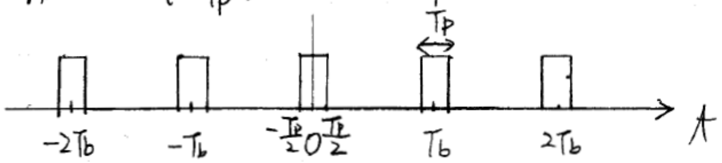
2021 Homework 2 Reference Solution

1.

(a) $f(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-T_b}{T_p}\right)$ $T_b = 5T_p$



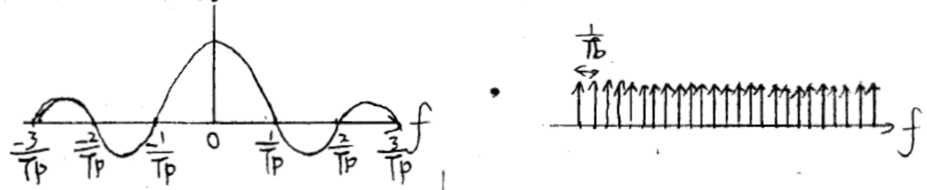
(b) We can consider $f(t)$ as a convolution of a rectangular function: $\text{rect}(\frac{t}{T_p})$ and a comb function: $\sum_{n=-\infty}^{\infty} \delta(t-nT_b)$.

$$f(t) = \text{rect}\left(\frac{t}{T_p}\right) \otimes \sum_{n=-\infty}^{\infty} \delta(t-nT_b)$$

$$\therefore F(f) = T_p \,\text{sinc}(fT_p) \cdot \sum_{m=-\infty}^{\infty} \frac{1}{T_b}\,\delta\left(f - m\frac{1}{T_b}\right)$$

$$= \frac{T_p}{T_b} \sum_{m=-\infty}^{\infty} \text{sinc}(fT_p)\cdot\delta\left(f-\frac{m}{T_b}\right)$$

(c) $\text{sinc}(fT_p)$ $\sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T_b})$



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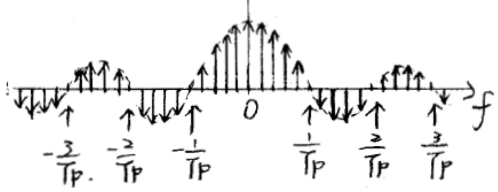
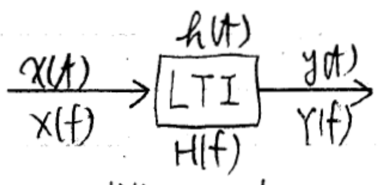


2. (a)



$y(t) = x(t) \otimes h(t) \Rightarrow Y(f) = X(f)\,H(f)$

$\therefore H(f) = \frac{Y(f)}{X(f)}$

$\because \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \frac{dx(t)}{dt} + 2x(t)$

Take the Fourier Transform on both sides:

$(j2\pi f)^2\, Y(f) + 4(j2\pi f)Y(f) + 3Y(f) = (j2\pi f)X(f) + 2X(f)$

$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{j2\pi f + 2}{-4\pi^2 f^2 + 4j2\pi f + 3}$$

(b)

Since the system is causal, we may assume $h(t) = 0$ for $t < 0$, then $H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = \int_0^{\infty} h(t) e^{-j2\pi ft} dt$

Let $s = j2\pi f$. Then $H(s) = \int_0^{\infty} h(t) e^{-st}\, dt$: a Laplace Transform.

(c)

$$\because H(s) = \frac{s+2}{s^2+4s+3} = \frac{1}{2}\left(\frac{1}{s+1}+\frac{1}{s+3}\right)$$

Then $h(t) = \begin{cases}\frac{1}{2}\left(e^{-t}+e^{-3t}\right) & \text{for } t\ge 0\\ 0 & t<0\end{cases}$

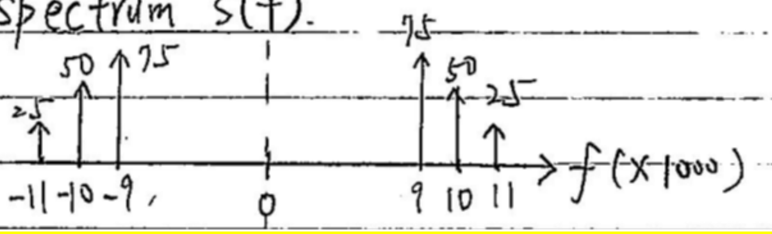
We may also express $h(t)=\frac{1}{2}\left(e^{-t}+e^{-3t}\right)u(t)$ where
$u(t)=\begin{cases}1 & \text{for } t\ge 0\\ 0 & t<0\end{cases}$

3. (a)

$$s(t) = 50\cos(2\pi\cdot 11000t) + 100\cos(2\pi\cdot 10000t) + 150\cos(2\pi\cdot 9000t)$$

Then
$$S(f) = 25\left[\delta(f-11000)+\delta(f+11000)\right] + 50\left[\delta(f-10000)+\delta(f+10000)\right] +75\left[\delta(f-9000)+\delta(f+9000)\right]$$

Then the spectrum $S(f)$.



(b) f_c = 10 kHz, can also be 9 kHz or 11 kHz, then answers for (d), (e), (f) and (g) need to be modified.

(c) Transmission Bandwidth B_T = 2 kHz

(d)

By definition: $s(t) = \mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\}$

Then
$$\begin{aligned} s(t) &= 50\cos(2\pi\cdot 11000t) + 100\cos(2\pi\cdot 10000t) + 150\cos(2\pi\cdot 9000t)\\ &= \mathrm{Re}\left\{50e^{j2\pi\cdot 11000t} + 100\cdot e^{j2\pi\cdot 10000t} + 150\cdot e^{j2\pi\cdot 9000t}\right\}\\ &= \mathrm{Re}\left\{\left(50\cdot e^{j2\pi\cdot 1000t} + 100 + 150e^{-j2\pi\cdot 1000t}\right)e^{j2\pi\cdot 10000t}\right\}\end{aligned}$$

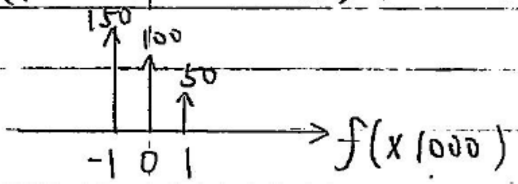
$$\begin{aligned}\therefore \tilde{s}(t) &= 50\cdot e^{j2\pi\cdot 1000t} + 100 + 150e^{-j2\pi\cdot 1000t}\\ &= 50\cos(2\pi\cdot 1000t) + j50\sin(2\pi\cdot 1000t)\\ &\quad + 100\\ &\quad + 150\cos(2\pi\cdot 1000t) - j150\sin(2\pi\cdot 1000t)\\ &= \left[100 + 200\cos(2\pi\cdot 1000t)\right] + j\left[-100\sin(2\pi\cdot 1000t)\right]\end{aligned}$$

$$\Rightarrow m_I(t) = 100 + 200\cos(2\pi\cdot 1000t) \qquad m_Q(t) = -100\sin(2\pi\cdot 1000t)$$

(e)

$\because\ \tilde{s}(t) = 50e^{j2\pi\cdot 1000t} + 100 + 150e^{-j2\pi\cdot 1000t}$

$\Rightarrow \tilde{S}(f) = 50\,\delta(f-1000) + 100\,\delta(f) + 150\,\delta(f+1000)$

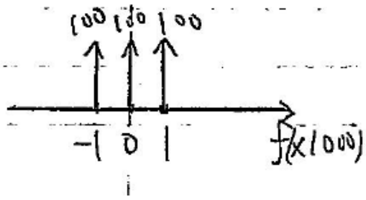


(f)

$m_I(t) = 100 + 200\cos(2\pi\cdot 1000t)$

$\Rightarrow M_I(f) = 100\,\delta(f) + 100[\delta(f-1000) + \delta(f+1000)]$

$|M_I(f)| = M_I(f)$



(g)

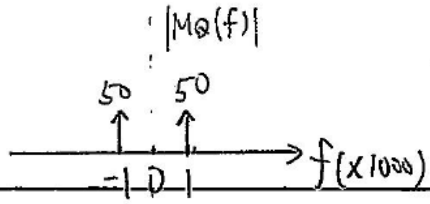
$m_Q(t) = -100\sin(2\pi\cdot 1000t)$

$M_Q(f) = -100 \times \frac{1}{2j}[\delta(f-1000) - \delta(f+1000)]$

$= j50[\delta(f-1000) - \delta(f+1000)]$

$|M_Q(f)| = |j50[\delta(f-1000) - \delta(f+1000)]|$

$= 50\,\delta(f-1000) + 50\,\delta(f+1000)$



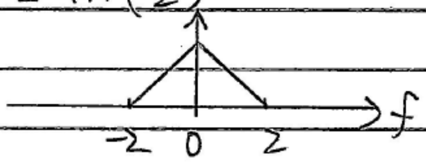
4.

(a) $h(t) = 2\,\mathrm{sinc}^2(2t) = 2\cdot \mathrm{sinc}(2t)\cdot \mathrm{sinc}(2t)$

$H(f) = 2\cdot \mathcal{F}^{-1}\{\mathrm{sinc}(2t)\} \otimes \mathcal{F}^{-1}\{\mathrm{sinc}(2t)\}$

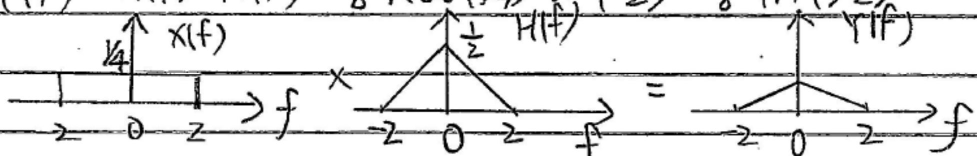
$= 2\cdot\left(\frac{1}{2}\mathrm{rect}\left(\frac{f}{2}\right)\right) \otimes \left(\frac{1}{2}\mathrm{rect}\left(\frac{f}{2}\right)\right)$

$= \frac{1}{2}\mathrm{tri}\left(\frac{f}{2}\right)$



(b) $x(t) = \mathrm{sinc}(4t) \Rightarrow X(f) = \frac{1}{4}\mathrm{rect}\left(\frac{f}{4}\right)$

$Y(f) = X(f)\cdot H(f) = \frac{1}{8}\mathrm{rect}(f/4)\,\mathrm{tri}\left(\frac{f}{2}\right) = \frac{1}{8}\mathrm{tri}(f/2)$



$\therefore\ y(t) = \mathcal{F}^{-1}\{\frac{1}{8}\mathrm{tri}(f/2)\} = \frac{1}{4}\times(2\,\mathrm{sinc}^2(2t)) = \frac{1}{2}\mathrm{sinc}^2(2t)$