Problem set #3 (Modern Physics)

05/24/2018 Provided by *Masahito Oh-e*

Solve the problems below. Describe the ways of thinking in English: only final solutions are not accepted. Make clear how you reach each solution.

As official homework, you are not forced to do the problem 3', but for those who are interested in deriving Schrödinger equation expressed by spherical polar coordinates, the problem 3' is prepared. If you work on problem 3' and properly solve it, you can get some bonus points.

Problem 1. (20 points)

Time-dependent Schrödinger equation: $\hat{\mathcal{H}}\Psi = i\hbar \frac{\partial}{\partial t}\Psi$ (1)

If $\Psi_1 = \psi_1(x,y,z)\exp(-2\pi\nu_1 t)$, $\Psi_2 = \psi_2(x,y,z)\exp(-2\pi\nu_2 t)$, ⋯⋯ are solutions of the Schrödinger equation above, show that the linear combination of Ψ_1, Ψ_2 , ⋯⋯is also the solution of the Schrödinger equation.

We have $\mathcal{H}\psi_1=E_1\psi_1$, $\mathcal{H}\psi_2=E_2\psi_2$, ⋯⋯

Linear combination of Ψ_1 and Ψ_2 ⋯⋯ can be expressed as $\Psi = c_1\Psi_1+c_2\Psi_2+\cdots\cdots$

The left side of the eq. (1): $\hat{\mathcal{H}}\Psi=\hat{\mathcal{H}}(c_1\Psi_1+c_2\Psi_2+\cdots\cdots)=c_1E_1\Psi_1+c_2E_2\Psi_2+\cdots\cdots$

where E_1=hν_1, E_2=hν_2, ⋯⋯

The right side of the eq. (1): $i\hbar \frac{\partial}{\partial t}\Psi$ =- $i\hbar\{c_1(2\pi\nu_1)\Psi_1+2\pi\nu_2)\Psi_2+\cdots\cdots\}=c_1E_1\Psi_1+c_2E_2\Psi_2+\cdots\cdots$

Since the left and right sides are equal, the linear combination of Ψ_1, Ψ_2 , ⋯⋯is also the solution of the Schrödinger equation (1).

Problem 2. (20 points)

Show that

(a) $[x, P_x] = [y, P_y] = [z, P_z] = i\hbar$

(b) $[x^2, p_x] = 2i\hbar x$

(Hints: The operator of the momentum $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, $[\hat{A}, \hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$)

(a) $[x, P_x]\psi=xP_x\psi-P_x x\psi=x(-i\hbar\frac{\partial}{\partial x})\psi+i\hbar\frac{\partial}{\partial x}(x\psi)=-i\hbar x\frac{\partial\psi}{\partial x}+i\hbar x\frac{\partial\psi}{\partial x}+i\hbar\psi=i\hbar\psi$ → $[x, P_x]= i\hbar$

(b) $[x^2, p_x]\psi=x^2(-i\hbar\frac{\partial}{\partial x})\psi+i\hbar\frac{\partial}{\partial x}(x^2\psi)=-i\hbar x^2\frac{\partial\psi}{\partial x}+i\hbar x^2\frac{\partial\psi}{\partial x}+i\hbar(2x)\psi=2i\hbar x\psi$ → $[x^2, P_x]=2i\hbar x$

Problem 3. (20 points)

(a) If $\psi(x) = \frac{N}{x^2+a^2}$, calculate the normalization constant N.

(b) Given $\psi(x) =\left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4}}\exp(-\frac{\alpha^2 x^2}{2})$, calculate the expectation values <x> and <x^2>.

(a) Normalization condition is $\int_{-\infty}^{\infty}|\psi|^2 dx = 1$

$N^2\int_{-\infty}^{\infty}(x^2+a^2)^{-2}dx = 1$

Put $x=a\cdot\tan\theta$; $dx=a\cdot\sec^2\theta\, d\theta$

$\left(\frac{2N^2}{a^3}\right)\int_0^{\pi/2}\cos^2\theta\, d\theta=\frac{\pi N^2}{2a^3}=1$

Therefore, N=$\left(\frac{2a^3}{\pi}\right)^{1/2}$;

$$\int_0^\infty (x^2+1^2)^{-2}\,dx$$

$$\frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} (\mathrm{Cos}[x])^2\,dx$$

$$\frac{\pi}{4}$$

(b) $\psi(x) = \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4}}\exp\left(-\frac{\alpha^2}{2}x^2\right)$

The expectation value <x>=$\int_{-\infty}^{\infty}\psi^* x\psi\, dx$=0, because ψ and ψ^* are even functions while x is an odd function. Therefore, <x>=0

$<x^2>=\left(\frac{\pi}{\alpha}\right)^{-\frac{1}{2}}\int_{-\infty}^{\infty}x^2\exp(-\alpha^2 x^2)dx$

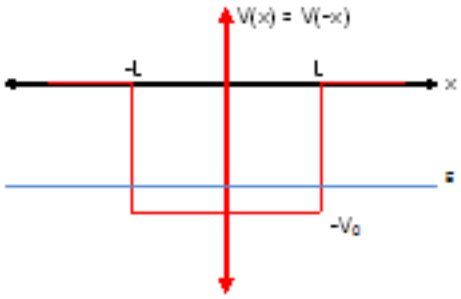
Put $\alpha^2 x^2$=y; dx = $\frac{dy}{2\alpha\sqrt{y}}$

$$<x^2> = (\pi\alpha^5)^{-\frac{1}{2}}\int_0^\infty y^{\frac{1}{2}} e^{-y}\, dy = (\pi\alpha^5)^{-\frac{1}{2}} \times \frac{\sqrt{\pi}}{2} = \frac{1}{2\sqrt{\alpha^5}}$$

$$\int_0^\infty \sqrt{y}\ \mathrm{Exp}[-y]\,dy$$

$$\frac{\sqrt{\pi}}{2}$$

Problem 4. (20 points)



(1) Write forms of Ψ(x) in the three domains for odd Ψ(x).
(2) Write a boundary condition for continuity of Ψ.
(3) Write a boundary condition for continuity of ∂Ψ.
(4) Show that you get k = -/cot(/L). $k=\frac{\sqrt{-2mE}}{\hbar}$, /=$\frac{\sqrt{2m(E+V_0)}}{\hbar}$

See the viewgraphs on Schrödinger equation for quantum well potential.
(1) x≤-L: Ψ(x) = Be^{kx}, -L<x<L: Ψ(x) = Csin(lx), x≥L: Ψ(x) = $-Be^{-kx}$ (B, C: constants, $/^2=(E+V_0)\,\frac{2m}{\hbar^2}$)
(2) Continuity of Ψ: $-Be^{-kL}$=Csin(lL) ····· (a)
(3) Continuity of ∂Ψ: kBe^{-kL}= lCcos(lL) ····· (b)
(4) (b) ÷ (a): -k = /cot(/L) ∴ k = -/cot(/L)

Problem 5. (20 points)
A particle is trapped in a one dimensional potential given by = $kx^2/2$. At a time t = 0 the state of the

particle is described by the wave function $\psi =$
$C_1\psi_1 + C_2\psi_2$, where ψ is the eigen function belonging to the eigen value E_n.
What is the expected value of the energy?

The form of potential corresponds to that of a linear Simple harmonic Oscillator.
The energy of the oscillator will be $E_1 = \frac{\hbar\omega}{2}$ and $E_2 = \frac{3\hbar\omega}{2}$.
$< E > = < \psi|\hat{\mathcal{H}}|\psi > = < (C_1\psi_1 + C_2\psi_2)|\hat{\mathcal{H}}|(C_1\psi_1 + C_2\psi_2) > = < (C_1\psi_1 + C_2\psi_2)|(C_1\hat{\mathcal{H}}\psi_1 + C_2\hat{\mathcal{H}}\psi_2) >$
$= < (C_1\psi_1 + C_2\psi_2)|(C_1E_1\psi_1 + C_2E_2\psi_2) > = C_1^2E_1 + C_2^2E_2 = \frac{\hbar\omega}{2}C_1^2 + \frac{3\hbar\omega}{2}C_2^2$
where $\omega = \left(\frac{k}{m}\right)^{1/2}$ (m: mass of the particle)

Problem 6. (20 points)
Consider a particle described by a wavefunction: $\psi = e^{ix} + 2ie^{3ix}$ $(-\pi \le x \le \pi)$
(a) Normalize the wavefunction.
(b) If you precisely measure the momentum of the state expressed by the wavefunction ψ, what values can you obtain and what probability cay you get, respectively? (Hints: See the wavefunction ψ as the superposition of the two waves. Deduce the momentum of each wave. The operator of the momentum $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$)

(a) $\psi^*\psi = (e^{-ix} - 2ie^{-3ix})(e^{ix} + 2ie^{3ix}) = 5 - 2ie^{-2ix} + 2ie^{2ix}$
$\int_{-\pi}^{\pi}\psi^*\psi\,dx = \int_{-\pi}^{\pi}(5 - 2ie^{-2ix} + 2ie^{2ix})\,dx = 10\pi$
Therefore, the normalized function is $\boldsymbol{\psi = \frac{1}{\sqrt{10\pi}}(e^{ix} + 2ie^{3ix})}$

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(e^-ix-2ie^(-3ix)) (e^ix+2ie^(3ix))  »
(E^(-(I*x)) - (2*I)/E^(3*I*x))*(E^(I*x) + 2*I*E^(3*I*x))
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$(e^{-ix} - 2ie^{-3ix})(e^{ix} + 2ie^{3ix})$

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Simplify[(e^(-i x) - 2 i e^(-3 i x)) (e^(i x) + 2 i e^(3 i x))]
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$5 - 2ie^{-2ix} + 2ie^{2ix}$

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Simplify[(e^(-i x) + 2 i e^(-3 i x)) (e^(i x) + 2 i e^(3 i x))]
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$-3 + 2ie^{-2ix} + 2ie^{2ix}$

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$$\int_{-\pi}^{\pi}(5 - 2ie^{-2ix} + 2ie^{2ix})\,dx$$

10π

(b) ψ can be seen as the superposition of the two waves: e^{ix} and e^{-3ix}. From the each wave, $\hat{p}_x e^{ix} = -i\hbar\frac{\partial}{\partial x}e^{ix} = \hbar e^{ix}$, $\hat{p}_x e^{-3ix} = -i\hbar\frac{\partial}{\partial x}e^{-3ix} = 3\hbar e^{-3ix}$. Therefore, **the values of the momentum are $\hbar$ and $3\hbar$**. Since the ratio of coefficients of the two wavefunctions is 1:2, the ratio of the probability is 1:4. Therefore, **the probability for the momentum $\hbar$ is $\frac{1}{5}$, and the probability for $3\hbar$ is $\frac{4}{5}$.**