Problem set #4 (Modern Physics)

05/24/2018 Provided by *Masahito Oh-e*

Solve the problems below. Describe the ways of thinking in English: only final solutions are not accepted. Make clear how you reach each solution.

As official homework, you are not forced to do the problem 1', but for those who are interested in deriving Schrödinger equation expressed by spherical polar coordinates, the problem 1' is prepared. If you work on problem 1' and properly solve it, you can get some bonus points.

Due is on June 4 (Mon). Delayed submission will not be accepted without any reasonable reason.

Problem 1.

Express $\frac{\partial^2}{\partial z^2}$ by spherical-polar coordinates. Use the relations below.

$$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

Problem 1'. (For bonus)

In the same way of problem 1, derive $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ by spherical coordinates and derive how kinetic energy hamiltonian $\hat{\mathcal{H}}=-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})$ can be expressed in spherical coordinates.

Problem 2.

Using the commutator $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, and its cyclic variants, prove that the total angular momentum squared and the individual components of angular momentum commute, i.e $[\hat{L}^2, \hat{L}_x] = 0$ etc.

Problem 3.

The Schrödinger wave function for the stationary state of an atom is $\psi = Af(r)\sin\theta\cos\theta\cdot e^{i\phi}$ where (r, θ, ϕ) are spherical polar coordinates.

(a) Find the z component of the angular momentum of the atom.

(b) Find the square of the total angular momentum of the atom.

Problem 4.

Calculate the expectation value <r> of an electron in the state of n=1 and l=0 of the hydrogen atom. r is the position from the nucleus. Use the wave functions appropriately in Table 6-1 of the textbook. You can use the integration of $\int_0^\infty x^n \exp(-ax)\,dx = \frac{n!}{a^{n+1}}$ (n>-1, a>0).

Problem 5.

Assume the Hamiltonian of a particle in three dimension is expressed by spherical-polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$) as below:

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\mathrm{Cos}\theta}{r^2\,\mathrm{Sin}\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\,\mathrm{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2}\right]-\frac{k}{r}\quad \text{(k: the positive real constant)}$$

If the wavefunction: $\psi = Ne^{-br}$ (N, b: the positive real constant) is given, determine the value of b so that ψ is the eigenfunction of $\hat{\mathcal{H}}$ and also derive the eigenvalue.