2021 Homework 4 Reference Solution

1.

(a) $s(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau)$

(b) $s(t) = Re\{\tilde{s}(t) e^{j2\pi f_c t}\} = Re\{A_c e^{j(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau)}\}$

$= Re\{A_c e^{j2\pi k_f \int_0^t m(\tau)d\tau} e^{j2\pi f_c t}\}$

$\therefore \tilde{s}(t) = A_c e^{j2\pi k_f \int_0^t m(\tau)d\tau}$

(c) Within weak modulation scheme, we can let $|2\pi k_f \int_0^t m(\tau)d\tau| << 1$.

$\therefore \tilde{s}(t) = A_c e^{j2\pi k_f \int_0^t m(\tau)d\tau} \simeq A_c[1 + j2\pi k_f \int_0^t m(\tau)\,d\tau]$

$\therefore s(t) = Re\{\tilde{s}(t) e^{j2\pi f_c t}\} \simeq Re\{A_c(1 + j2\pi k_f \int_0^t m(\tau)\,d\tau) e^{j2\pi f_c t}\}$

$= A_c \cos(2\pi f_c t) - A_c \cdot 2\pi k_f \int_0^t m(\tau)\,d\tau \cdot \sin(2\pi f_c t)$

(d) Since the information is $m(t)$ is carried by $\sin(2\pi f_c t)$, we have to apply $c(t) = -2\sin(2\pi f_c t)$ to the product modulator.

Then $v(t) = s(t)\cdot c(t)$

$= [A_c \cos(2\pi f_c t) - A_c 2\pi k_f \int_0^t m(\tau)d\tau \cdot \sin(2\pi f_c t)] \times (-2)\sin(2\pi f_c t)$

$= -2A_c \sin(2\pi f_c t)\cos(2\pi f_c t) + 2A_c \cdot 2\pi k_f \int_0^t m(\tau)d\tau \cdot \sin^2(2\pi f_c t)$

$= -A_c \cdot \sin(4\pi f_c t) + A_c \cdot 2\pi k_f \int_0^t m(\tau)d\tau$

$- A_c \cdot 2\pi k_f \int_0^t m(\tau)d\tau \cdot \cos(4\pi f_c t)$

After the LPF, we have $A_c \cdot 2\pi k_f \int_0^t m(\tau)d\tau = v_0$

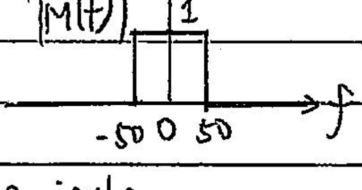
After the differentiator, we have $A_c \cdot 2\pi k_f \cdot m(t)$

(e) $m(t) = \frac{\sin(100\pi t)}{\pi t} = 100\,\mathrm{sinc}(100t)$ bandwidth = 50 Hz.

Take Fourier Transform on $m(t)$ we get $M(f) = \mathrm{rect}(\frac{f}{100})$

(f) From the Rayleigh's Energy theorem $\int_{-\infty}^{\infty} |m(t)|^2 dt = \int_{-\infty}^{\infty} |M(f)|^2 df$

$\because M(f) = \mathrm{rect}(\frac{f}{100}) \Rightarrow$



$\therefore E = \int_{-\infty}^{\infty} |m(t)|^2 dt$

$= \int_{-\infty}^{\infty} |M(f)|^2 df$

$= \int_{-\infty}^{\infty} |\mathrm{rect}(\frac{f}{100})|^2 df = 100$ joule.

(g) $s(t) = 3\cos(2\pi 10^6 t + 100\pi \int_0^t m(\tau)\,d\tau)$

$= A_c \cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau) \Rightarrow 2\pi k_f = 100\pi \Rightarrow k_p = 50$

$m(t) = 100\,\mathrm{sinc}(100t) \Rightarrow \max\{m(t)\} = 100$

$\therefore \Delta f = k_f \cdot \max\{m(t)\} = 50 \times 100 = 5000$,

maximum phase deviation $\beta = \frac{\Delta f}{W} = \frac{5000}{50} = 100 \gg 1$.

It's not a weak FM signal.

(h) Carson's rule: $B_T = 2(\Delta f + W) = 2(5000 + 50) = 10.1$ KHz

2.

(a) $\theta_i(t) = 2\pi f_c t + k_p m(t) = 2\pi f_c t + k_p A_m \cos(2\pi f_m t)$

$= 2\pi f_c t + \beta_p \cos(2\pi f_m t)$ where β_p is phase deviation.

Then the instantaneous frequency is

$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} = f_c - \beta_p f_m \sin(2\pi f_m t)$

$\therefore$ the frequency deviation $|\Delta f| = \beta_p f_m$

By Carson's rule: $B_T = 2\Delta f(1 + \frac{1}{\beta_p})$

$\because \beta_p \gg 1 \Rightarrow \frac{1}{\beta_p} \ll 1 \Rightarrow B_T \simeq 2\Delta f = 2\beta_p f_m \propto f_m$

$\therefore$ The bandwidth is linearly proportional to the modulation frequency f_m

(b) In FM, $\Delta f = k_f A_m$, $B_T = 2\Delta f + 2f_m$. Since typically Δf is much larger than f_m, the transmission bandwidth is mainly dominated by the amplitude A_m or frequency sensitivity k_f and is not related to modulation frequency.

3.

$s(t) = 10\cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau)$

$f_c = 100$ MHz, $m(t) = 10\cos(2\pi f_m t)$, $f_m = 3$ KHz

(a) After the differentiator:

$s'(t) = -10(2\pi f_c + 2\pi k_f m(t))\sin(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau)$

For the envelope detector, $2\pi f_c + 2\pi k_f m(t)$ should be always positive $\Rightarrow f_c + k_f 10\cos(2\pi f_m t) \geq 0 \Rightarrow f_c - 10k_f \geq 0$

$\Rightarrow k_f \leq f_c/10 = 100 \times 10^6/10 = 10^7$ Hz/Volt.

(b) $k_f = 10$ From Carson's rule $B_T = 2\Delta f + 2 f_m$

$\Delta f = k_f \cdot A_m = 10 \cdot 10 = 100$ and $f_m = 3k$ Hz $\Rightarrow B_T = 6.2$ KHz.

$\beta = \Delta f / f_m = 100/3000 \ll 1 \Rightarrow$ This is NBFM.

(c) The instantaneous frequency is $f_c + k_f m(t) = 10^8 + 100\cos(2\pi f_m t)$

$\therefore$ The maximum frequency is $10^8 + 100$ Hz.

The minimum frequency is $10^8 - 100$ Hz

(d) For an envelope detector to work properly, the charging time constant $R_s C$ must be short compared to the carrier period $1/f_c$. Thus $R_s \ll 1/f_c C = 1/10^8 \cdot 10^{-9} = 10\Omega$. Choose $R_s = 0.1\Omega$. On the other hand, the discharging time constant should be much greater than the carrier period $1/f_c = 10^{-8}$ sec but much smaller than the message period of $1/3000$ sec So we may pick R_e to be 1000Ω, so that the discharging time constant is $R_e \cdot C = 10^3 \cdot 10^{-9} = 10^{-6}$ sec.

(e) To find the minimum interval T such that there are at least 4 zero crossings, we must consider the minimum value of the instantaneous frequency $f_i(t)$. The minimum value is $10^8 - 100$. Since there are two zero crossing is every carrier period,

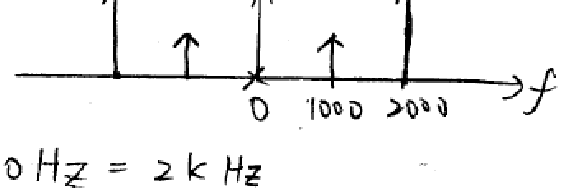
$$\frac{4}{2T} = 10^8 - 100 \Rightarrow T = \frac{2}{10^8 - 100} = 2.000002 \times 10^{-8} \simeq 2 \times 10^{-8} \text{ sec.}$$

4.

(a)
$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right)$$
$$= 10\cos(2\pi f_c t + 10\sin(2\pi \cdot 1000t) + 20\sin(2\pi \cdot 2000t))$$
$$\therefore 2\pi k_f \int_0^t m(\tau)\,d\tau = 10\sin(2\pi \cdot 1000t) + 20\sin(2\pi \cdot 2000t)$$

Take the derivative on both sides

$$2\pi k_f m(t) = 2\pi \cdot 10^4 \cos(2\pi \cdot 1000t) + 2\pi \cdot 4\times10^5 \cos(2\pi \cdot 2000t).$$
$$\Rightarrow m(t) = \frac{10^4}{k_f}\left(\cos(2\pi \cdot 1000t) + 4\cos(2\pi \cdot 2000t)\right)$$
$$\Rightarrow M(f) = \frac{1}{2}\frac{10^4}{k_f}[\delta(f-1000) + \delta(f+1000) + 4\delta(f-2000) + 4\delta(f+2000)].$$



$\therefore W = 2000$ Hz $= 2$ k Hz

(b) $P_{av} = \frac{1}{2} A_c^2 = \frac{1}{2} \cdot 10^2 = 50\,W$

(c)

The instantaneous frequency is

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \left[2\pi f_c t + 10 \sin(2\pi \cdot 1000 t) + 20 \sin(2\pi \cdot 2000 t) \right]$$
$$= 10^6 + 10^4 \cos(2\pi \cdot 1000 t) + 4 \times 10^4 \cos(2\pi \cdot 2000 t).$$

(d)

When $\cos(2\pi \cdot 1000 t) = \cos(2\pi \cdot 2000 t) = 1$, we have maximum frequency deviation $10^4 + 4 \times 10^4 = 50$ kHz.

and when $2\pi \cdot 1000\, t_n = k \cdot 2\pi$, $2\pi \cdot 2000 = l \cdot 2\pi$, k, l: integer

$$t_n = \frac{k}{1000} = \frac{l}{2000} \Rightarrow l = 2k$$

$\Rightarrow$ when $t_n = k \cdot 10^{-3}$ sec k = integer, we have maximum Δf.

(e)

$$B_T = 2(\Delta f + W) = 2(50k + 2k) = 104 \text{ kHz}.$$