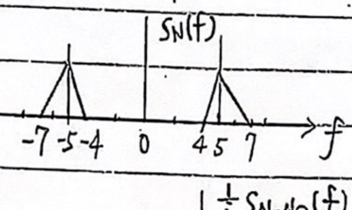
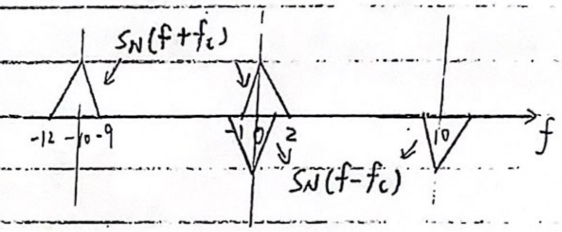
2021 Final Exam Reference Solution.

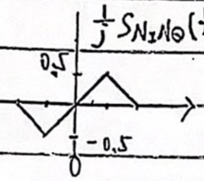
1. (a) No. For this result, the probability of H is 6/10 and T is 4/10, but they should be 1/2 for a fair coin.

(b) $S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] & -B \le f \le B \\ 0, & \text{elsewhere} \end{cases}$



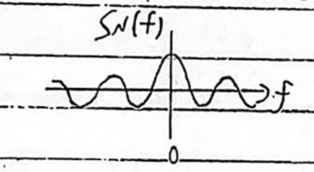
⇒





(C) $R(\tau) = \begin{cases} 1 & -1 \le \tau \le 1 \\ 0 & \text{elsewhere} \end{cases}$ $\Rightarrow S_N(f) = \mathcal{F}\{R(\tau)\} = \int_{-1}^{1} e^{-j2\pi f t} dt$

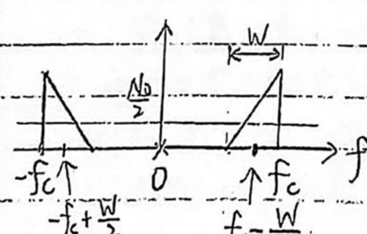
$= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$



This power spectral density function exist negative values at some frequencies, which is against that PSD ≥ 0 for all frequencies. ∴ It's not a valid autocorrelation function for a communication signal.

(d) No. The f^2 term comes from $\frac{d}{dt}$ function in FM discriminator. For PM detector, we don't have a slope circuit in it, so PM doesn't have f^2 noise reduction advantage.

2. (a)



Since it's a LSB signal, the BPF is with bandwidth W centered at $f_c - \frac{W}{2}$

∴ The filtered narrow band noise

$n(t) = n_I \cos(2\pi(f_c - \frac{W}{2})t) - n_Q \sin(2\pi(f_c - \frac{W}{2})t)$

$\therefore x(t) = s(t) + n(t) = A_c m(t) \cos(2\pi f_c t) + A_c \hat{m}(t) \sin(2\pi f_c t)$

$+ n_I(t) \cos(2\pi(f_c - \frac{W}{2})t) - n_Q(t) \sin(2\pi(f_c - \frac{W}{2})t)$ (4%)

$v(t) = x(t)\cos(2\pi f_c t)$

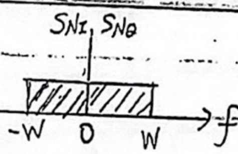
$= [A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t) + n_I \cos(2\pi(f_c - \frac{W}{2})t)$
$- n_Q(t)\sin(2\pi(f_c - \frac{W}{2})t)]\cos(2\pi f_c t)$

$= \frac{1}{2}A_c m(t) + \frac{1}{2}A_c m(t)\cos(4\pi f_c t) + \frac{1}{2}A_c \hat{m}(t)\sin(4\pi f_c t)$
$+ \frac{1}{2}n_I(t)\cos(2\pi\cdot\frac{W}{2}t) + \frac{1}{2}n_I(t)\cos(2\pi(2f_c - \frac{W}{2})t)$
$+ \frac{1}{2}n_Q(t)\sin(2\pi\frac{W}{2}t) - \frac{1}{2}n_Q(t)\sin(2\pi(2f_c - \frac{W}{2})t)$ (4%)

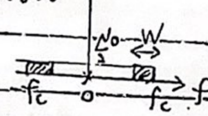
After LPF, $y(t) = \frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)\cos(2\pi\cdot\frac{W}{2}t) + \frac{1}{2}n_Q(t)\sin(2\pi\cdot\frac{W}{2}t)$

(b) At the output, the demodulated message power is $\frac{1}{4}A_c^2 P$. (4%)
the demodulated noise power is $\frac{1}{8}\times\frac{N_0}{2}\times 2W + \frac{1}{8}\times\frac{N_0}{2}\times 2W = \frac{1}{4}N_0 W$.

$\therefore (SNR)_O = \frac{\frac{1}{4}A_c^2 P}{\frac{1}{4}N_0 W} = \frac{A_c^2 P}{N_0 W}$



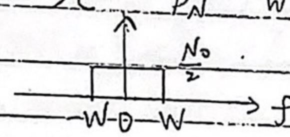
(c) $(SNR)_I = \frac{P_S}{P_N} = \frac{A_c^2 P}{N_0 W}$



Since $m(t)$ and $\hat{m}(t)$ carry the same power $\Rightarrow P_S = \frac{1}{2}A_c^2 P + \frac{1}{2}A_c^2 P = A_c^2 P$

$P_N = \frac{N_0}{2}\times W\times 2 = N_0 W$

(d) $(SNR)_C = \frac{P_S}{P_N}$ where $P_S = A_c^2 P$.



$P_N = \frac{N_0}{2}\times 2W = N_0 W$

$\therefore (SNR)_C = \frac{A_c^2 P}{N_0 W}$.

(e) $(FOM) = \frac{(SNR)_O}{(SNR)_C} = \frac{A_c^2 P / N_0 W}{A_c^2 P / N_0 W} = 1$.

3.(a) Θ is not a random variable $\Rightarrow Z(t) = X(t)\cos(2\pi f_c t + \theta_1)$

$\mu_Z = E[X(t)\cos(2\pi f_c t + \theta_1)] = E[X(t)]\cos(2\pi f_c t + \theta_1)$
$= \mu_X\cdot\cos(2\pi f_c t + \theta_1) = 0$. $\leftarrow$ stationary to the 1st order

$R_Z(t_2, t_1) = E[Z(t_2)Z(t_1)] = E[X(t_2)\cos(2\pi f_c t_2 + \theta_1)X(t_1)\cos(2\pi f_c t_1 + \theta_1)]$
$= E[X(t_2)X(t_1)]\cdot\frac{1}{2}[\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_2 + t_1) + 2\theta_1)]$
$= R_X(t_2 - t_1)\cdot\frac{1}{2}[\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_2 + t_1) + 2\theta_1)]$

$\because \cos(2\pi f_c(t_2 + t_1) + 2\theta_1)$ is not a function of $t_2 - t_1$, thus
$R_Z(t_2, t_1)$ is not simply a function of $t_2 - t_1 = \tau$.
$\therefore Z(t)$ is not wide sense stationary.

(b) If Θ is also a random variable and is independent of $X(t)$
Then $\mu_Z = E[Z(t)] = E[X(t)\cos(2\pi f_c t + \Theta)] = E[X(t)]E[\cos(2\pi f_c t + \Theta)]$

$= \mu_X \, \mathbb{E}[\cos(2\pi f_c t + \Theta)] = 0 \cdot 0 = 0$

$\therefore \mu_Z$ is a constant.

$R_Z(t_2, t_1) = \mathbb{E}[Z(t_2) Z(t_1)] = \mathbb{E}[X(t_2)\cos(2\pi f_c t_2 + \Theta) \cdot X(t_1)\cos(2\pi f_c t_1 + \Theta)]$

$= \mathbb{E}[X(t_2)X(t_1)]\,\mathbb{E}[\cos(2\pi f_c t_2 + \Theta)\cos(2\pi f_c t_1 + \Theta)]$

$= R_X(t_2 - t_1) \cdot \frac{1}{2}\cos(2\pi f_c(t_2 - t_1))$

$\because R_X(t_2 - t_1) = R_X(\tau)$ is a function of $\tau = t_2 - t_1$

and $\frac{1}{2}\cos(2\pi f_c(t_2 - t_1)) = \frac{1}{2}\cos(2\pi f_c \tau)$ is also a function of $\tau = t_2 - t_1$ only $\Rightarrow R_Z(t_2 - t_1) = R_Z(\tau)$ is a function of τ.

$\therefore Z(t)$ is wide sense stationary.

4.(a) $(SNR)_0 = \frac{3}{2}\frac{A_c^2 k_f^2 P}{N_0 W^3}$ $\quad CNR = \frac{\frac{1}{2}A_c^2}{2 \times \frac{1}{2} N_0 B_T} = \frac{A_c^2}{2 N_0 B_T} = \rho$

$\because \Delta f = k_f A_m \Rightarrow A_m = \Delta f / k_f \quad P = \frac{1}{2}A_m^2 = \frac{1}{2}\left(\frac{\Delta f}{k_f}\right)^2, \quad f_m = W$

$\because B_T = 2(\Delta f + W) \simeq 2\Delta f \Rightarrow \Delta f \simeq B_T/2 \Rightarrow P \simeq \frac{1}{2}\left(\frac{B_T}{2k_f}\right)^2$

$\Rightarrow (SNR)_0 = \frac{A_c^2}{2 N_0 B_T} \times 3 \times \frac{1}{8}\left(\frac{B_T}{W}\right)^3 = 3\rho \cdot \left(\frac{B_T}{2W}\right)^3$

(b) To enhance $(SNR)_0$, we can either increase ① CNR, ρ or ② $\frac{B_T}{2W}$, that is transmission bandwidth (4%)

Since $(SNR)_0$ is linearly proportional to CNR, ρ, but is cubically proportional to B_T, so increasing B_T is more efficient. (4%)

(c) For AM, $\max\{FOM\} = 1/3$

$(FOM)_{FM} = \frac{3 k_f P}{W} = \frac{3}{2}\left(\frac{\Delta f}{W}\right)^2 \simeq \frac{3}{2}\left(\frac{B_T}{2W}\right)^2$

If FM outperforms AM: $\frac{3}{2}\left(\frac{B_T}{2W}\right)^2 \geq 1/3. \Rightarrow B_T \geq 2\sqrt{2}/3\, W$

$\therefore B_T \geq 0.943 W$

$\Rightarrow \because B_T = 2(\Delta f + W) > 2W$, So we can guarantee FM is always better than AM.

5.(a) $h(t) = 20 e^{-5t} u(t) \Rightarrow H(f) = \int_0^\infty 20 e^{-5t} e^{-j2\pi f t} dt = \frac{20}{5 + j2\pi f}$

(b) $S_N(f) = |H(f)|^2 S_W(f) = \frac{400}{25 + (2\pi f)^2} \cdot S_W(f)$

$\because N_0 = 2 \times 10^{-2}\ W/Hz \Rightarrow N_0/2 = 10^{-2}\ W/Hz$

$\therefore S_N(f) = \frac{N_0}{2}|H(f)|^2 = \frac{400}{25 + (2\pi f)^2} \times 10^{-2} = \frac{4}{25 + (2\pi f)^2}\ W/Hz.$

(c) $R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\} = \mathcal{F}^{-1}\left\{\frac{4}{25 + (2\pi f)^2}\right\}$

From $e^{-a|t|} \Leftrightarrow \frac{2a}{a^2 + (2\pi f)^2} \Rightarrow a = 5 \Rightarrow S_N(f) = \frac{4}{10} \times \frac{10}{25 + (2\pi f)^2}$

$\Rightarrow R_N(\tau) = \frac{2}{5} \times e^{-5|\tau|}$

(d) $n(t) = w(t) \otimes h(t)$

$\therefore \mu_N(t) = \mathbb{E}[n(t)] = \mathbb{E}[w(t) \otimes h(t)] = \mathbb{E}[w(t)] \otimes h(t) = \mu_w \otimes h(t)$

$= 0 \otimes h(t) = 0.$

(e) $\sigma_N^2 = \mathbb{E}[(N(t) - \mu_N)^2] = \mathbb{E}[N(t)^2] = \mathbb{E}[N(t)N(t)] = R_N(0)$

$= \frac{2}{5} \times e^{-5|\tau|}\Big|_{\tau=0} = 2/5$

(f) $n(t)$ is not a white noise because $R_N(\tau) = \frac{2}{5} e^{-5|\tau|} \neq \delta(\tau)$, or because $S_N(f)$ is not a constant at all frequencies.

So, the answer is NO.

(g) Yes, $n(t)$ is still a Gaussian Noise. This is because, according to the property of Gaussian process, any Gaussian processed signal passes a linear time invariant system, the output is also Gaussian.

$\therefore$ $w(t)$: additive white Gaussian noise

$h(t)$: linear time invariant

$n(t) = w(t) \otimes h(t)$: still Gaussian

(h) $\mu_N = 0 \quad \sigma_N^2 = 2/5.$

$f_N(n) = \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{(n-\mu_N)^2}{2\sigma_N^2}} = \frac{1}{\sqrt{2\pi} \times \sqrt{2/5}} \times e^{-\frac{n^2}{2 \times 2/5}} = \sqrt{\frac{5}{4\pi}}\, e^{-\frac{5n^2}{4}}$