Problem set #4 (Modern Physics)

06/04/2018 Provided by *Masahito Oh-e*

Solve the problems below. Describe the ways of thinking in English: only final solutions are not accepted. Make clear how you reach each solution.

<u>As official homework, you are not forced to do the problem 3', but for those who are interested in deriving Schrödinger equation expressed by spherical polar coordinates, the problem 3' is prepared. If you work on problem 3' and properly solve it, you can get some bonus points.</u>

<u>Problem 1.</u> (20 points)

Express $\frac{\partial^2}{\partial z^2}$ by spherical-polar coordinates. Use the relations below.

$$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial z^2}=\left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)=\left(\cos\theta\frac{\partial}{\partial r}\right)\left(\cos\theta\frac{\partial}{\partial r}\right)-\left(\cos\theta\frac{\partial}{\partial r}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)-\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}\right)+\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}-\cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right)+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}-\cos\theta\sin\theta\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}\right\}-\frac{\sin\theta}{r}\left\{\frac{\partial}{\partial\theta}(\cos\theta)\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\right\}+\frac{\sin\theta}{r^2}\left\{\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\right\}$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}-\cos\theta\sin\theta\left\{-\frac{1}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}\right\}-\frac{\sin\theta}{r}\left\{-\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\right\}+\frac{\sin\theta}{r^2}\left\{\cos\theta\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial^2}{\partial\theta^2}\right\}$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

<u>Problem 1'.</u> (Bonus)

In the same way of problem 1, derive $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ by spherical coordinates and derive how kinetic energy hamiltonian $\hat{\mathcal{H}}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$ can be expressed in spherical coordinates.

See the following pages.

<u>Problem 2.</u> (20 points)

Using the commutator $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, and its cyclic variants, prove that total angular momentum squared and the individual components of angular momentum commute, i.e $[\hat{L}^2, \hat{L}_x] = 0$ etc.

Using $\hat{L}^2 = \hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$, the commutator with total angular momentum squared can be evaluated

$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2, \hat{L}_z] = [\hat{L}_x^2+\hat{L}_y^2, \hat{L}_z] = \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y = -i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_x\hat{L}_y = 0$

Similarly $[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}] = 0$

<u>Problem 3.</u> (30 points)

The Schrödinger wave function for a stationary state of an atom is ψ = Af(r)sinθ cos$\theta \cdot e^{i\phi}$ where (r, θ, ϕ) are spherical polar coordinates.

(a) Find the z component of the angular momentum of the atom.

(b) Find the square of the total angular momentum of the atom.

(a) $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

$\hat{L}_z\psi = -i\hbar\frac{\partial}{\partial\phi}$Af(r)sin$\theta$ cos$\theta \cdot e^{i\phi}$=i($-i\hbar$Af(r)sinθ cos$\theta\cdot e^{i\phi}$) = $\hbar$Af(r)sinθ cos$\theta\cdot e^{i\phi}$=$\hbar\psi$

Therefore, the z-component of the angular momentum is $\hbar$.

(b) $\hat{L}^2 = -\hbar^2[\frac{\partial^2}{\partial\theta^2} + \frac{\mathrm{Cos}\theta}{\mathrm{Sin}\theta}\frac{\partial}{\partial\theta} + \frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2}]$

$\hat{L}^2\psi = -\hbar^2[\frac{\partial^2}{\partial\theta^2} + \frac{\mathrm{Cos}\theta}{\mathrm{Sin}\theta}\frac{\partial}{\partial\theta} + \frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2}]$ Af(r)sinθ cos$\theta\cdot e^{i\phi}$=$-\hbar^2$Af(r)$e^{i\phi}${-4sinθcosθ + cotθ(cos$^2\theta$ $-$sin$^2\theta$) $-\frac{\sin\theta\cos\theta}{\sin^2\theta}$}

= $6\hbar^2$Af(r)sinθ cos$\theta\cdot e^{i\phi}$ = $6\hbar^2\psi$

Thus, L^2= $6\hbar^2$. But L^2 = l(l+1) = $6\hbar^2$. Therefore, l=2

<u>Problem 4.</u> (20 points)

Calculate the expectation value <r> of an electron in a state of n=1 and l=0 of hydrogen atom. r is the position from the nucleus. Use the wave functions appropriately in Table 6-1 of the textbook. You can use the integration of $\int_0^\infty x^n \exp(-ax)\,dx = \frac{n!}{a^{n+1}}$ (n>-1, a>0).

What we are interested in is the radial part of the wavefunction of the state, n=1 and l=0.

We use the $R_{1,0}(r) = \frac{2}{a_0^{3/2}}e^{-r/a_0}$

<r>=$\int_0^\infty R^*_{1,0}(r) r R_{1,0}(r) r^2 dr = 4(\frac{1}{a_0})^3 a_0^4 \int_0^\infty \rho^3 \exp(-2\rho)d\rho = \frac{3}{2}a_0$ $\quad(\rho = \frac{r}{a_0})$

<u>Problem 5.</u> (30 points)

Assume the Hamiltonian of a particle in three dimension is expressed by spherical-polar coordinates (x = r sinθcosϕ, y = r sinθsinϕ, y = r cosθ) as below:

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\mathrm{Cos}\theta}{r^2\,\mathrm{Sin}\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\,\mathrm{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2}] - \frac{k}{r}$$ **(k: the positive real constant)**

If a wavefunction: ψ = Ne^{-br} (N, b: the positive real constant) is given, determine the value of b so that ψ is the eigenfunction of $\hat{\mathcal{H}}$ and also derive the eigenvalue.

The wavefunction ψ = Ne^{-br} depends only on r.

$\hat{\mathcal{H}}\psi = [-\frac{\hbar^2}{2m}\nabla^2 - \frac{k}{r}]\psi = -\frac{\hbar^2}{2m}[b^2 - \frac{2b}{r}]Ne^{-br} - \frac{k}{r}Ne^{-br} = -\frac{\hbar^2 b^2}{2m}\psi + [\frac{\hbar^2 b}{m} - k]\frac{1}{r}\psi$. Here, we use $\frac{d}{dr}e^{-br}$= -b e^{-br}, $\frac{d^2}{dr^2}e^{-br}=\frac{d}{dr}$(-b e^{-br}) = b^2e^{-br}.

In order for ψ to be an eigenfunction of $\hat{\mathcal{H}}$ for any r,

$\frac{\hbar^2 b}{m}$ - k = 0

$\therefore$ **b = $\frac{mk}{\hbar^2}$**

The eigenvalue for this,

$-\frac{\hbar^2 b^2}{2m} = -\frac{\hbar^2}{2m}\frac{m^2k^2}{\hbar^4} = -\frac{mk^2}{2\hbar^2}$

✓ Second partial derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f = \left(\frac{\partial}{\partial x}\right)^2 f = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right) f.$$

$$\frac{\partial^2}{\partial z^2} = \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta \left(\frac{\partial}{\partial r}\right)^2 - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r^2}\left(\frac{\partial}{\partial \theta}\right)^2 \qquad \underline{\text{N/G}}$$

$$\frac{\partial}{\partial x}(gf) = \frac{\partial g}{\partial x} f + g\frac{\partial f}{\partial x} \Rightarrow \frac{\partial}{\partial x} g = \frac{\partial g}{\partial x} + g\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial z^2} = \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right)$$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \left(\cos\theta\frac{\partial}{\partial r}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) - \left(\cos\theta\frac{\partial}{\partial r}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$- \left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) + \left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta}\right\}$$

$$- \frac{\sin\theta}{r}\left\{\frac{\partial}{\partial \theta}(\cos\theta)\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\right\} + \frac{\sin\theta}{r^2}\left\{\frac{\partial}{\partial \theta}(\sin\theta)\frac{\partial}{\partial \theta} + \sin\theta\frac{\partial}{\partial \theta}\frac{\partial}{\partial \theta}\right\}$$

$$= \cos^2\frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\left\{-\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta}\right\}$$

$$- \frac{\sin\theta}{r}\left\{-\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial^2}{\partial r\partial \theta}\right\} + \frac{\sin\theta}{r^2}\left\{\cos\theta\frac{\partial}{\partial \theta} + \sin\theta\frac{\partial^2}{\partial \theta^2}\right\}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial \theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

— (1)

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2}\;{}^{1} + \sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)^{2} - \sin\phi\cos\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\phi}\right)\checkmark$$

$$+ \frac{\cos\theta\cos^2\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\right)^{3} + \frac{\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)^{4} - \frac{\cos\theta\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\checkmark$$

$$- \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial r}\right)^{5} - \frac{\cos\theta\sin\phi}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right)^{7} + \frac{\sin\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)^{9}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2}\;{}^{1} + \sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)^{2} + \sin\phi\cos\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\phi}\right)\checkmark$$

$$+ \frac{\cos\theta\sin^2\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\right)^{3} + \frac{\cos\theta\sin^2\phi}{r^2}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)^{4} + \frac{\cos\theta\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\checkmark$$

$$+ \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial r}\right)^{6} + \frac{\cos\theta\cos\phi}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right)^{8} + \frac{\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)^{10}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \sin^2\theta \frac{\partial^2}{\partial r^2}{}^{1} + \sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)^{2} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial r}\right)^{3} + \frac{\cos\theta}{r^2}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial \theta}\right)^{4}$$

$$- \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial r}\right)^{5} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial r}\right)^{6}$$

$$- \frac{\cos\theta\sin\phi}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial \theta}\right)^{7} + \frac{\cos\theta\cos\phi}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \theta}\right)^{8}$$

$$+ \frac{\sin\phi}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right)^{9} + \frac{\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial \phi}\right)^{10}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial \theta}$$

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r}{}^{*} - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial \phi}{}^{**}$$

$$+ \frac{\cos^2\phi}{r}\frac{\partial}{\partial r}{}^{*} + \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial \phi}{}^{**} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial}{\partial \theta}{}^{***} - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial \theta\partial \phi}{}^{\triangle}$$

$$+ \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\frac{\partial}{\partial \theta}{}^{***} + \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial \theta\partial \phi}{}^{\triangle} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}{}^{\triangle\triangle} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}{}^{\triangle\triangle\triangle}$$

$$- \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}{}^{\triangle\triangle} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}{}^{\triangle\triangle\triangle}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$+ \left(\frac{1}{r}\right)\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \quad \text{— (2)}$$

(1) + (2)

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= (\cos^2\theta + \sin^2\theta)\frac{\partial^2}{\partial r^2} + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial}{\partial r} + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)\frac{\partial^2}{\partial \theta^2}$$

$$+ \frac{1}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$