pi}{4}+\Theta\right)\right] \\
&= 2R_{N_1}(\tau)\cdot\tfrac{1}{2}\cos(2\pi f_c\tau) = R_{N_1}(\tau)\cdot\cos(2\pi f_c\tau).
\end{aligned}
$$

(e)
$$
\begin{aligned}
S_{N_2}(f) &= \mathcal{F}\{R_{N_2}(\tau)\} = \mathcal{F}\{R_{N_1}(\tau)\cos(2\pi f_c\tau)\} \\
&= \tfrac{1}{2}S_{N_1}(f-f_c) + \tfrac{1}{2}S_{N_1}(f+f_c)
\end{aligned}
$$



2. For a PM signal: $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = k_p m(t)$

$x(t) = s(t) + n(t) = A_c \cos(2\pi f_c t + \phi(t)) + r(t) \cos(2\pi f_c t + \psi(t))$

where $r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$, $\psi(t) = \tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right)$

the received phase $\theta(t) = \phi(t) + \tan^{-1}\left[\frac{r(t)\sin(\psi(t) - \phi(t))}{A_c + r(t)\cos(\psi(t) - \phi(t))}\right]$

For large CNR, $A_c \gg r(t)$

$$\theta(t) \simeq \phi(t) + \frac{r(t)}{A_c}\sin(\psi(t) - \phi(t))$$
$$= k_p m(t) + \frac{r(t)}{A_c}\sin(\psi(t) - \phi(t))$$

Since $\psi(t)$ is uniformly distributed over $[0, 2\pi]$, so is $\psi(t) - \phi(t)$, $\Rightarrow$ The noise after phase detector is independent of $\phi(t)$ $\Rightarrow$ The output $y(t) = k_p m(t) + \frac{r(t)}{A_c}\sin(\psi(t)) = k_p m(t) + \frac{n_Q(t)}{A_c}$

(a) The output signal is $k_p m(t)$ $\Rightarrow$ signal power: $P_S = k_p^2 P$.

Noise power: $P_N = \frac{1}{A_c^2} \cdot N_0 \cdot 2W$ $\Rightarrow$ $(SNR)_0 = A_c^2 k_p^2 P / 2N_0 W$

(b) The channel SNR: the modulated signal power: $P_S = \frac{1}{2} A_c^2$

The noise power $= \frac{N_0}{2} \times 2W = N_0 W$.

$\therefore (SNR)_C = A_c^2 / 2N_0 W$

$\therefore FOM = (SNR)_0 / (SNR)_C = k_p^2 P$

(c) If the message is a sinusoidal wave: $m(t) = A_m \cos(2\pi f_m t)$

$P = \frac{1}{2} A_m^2$, Then $(FOM)_{PM} = \frac{1}{2}(k_p A_m)^2 = \frac{1}{2}\beta_p^2$, where β_p is phase deviation in PM.

$(FOM)_{FM} = \frac{3}{2}\beta^2$, where β is the phase deviation in FM.

$\therefore$ For the same phase deviation, FM is 3 times better than PM.

3. $s(t) = A_c[1 + \mu m(t)] \cos(2\pi f_c t)$ and $m(t) = \cos(2\pi f_m t)$

then signal power: $P_S = \frac{1}{2} A_c^2 (1 + \mu^2/2)$

(a) $$(SNR)_C = \frac{\frac{1}{2} A_c^2 (1 + \mu^2/2)}{N_0 W} = \frac{A_c^2 (2 + \mu^2)}{4 N_0 W}$$

(b) $$(SNR)_I = \frac{\frac{1}{2} A_c^2 (1 + \mu^2/2)}{2 N_0 W} = \frac{A_c^2 (2 + \mu^2)}{8 N_0 W}$$

(c) $v(t) = LPF\{x^2(t)\} = LPF\{(s(t)+n(t))^2\}$, if $\mu|m(t)| << 1$

$$\simeq \frac{A_c^2}{2}[1+2\mu m(t)] + A_c[1+\mu m(t)]n_I(t) + \frac{1}{2}n_I^2(t) + \frac{1}{2}n_Q^2(t)$$

(d) $y(t)$ is the time varying term of $v(t)$

$$\Rightarrow y(t) = A_c^2 \mu m(t) + A_c[1+\mu m(t)]n_I(t) + \frac{1}{2}n_I^2(t) + \frac{1}{2}n_Q^2(t)$$

(e) demodulated message power: $P_M = \frac{1}{2}A_c^4 \mu^2$

demodulated noise: $n_0 = A_c[1+\mu m(t)]n_I(t) + \frac{1}{2}n_I^2(t) + \frac{1}{2}n_Q^2(t)$

The corresponding noise power: $P_N = \sigma_{n_0}^2 = \mathbb{E}[n_0^2] - (\mathbb{E}[n_0])^2$

$$\Rightarrow P_N = A_c^2\left(1+\frac{\mu^2}{2}\right)\sigma_N^2 + \sigma_N^4$$

where $\sigma_N^2 = 2N_0W$

$$\Rightarrow (SNR)_0 = \frac{\frac{1}{2}A_c^4\mu^2}{A_c^2\left(1+\frac{\mu^2}{2}\right)\sigma_N^2 + \sigma_N^4} = \frac{2A_c^4\mu^2}{(A_c^2(2+\mu^2) + 2\sigma_N^2)2\sigma_N^2}$$

$$= 2\left(\frac{\mu}{2+\mu^2}\right)^2\left(\frac{(SNR)_c}{1+\frac{1}{(SNR)_c}}\right) \text{ where } (SNR)_c = \frac{A_c^2(2+\mu^2)}{4N_0W}$$

(f) if $(SNR)_c >> 1$, $(SNR)_0 \sim 2\left(\frac{\mu}{2+\mu^2}\right)^2 (SNR)_c$.

(g) if $(SNR)_c << 1$ $(SNR)_0 \sim 2\left(\frac{\mu}{2+\mu^2}\right)^2 (SNR)_c^2$

Note: $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) = r(t)\cos(2\pi f_c t + \psi(t))$

where $r^2(t) = n_I^2(t) + n_Q^2(t)$, $\psi(t) = \tan^{-1}\frac{n_Q(t)}{n_I(t)}$.

$$Var\left[\frac{1}{2}n_I^2 + \frac{1}{2}n_Q^2\right] = Var\left[\frac{1}{2}r^2\right] = \mathbb{E}\left[\left(\frac{1}{2}r^2\right)^2\right] - \left(\mathbb{E}\left[\frac{1}{2}r^2\right]\right)^2 = \frac{1}{4}\left[\mathbb{E}[r^4] - (\mathbb{E}[r^2])^2\right]$$

The pdf of R is $f_R(r) = \frac{r}{\sigma_N^2}e^{-\frac{r^2}{2\sigma_N^2}}$ and $0 \le r \le \infty$

From Gaussian Integral: $\int_0^\infty x^{2n+1}e^{-\frac{x^2}{a^2}}dx = \frac{n!}{2}a^{2n+2}$

$\therefore \mathbb{E}[r^4] = \int_0^\infty r^4 \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma_N^2}}dr = \frac{1}{\sigma_N^2}\int_0^\infty r^5 e^{-\frac{r^2}{2\sigma_N^2}}dr$, take $n=2$, $a=\sqrt{2}\sigma_N$

$$= \frac{1}{\sigma_N^2}\cdot\frac{2!}{2}\cdot(\sqrt{2}\sigma_N)^6 = 8\sigma_N^4$$

$$\mathbb{E}[r^2] = \int_0^\infty r^2 \frac{r}{\sigma_N^2}e^{-\frac{r^2}{2\sigma_N^2}}dr = \frac{1}{\sigma_N^2}\frac{1!}{2}(\sqrt{2}\sigma_N)^4 = 2\sigma_N^2.$$

$$\therefore Var\left[\frac{1}{2}n_I^2 + \frac{1}{2}n_Q^2\right] = \frac{1}{4}\left[\mathbb{E}[r^4] - (\mathbb{E}[r^2])^2\right] = \frac{1}{4}\left[8\sigma_N^4 - (2\sigma_N^2)^2\right]$$

$$= \frac{1}{4}\times 4\sigma_N^4 = \sigma_N^4$$