Final (Modern Physics) (6 problems on both sides of this paper)

06/11/2018 Provided by Masahito Oh-e

Instructions:

- You are not allowed to open any textbook, copies of my lecture notes and ppt files, but allowed to see **your own notebook or memo**. You can also use **a simple calculator**.
- Do not use the Internet. If anyone is found who is cheating on the exam, she/he will be immediately failed in this course.
- Solve the problems below. Describe the ways of thinking in English: only final solutions are not accepted. Make clear how you reach each solution.
- When 30 minutes pass after the test starts, if you think you have completed the test, you can leave the room by submitting the answer sheets, except the 10 minutes before the exam finishes.
- Several physical constants are listed in the end of the sheet.
- If answers are decimal numbers, calculate them to one places of decimals at least.

Problem 1. (25 points)

Consider the particle described by a wavefunction: $\psi = e^{ix} + 2ie^{3ix}$ $(-\pi \le x \le \pi)$

(a) Normalize the wavefunction.

(b) If you precisely measure the momentum of the state expressed by the wavefunction ψ, what values can you obtain and what probability cay you get, respectively?

(c) Calculate the probability of finding the particle as a function of the position.

(d) Draw the graph of the probability density.

(e) Calculate the expectation value of the position.

(a) $\psi^* \psi = (e^{-ix} - 2ie^{-3ix})(e^{ix} + 2ie^{3ix}) = 5 - 2ie^{-2ix} + 2ie^{2ix}$

$$\int_{-\pi}^{\pi} \psi^* \psi \, dx = \int_{-\pi}^{\pi} (5 - 2ie^{-2ix} + 2ie^{2ix})\, dx = 10\pi$$

Therefore, the normalized function is $\boldsymbol{\psi = \frac{1}{\sqrt{10\pi}}(e^{ix} + 2ie^{3ix})}$

(ⅇ^-ix-2iⅇ^(-3ix)) (ⅇ^ix+2iⅇ^(3ix)) »

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(E^(-(I*x)) - (2*I)/E^(3*I*x))*
 (E^(I*x) + 2*I*E^(3*I*x))
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$\left(e^{-i x}-2 i e^{-3 i x}\right)\left(e^{i x}+2 i e^{3 i x}\right)$

Simplify$\left[\left(e^{-i x}-2 i e^{-3 i x}\right)\left(e^{i x}+2 i e^{3 i x}\right)\right]$

$5-2 i e^{-2 i x}+2 i e^{2 i x}$

Simplify$\left[\left(e^{-i x}+2 i e^{-3 i x}\right)\left(e^{i x}+2 i e^{3 i x}\right)\right]$

$-3+2 i e^{-2 i x}+2 i e^{2 i x}$

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ReleaseHold[HoldComplete[Null]]
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$$\int_{-\pi}^{\pi}\left(5-2 i e^{-2 i x}+2 i e^{2 i x}\right) d x$$

10π

(b) ψ can be seen as the superposition of the two waves: e^{ix} and e^{-3ix}. From the each wave, $\hat{p}_x e^{ix}=-i\hbar\frac{\partial}{\partial x}e^{ix}=\hbar e^{ix}$, $\hat{p}_x e^{-3ix}=-i\hbar\frac{\partial}{\partial x}e^{-3ix}=3\hbar e^{-3ix}$. Therefore, **the values of the momentum are $\hbar$ and $3\hbar$**. Since the ratio of coefficients of the two wavefunctions is 1:2, the ratio of the probability is 1:4. Therefore, **the probability for the momentum $\hbar$ is $\frac{1}{5}$, and the probability for $3\hbar$ is $\frac{4}{5}$.**

(c) The probability of finding the particle is expressed as $\psi^*\psi=|\psi|^2$.

$$\psi^*\psi=\frac{1}{2\pi}+\frac{i}{5\pi}\left(e^{2ix}-e^{-2ix}\right)=\frac{1}{2\pi}-\frac{2}{5\pi}\sin 2x$$

Simplify$\left[\frac{1}{\sqrt{10\pi}}\left(e^{-ix}-2ie^{-3ix}\right)\frac{1}{\sqrt{10\pi}}\left(e^{ix}+2ie^{3ix}\right)\right]$

$$\frac{\left(e^{-ix}-2ie^{-3ix}\right)\left(e^{ix}+2ie^{3ix}\right)}{10\pi}$$

(d) The graph is below:

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Plot[1/(2*Pi) - (2*Sin[2*x])/(5*Pi), {x, -Pi, Pi}]
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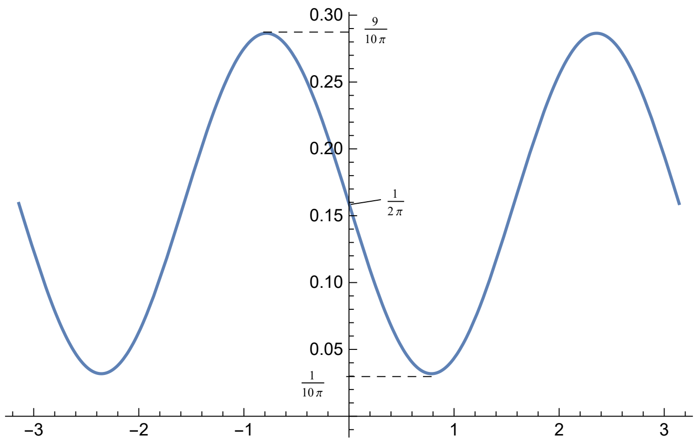

(e) The expectation value of the position is expressed as $\int_{-\pi}^{\pi} \psi^* x \psi \, dx$

$\int_{-\pi}^{\pi} \psi^* x \psi \, dx = \int_{-\pi}^{\pi} x\left(\frac{1}{2\pi} - \frac{2}{5\pi}\sin 2x\right) dx = \frac{2}{5}$

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Integrate[x (1/(2 π) - 2/(5 π) Sin[2 x]), {x, -Pi, Pi}]
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$\frac{2}{5}$

Problem 2. (20 points)

Show that the velocity of an electron in the n_{th} Bohr's orbit of the hydrogen atom is described as $V = \frac{e^2}{2\,\epsilon_0\, nh}$. And calculate the velocity and de Broglie's wavelengths of electrons in the first three Bohr's orbits.

The photon wavelength λ is described as $\lambda = \frac{h}{p}$ with Planck's constant h and the momentum p. With this, de Broglie's wavelength is given by $\lambda = \frac{h}{mv}$ (1). Bohr's condition for orbit stability is $2\pi r = n\lambda$ (2), where r is the radius of the orbit that contain n wavelengths, and n is called the quantum number of the orbit. With eqs. (1) and (2), the angular momentum is quantized as $mvr = \frac{nh}{2\pi} = n\hbar$ (3). From the balance between the coulomb force and the centrifugal force, we have $\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r}$ (4). With eqs. (3) and (4), r can be deduced to be $r = \frac{\epsilon_0 h^2 n^2}{\pi m e^2}$ (5). Therefore, plugging-in (5) for (3), we have $v = \frac{nh}{2\pi r m} = \frac{nh\pi m e^2}{2\pi m\,\epsilon_0 h^2 n^2} = \frac{e^2}{2\,\epsilon_0\, nh}$ (6).

With eqs. (1) and (6), de Broglie's wavelength is given by $\lambda = \frac{h}{mv} = \frac{h 2\,\epsilon_0\, nh}{me^2} = \frac{2\,\epsilon_0\, nh^2}{me^2}$ (7).

The velocity and de Broglie's wavelength for n=1 are calculated as

$$v_1 = \frac{e^2}{2\,\epsilon_0\, h} = \frac{(1.602\times10^{-19})^2}{2\times8.854\times10^{-12}\times6.626\times10^{-34}} = 2.187\times10^6\ m/s$$

$$v_n = \frac{v_1}{n},\ \text{therefore},\ v_2 = 1.094\times10^6\ m/s,\ \text{ and } v_3 = 0.729\times10^6\ m/s$$

$\lambda_1 = \frac{h}{mv_1} = \frac{6.626\times10^{-34}}{9.109\times10^{-31}\times2.187\times10^6} = 0.3326\times10^{-9}\ m = 333$ pm

$\lambda_n = n\,\lambda_1$, therefore, $\lambda_2 = 665$ pm, and $\lambda_3 = 997$ pm.

Problem 3. (20 points)

Consider one dimension and answer the questions below.

(a) If the wavefunction of a particle: $\psi(x) = \sqrt{a}\, e^{-a|x|}$ (a: the real and positive constant) is given, calculate the probability of finding the particle in the region of $|x| \leq L$.

(b) Defining two operators $\hat{a}$ and $\hat{b}$ as $\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{p} - i\hat{x})$, $\hat{b} = \frac{1}{\sqrt{2\hbar}}(\hat{p} + i\hat{x})$, respectively, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, calculate the commutation relation $[\hat{a}, \hat{b}]$.

(a) The probability can be written as $\int_{-L}^{L} \psi^* \psi\, dx$

$$\int_{-L}^{L} \psi^* \psi\, dx = \int_{-L}^{L} \left(\sqrt{a}\right)^2 e^{-2a|x|}\, dx = a\int_{-L}^{0} e^{2ax}\, dx dx + a\int_{0}^{L} e^{-2ax} dx = \mathbf{1 - e^{-2aL}}$$

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Integrate[E^(2*a*x), {x, -L, 0}] + Integrate[E^(-2*a*x), {x, 0, L}]
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$$-\frac{-1 + e^{-2aL}}{a}$$

(b) $[\hat{a},\hat{b}] = [\frac{1}{\sqrt{2\hbar}}(\hat{p} - i\hat{x}), \frac{1}{\sqrt{2\hbar}}(\hat{p} + i\hat{x})] = \frac{1}{2\hbar}\{[\hat{p},\hat{p}]-[i\hat{x},\hat{p}]+[\hat{p},i\hat{x}]-[-i\hat{x},i\hat{x}]\} = -\frac{i}{\hbar}[\hat{x},\hat{p}] = -\frac{i}{\hbar} i\hbar = 1$

Problem 4. (30 points)

Consider a wavefunction $\psi(\vec{r}, t) = \sum_n c_n(t)\phi_n(\vec{r})$ with the orthonormalized functions $\phi_n(\vec{r})$ that satisfy $\hat{\mathcal{H}}\phi_n = \hbar\omega_n\phi_n$ (n: natural number).

(a) Using the time-dependent Schrödinger equation, derive a derivative (differential) equation that the time-dependent coefficients $c_n(t)$ satisfy.

(Obtain a derivative (differential) equation of $c_n(t)$ from the time-dependent Schrödinger equation.)

(b) Obtain the wavefunction at time t. The initial conditions are $c_1(0) = \alpha$, $c_2(0) = \beta$ and others $c_n(0) = 0$ (α, β : the positive real numbers, and $\alpha^2 + \beta^2 = 1$)

(c) With the wavefunction obtained in (b), calculate the expectation value of the energy of that state.

(a) Plugging in $\psi(\vec{r}, t) = \sum_n c_n(t)\phi_n(\vec{r})$ for the time-dependent Schrödinger equation: $\hat{\mathcal{H}}\Psi = i\hbar \frac{\partial}{\partial t}\Psi$,

$$\hat{\mathcal{H}}\sum_n c_n(t)\phi_n(\vec{r}) = i\hbar \frac{\partial}{\partial t}\sum_n c_n(t)\phi_n(\vec{r})$$

$$\sum_n c_n(t)\hbar\omega_n\phi_n(\vec{r}) = i\hbar\sum_n \frac{d}{dt}c_n(t)\phi_n(\vec{r})$$

Multiplying the both sides by $\phi_k^*(\vec{r})$,

$$\sum_n c_n(t)\hbar\omega_n \int \phi_k^*(\vec{r})\, \phi_n(\vec{r})\, dV = i\hbar\sum_n \frac{d}{dt}c_n(t)\int \phi_k^*(\vec{r})\, \phi_n(\vec{r})\, dV .$$

Due to the orthogonality of the eigenfunctions $\int \phi_k^*(\vec{r})\, \phi_n(\vec{r})\, dV = \delta_{kn}$,

$\sum_n c_n(t)\omega_n\delta_{kn} = i\sum_n \frac{d}{dt}c_n(t)\delta_{kn}$

Therefore, the required derivative equation is expressed as

$\mathbf{i\frac{d}{dt}c_n(t) = \omega_n\, c_n(t)}$

(b) The solution of the derivative equation obtained in (a) is given by

$c_n(t) = c_n(0)e^{-i\omega_n t}$

Since the initial conditions are $c_1(0) = \alpha$, $c_2(0) = \beta$ and others $c_n(0) = 0$, the wavefunction at time t can be expressed as

$$\boldsymbol{\psi(\vec{r}, t) = \alpha e^{-i\omega_1 t}\phi_1(\vec{r}) + \beta e^{-i\omega_2 t}\phi_2(\vec{r})}$$

(c) The expectation value of the energy of that state is $<\hat{\mathcal{H}}>$, and with the relation of $\int \phi_k^*(\vec{r})\,\phi_n(\vec{r})\,dV = \delta_{kn}$, it can be calculated as follows:

$$<\hat{\mathcal{H}}>=\int \psi^*(\vec{r},t)\hat{\mathcal{H}}\,\psi(\vec{r},t)\,dV =$$

$$\int\{\alpha e^{-i\omega_1 t}\phi_1(\vec{r}) + \beta e^{-i\omega_2 t}\phi_2(\vec{r})\}^*\hat{\mathcal{H}}\{\alpha e^{-i\omega_1 t}\phi_1(\vec{r}) + \beta e^{-i\omega_2 t}\phi_2(\vec{r})\}\,dV$$

$$= \int\{\alpha e^{i\omega_1 t}\phi_1^*(\vec{r}) + \beta e^{\omega_2 t}\phi_2^*(\vec{r})\}\{\alpha e^{-i\omega_1 t}\hbar\omega_1\phi_1(\vec{r}) + \beta e^{-i\omega_2 t}\hbar\omega_2\phi_2(\vec{r})\}\,dV$$

$$= \boldsymbol{\alpha^2\hbar\omega_1 + \beta^2\hbar\omega_2}$$

Problem 5. (20 points)

Assume the Hamiltonian of a particle in three dimension is expressed by spherical-polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $y = r\cos\theta$) as below:

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\text{Cos}\theta}{r^2\,\text{Sin}\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\,\text{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2}\right] - \frac{k}{r}$$ (k: the positive real constant)

(a) If a wavefunction: $\psi = Ne^{-br}$ (N, b: the positive real constant) is given, determine the value of b so that ψ is the eigenfunction of $\hat{\mathcal{H}}$ and also derive the eigenvalue.

(b) Normalize ψ and deduce N that is expressed by b.

(a) The wavefunction $\psi = Ne^{-br}$ depends only on r.

$$\hat{\mathcal{H}}\psi = [-\frac{\hbar^2}{2m}\nabla^2 - \frac{k}{r}]\psi = -\frac{\hbar^2}{2m}[b^2 - \frac{2b}{r}]Ne^{-br} - \frac{k}{r}Ne^{-br} = -\frac{\hbar^2 b^2}{2m}\psi + [\frac{\hbar^2 b}{m} - k]\,\frac{1}{r}\psi\,.$$

Here, we use $\frac{d}{dr}e^{-br} = -b\,e^{-br}$, $\frac{d^2}{dr^2}e^{-br} = \frac{d}{dr}(-b\,e^{-br}) = b^2e^{-br}$.

In order for ψ to be an eigenfunction of $\hat{\mathcal{H}}$ for any r,

$\frac{\hbar^2 b}{m} - k = 0$

$\therefore\ \mathbf{b = \frac{mk}{\hbar^2}}$

The eigenvalue for this,

$-\frac{\hbar^2 b^2}{2m} = -\frac{\hbar^2}{2m}\frac{m^2k^2}{\hbar^4} = \mathbf{-\frac{mk^2}{2\hbar^2}}$

(b) Plugging in $\psi = Ne^{-br}$ for $\int\psi^*\psi\,dV = 1$, and using partial integral for $\int_0^\infty e^{-2br}r^2dr$,

$$\int_0^\infty\int_0^\pi\int_0^{2\pi} N^2e^{-2br}r^2\sin\theta\,dr\,d\theta\,d\phi = 4\pi N^2\int_0^\infty e^{-2br}r^2dr = 4\pi N^2\frac{1}{4b^3} = \frac{\pi N^2}{b^3} = 1$$

Therefore, **N =** $\sqrt{\frac{b^3}{\pi}}$

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Integrate[r^2 Exp[-2 b r], {r, 0, ∞}]
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ConditionalExpression[1/(4 b^3), Re[b] > 0]
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Problem 6. (25 points)

An electron trapped in an infinite depth well of width L=1nm. Consider the transition from the excited state n = 2 to the ground state n = 1. Calculate the wavelength of light emitted.

Describe and explain all the processes of how to derive energy and wavelength emitted. (Key points and processes): Create Schrödinger's equation. Derive a general solution. Make clear boundary conditions. Discuss energy states.

The one-dimensional Schrödinger equation along the x direction is expressed as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \ ,$$

where ψ is the wavefunction of the particle, E is the total energy of the particle, and V is the potential energy. We consider the potential energy V is ∞ in the region of $x \leqslant 0$, and $x \geqslant L$, and V is 0 in the range of $0 < x < L$.

Therefore, in the region of $x \leqslant 0$, and $x \geqslant L$, $\psi(x) = 0$, and in the region of $0 < x < L$, The Schrödinger equation is rewritten by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \ .$$

With $k = \frac{\sqrt{2mE}}{\hbar}$, $\frac{d^2\psi}{dx^2} = -k^2\psi$. The general solution of this differential equation is $\psi(x) = A\sin(kx) + B\cos(kx)$, where A and B are the constants. To determine A, B and k, we use the boundary conditions; those are $\psi(0) = 0$ and $\psi(L) = 0$. From the former, we obtain B = 0, and from the latter, $\psi(L) = A\sin(kL) = 0$. If A=0, $\psi(x) = 0$. This means no particle exists, which is unreasonable. Therefore, A≠0, and the possible values of KL are $kL = 0, \pm\pi, \pm 2\pi, \cdots$. However, kL=0 means $\psi(x)=0$ since E becomes 0, and the constant A absorbs the sign since sin(-x) = -sinx. Therefore, we obtain the condition as below: $kL = \pi, 2\pi, 3\pi, \cdots = n\pi$ (n = 1, 2, 3, ⋯)

We now obtain,

$$\frac{\sqrt{2mE}}{\hbar} = n\pi \ (n = 1, 2, 3, \cdots) \ .$$

As a result,

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2} \ (n = 1, 2, 3, \cdots)$$

This means the energy states are not continuous but discrete depending on natural number n.

Using the expression of the energy E, the transition from the excited state n = 2 to the ground state n = 1 is given by

$$E_2 - E_1 = \frac{3\pi^2\hbar^2}{2\,mL^2} = \frac{3\,h^2}{8\,mL^2}$$

This energy gap corresponds to the emitted photon energy hν. Therefore,

$$h\nu = h\,\frac{c}{\lambda} = \frac{3\,h^2}{8\,mL^2} = \frac{3\times(6.626\times10^{-34})^2}{8\times(9.1095\times10^{-31})\times(10^{-9})^2} = 1.80734\times10^{-19}\ \text{J}$$

The wavelength emitted is obtained as below:

$$\lambda = \frac{hc}{1.80734\times10^{-19}} = \frac{6.626\times10^{-34}\times2.998\ X\ 10^8}{1.80734\times10^{-19}} = 1.09912\times10^{-6}\ \text{m} = 1.1\ \mu\text{m}$$

$$\frac{3\,(6.626\ 10^{-34})^2}{8\,(9.1095\ \ 10^{-31})\,(10^{-9})^2}$$

1.80734×10^{-19}

$$\frac{6.626\ 10^{-34}\ 2.998\ 10^8}{1.80734\times10^{-19}}$$

1.09912×10^{-6}

* Physical constants and some formula:

Speed of light: c = 2.998 X 10^8 m/s = 3.0 X 10^8 m/s

Planck's constant: h = 6.626 X 10^{-34} J·s = 6.6 X 10^{-34} J·s

Electron charge: e =1.602 X 10^{-19} C = 1.6 X 10^{-19} C

Electron rest mass: m_e =9.1095 X 10^{-31} Kg = 9.1 X 10^{-31} Kg

Dielectric constant in vacuum: ϵ_0 =8.854 X 10^{-12} $C^2 N^{-1} m^{-2}$ = 8.9 X 10^{-12} $C^2 N^{-1} m^{-2}$