2021 Homework 5 Reference Solution

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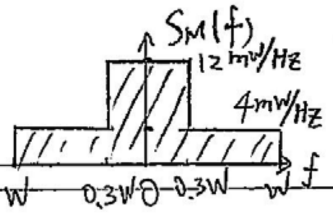
(a)



Total power:

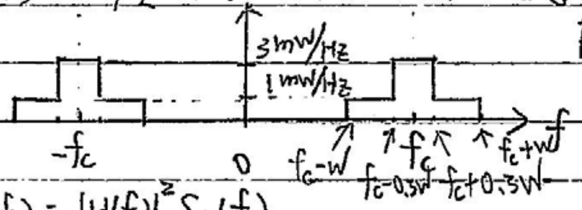
$$P_M = \int_{-\infty}^{\infty} S_M(f)\,df = 12\,\mathrm{mW/Hz} \times 0.6W + 4\,\mathrm{mW/Hz} \times 1.4W$$
$$= 7.2W + 5.6W = 12.8W\ (\mathrm{mW})$$

(b)

$$s(t) = m(t)\cdot\cos(2\pi f_c t) \Rightarrow R_S(\tau) = \frac{1}{2}R_M(\tau)\cdot\cos(2\pi f_c \tau)$$

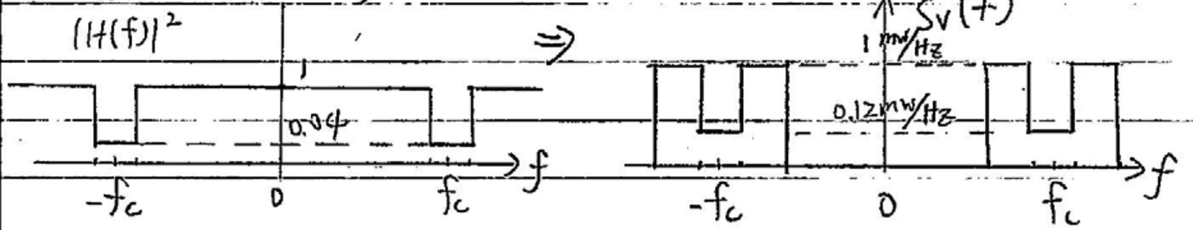
$$S_S(f) = \frac{1}{4}\left[S_M(f-f_c) + S_M(f+f_c)\right]$$

$$P_S = \frac{1}{2}P_M = 6.4W\ (\mathrm{mW})$$



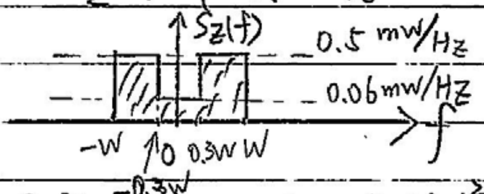
(c)

$$S_V(f) = |H(f)|^2 S_S(f)$$



d)

$$S_Z(f) = LPF\{S_Y(f)\} = LPF\left\{\frac{1}{4}\left(S_V(f-f_c) + S_V(f+f_c)\right)\right\}$$
$$= \frac{1}{2} LPF\{S_V(f-f_c)\}$$



(g)

If $G(f) \neq 1$, $S_V(f) = |H(f)|^2 S_S(f) \Rightarrow S_Y(f) = |G(f)|^2 |H(f)|^2 S_S(f)$

$S_Z(f) = LPF\left\{\frac{1}{4}|G(f)|^2 |H(f)|^2 S_S(f)\right\} = \frac{1}{2} LPF\left\{|G(f-f_c)|^2 |H(f-f_c)|^2 S_S(f-f_c)\right\}$

If $z(t) = m(t) \Rightarrow S_M(f) = S_Z(f)$

$\therefore$ For $|f| \le 0.3W \Rightarrow S_M(f) = 12\,\mathrm{mW/Hz} = S_Z(f)$, $S_S(f-f_c) = 3\,\mathrm{mW/Hz}$

$H(f-f_c) = 0.2 \Rightarrow 12 = \frac{1}{2} \times |G(f-f_c)|^2 \times 0.04 \times 3$

$G(f-f_c) = \sqrt{200} \Rightarrow G(f) = \sqrt{200}$ for $|f-f_c| \le 0.3W$

For $0.3W \le |f| \le W \Rightarrow S_M(f) = S_Z(f) = 4\,\mathrm{mW/Hz}$, $S_S(f-f_c) = 1\,\mathrm{mW/Hz}$

$H(f-f_c) = 1 \Rightarrow 4 = \frac{1}{2} \times |G(f-f_c)|^2 \times 1 \times 1$

$G(f-f_c) = \sqrt{8} \Rightarrow G(f) = \sqrt{8}$ for $0.3W \le |f-f_c| \le W$

$$\therefore G(f) = \begin{cases} \sqrt{200}, & |f-f_c| \le 0.3W \\ \sqrt{8}, & 0.3W \le |f-f_c| \le W \\ 0, & \text{elsewhere} \end{cases}$$

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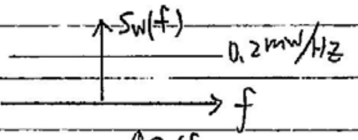
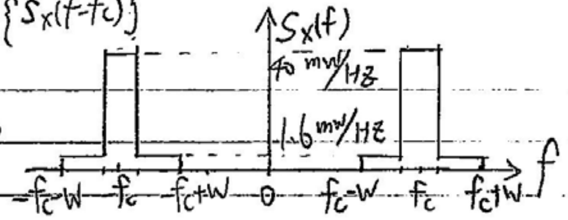
(f) $S_W(f) = \frac{N_0}{2} = 0.2\ \mathrm{mW/Hz} \Rightarrow S_X(f) = |G(f)|^2 S_W(f) \Rightarrow S_Y(f) = \frac{1}{4}\{S_X(f-f_c)+S_X(f+f_c)\}$

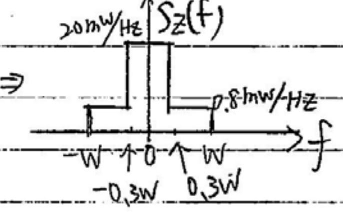
$S_Z(f) = \mathrm{LPF}\{S_Y(f)\} = \frac{1}{2}\mathrm{LPF}\{S_X(f-f_c)\}$



$\Rightarrow$



$\Rightarrow$



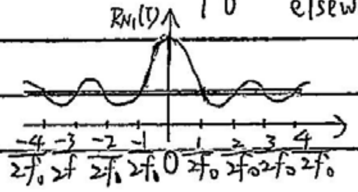
$$S_Z(f) = \begin{cases} 20\ \mathrm{mW/Hz}, & |f| \le 0.3W \\ 0.8\ \mathrm{mW/Hz}, & 0.3W \le |f| \le W \\ 0, & \text{elsewhere} \end{cases}$$

2.

(a) $S_{N_1}(f) = \delta(f) + \begin{cases} 1 & |f| \le f_0 \\ 0 & \text{elsewhere.} \end{cases}$, $R_{N_1}(\tau) = \mathcal{F}^{-1}\{S_{N_1}(f)\} = \int_{-\infty}^{\infty} S_{N_1}(f)\, e^{j2\pi f\tau}\, df$

$= \int_{-\infty}^{\infty} \delta(f) e^{j2\pi f\tau} df + \int_{-f_0}^{f_0} 1 \cdot e^{j2\pi f\tau} df$

$= 1 + 2f_0 \cdot \mathrm{sinc}(2f_0\tau)$. (5%)



(5%)

(b) As $\tau \to \pm\infty$, $R_{N_1}(\tau) = 1$, So when $\tau = \frac{n}{2f_0}$, n: integer, $R_{N_1}(\tau) = 1$
Then when the sampling rate is at $\frac{2f_0}{n}$, the sampled signals are uncorrelated.

(c) $R_{N_1N_2}(\tau) = \mathbb{E}[N_1(t+\tau)N_2(t)] = \mathbb{E}[N_1(t+\tau)\cdot(N_1(t)\cos(2\pi f_c t+\Theta) - N_1(t)\sin(2\pi f_c t+\Theta))]$

$= \mathbb{E}[N_1(t+\tau)N_1(t)]\,\mathbb{E}[\cos(2\pi f_c t+\Theta)] - \mathbb{E}[N_1(t+\tau)N_1(t)]\,\mathbb{E}[\sin(2\pi f_c t+\Theta)]$

$= R_{N_1}(\tau)\cdot 0 - R_{N_1}(\tau)\cdot 0 = 0$

(d) $n_2(t) = n_1(t)\cos(2\pi f_c t+\theta) - n_1(t)\sin(2\pi f_c t+\theta)$

$= \sqrt{2}\, n_1(t)\left[\frac{1}{\sqrt{2}}\cos(2\pi f_c t+\theta) - \frac{1}{\sqrt{2}}\sin(2\pi f_c t+\theta)\right]$

$= \sqrt{2}\, n_1(t)\left[\cos\frac{\pi}{4}\cos(2\pi f_c t+\theta) - \sin\frac{\pi}{4}\sin(2\pi f_c t+\theta)\right]$

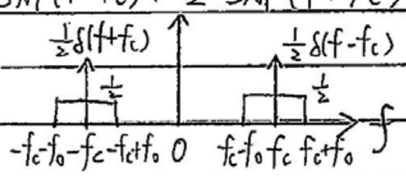
$= \sqrt{2}\, n_1(t)\cos(2\pi f_c t + \pi/4 + \theta)$

(e) $R_{N_2}(\tau) = \mathbb{E}[N_2(t+\tau)N_2(t)] = 2\mathbb{E}[N_1(t+\tau)N_1(t)]\,\mathbb{E}[\cos(2\pi f_c(t+\tau)+\frac{\pi}{4}+\theta)\cdot\cos(2\pi f_c t+\frac{\pi}{4}+\theta)]$

$= 2R_{N_1}(\tau)\cdot\frac{1}{2}\cos(2\pi f_c\tau) = R_{N_1}(\tau)\cos(2\pi f_c\tau)$

(f) $S_{N_2}(f) = \mathcal{F}\{R_{N_2}(\tau)\} = \mathcal{F}\{R_{N_1}(\tau)\cdot\cos(2\pi f_c\tau)\}$

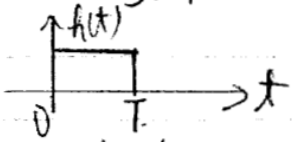
$= \frac{1}{2}S_{N_1}(f-f_c) + \frac{1}{2}S_{N_1}(f+f_c)$



3.

(a) $Y(t) = h(t) \otimes x(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\, d\tau = \int_{t-T}^{t} x(\tau)\, d\tau$

Thus $h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$



or you may find it by sending an impulse signal $\delta(t) = x(t)$

Then $Y(t) = h(t) \otimes \delta(t) = h(t) = \int_{t-T}^{t} \delta(\tau)\, d\tau = u(t) - u(t-T)$

$\therefore h(t) = u(t) - u(t-T)$

(b) $R_X(\tau) = E[X(t)X(t-\tau)] = E[A(t)\cos(2\pi f_c t+\theta)\, A(t-\tau)\cos(2\pi f_c(t-\tau)+\theta)]$

$= E[A(t)A(t-\tau)]\, E[\cos(2\pi f_c t+\theta)\cos(2\pi f_c(t-\tau)+\theta)]$

$= R_A(\tau)\cdot \frac{1}{2}\cos(2\pi f_c \tau)$

$S_X(f) = \frac{1}{4}[S_A(f-f_c) + S_A(f+f_c)]$

From (a) $h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \Rightarrow H(f) = T\operatorname{sinc}(Tf)\, e^{-j2\pi f(\frac{T}{2})}$

then $S_Y(f) = |H(f)|^2 S_X(f) = \frac{1}{4}T^2 \operatorname{sinc}^2(Tf)[S_A(f-f_c) + S_A(f+f_c)]$

(c) $A(t) = A$, $\theta = 0$ and the pdf of A is $N(0, \sigma_A^2)$

$X(t) = A\cos(2\pi f_c t)$

$Y(t) = \int_{t-T}^{t} x(\tau)\, d\tau = \int_{t-T}^{t} A\cos(2\pi f_c \tau)\, d\tau = \frac{A}{2\pi f_c}[\sin(2\pi f_c t) - \sin(2\pi f_c(t-T))]$

At $t = t_K$, $Y(t_K) = \frac{A}{2\pi f_c}[\sin(2\pi f_c t_K) - \sin(2\pi f_c(t_K - T))]$

Let $\alpha = [\sin(2\pi f_c t_K) - \sin(2\pi f_c(t_K-T))]/2\pi f_c$, a constant.

then $Y(t_K) = \alpha A \Rightarrow \mu_{Y(t_K)} = E[Y(t_K)] = E[\alpha A] = \alpha \mu_A = 0$.

$\sigma_{Y(t_K)}^2 = E[Y^2(t_K)] - (E[Y(t_K)])^2 = E[(\alpha A)^2] - (E[\alpha A])^2$

$= \alpha^2 E[A^2] - \alpha^2 (E[A])^2 = \alpha^2[E[A^2] - (E[A])^2]$

$= \alpha^2 \sigma_A^2 = \left(\frac{\sin(2\pi f_c t_K) - \sin(2\pi f_c(t_K - T))}{2\pi f_c}\right)^2 \cdot \sigma_A^2$

(d) From (C), $Y(t) = \frac{A}{2\pi f_c}[\sin(2\pi f_c t) - \sin(2\pi f(t-T))]$

If $f_c = \frac{n}{T}$, where n is an integer, then

$\sin(2\pi f_c t) - \sin(2\pi f_c(t-T)) = 0$, then $Y(t) = 0$, a constant.

If $f_c \neq \frac{n}{T}$, then $\alpha = \frac{\sin(2\pi f_c t) - \sin(2\pi f_c(t-T))}{2\pi f_c}$ is a function of t.

Thus $\sigma_{Y(t)}^2 = \alpha^2(t)\sigma_A^2$ is a time varying function, which means the pdf of $Y(t)$ is also a time varying function, thus $Y(t)$ is not a W.S.S. random variable

4.

Since $X(t)$ is a Gaussian process, we have to find the mean and variance of $X(t=3)$ to get the pdf as:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

$\because X(t)$ is zero-mean $\Rightarrow \mu_X = E[X(t=3)] = 0$

To find $\sigma^2_{X(t=3)} = Var[X(t=3)] = E[X(t=3)^2] - (E[X(t=3)])^2$

$= E[X(t=3)X(t=3)] = R_X(\tau=0)$

$\because S_X(f) = 5\,\mathrm{rect}\left(\frac{f}{1000}\right)$

$\therefore R_X(\tau=0) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau} df \Big|_{\tau=0} = \int_{-\infty}^{\infty} S_X(f) \cdot df$

$= 5 \cdot \int_{-500}^{500} 1\, df = 5000$

$\therefore \sigma_X^2 = 5000$

Then $f_{X(t=3)}(x) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{5000}} \times e^{-\frac{(x-0)^2}{2\times 5000}} = \frac{1}{\sqrt{10000\pi}} e^{\frac{-x^2}{10000}}$