Atomic structure

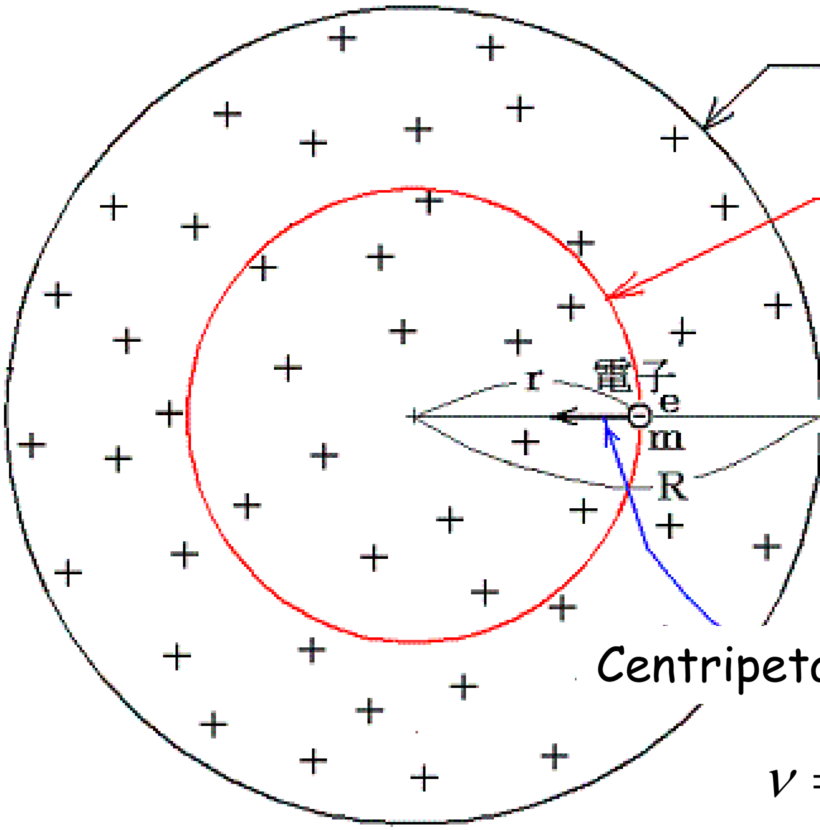
✔ In the 19th century, it was known that matter was made of different chemical elements consisting of individual atoms. Not very much was known about the constituents of the atoms.

✔ With the discovery of the electron, it became clear that atoms would contain negatively charged electrons and that some other part of the atom would need to contain positive charges to realize a neutral atom.

✔ Realizing that electrons are much lighter than any atoms it was found that most of mass of the atom should be carried by its positively charged components.

✔ Thomson (1898) model of the atom: homogeneously distributed positively charged matter with interspersed electrons.

J.J.Thomson's model (Year 1904)



Total of + charge

$$\frac{4}{3}\pi R^3 \times \rho = eZ$$

ρ : + density [C/m³]

+ charge within the radius r

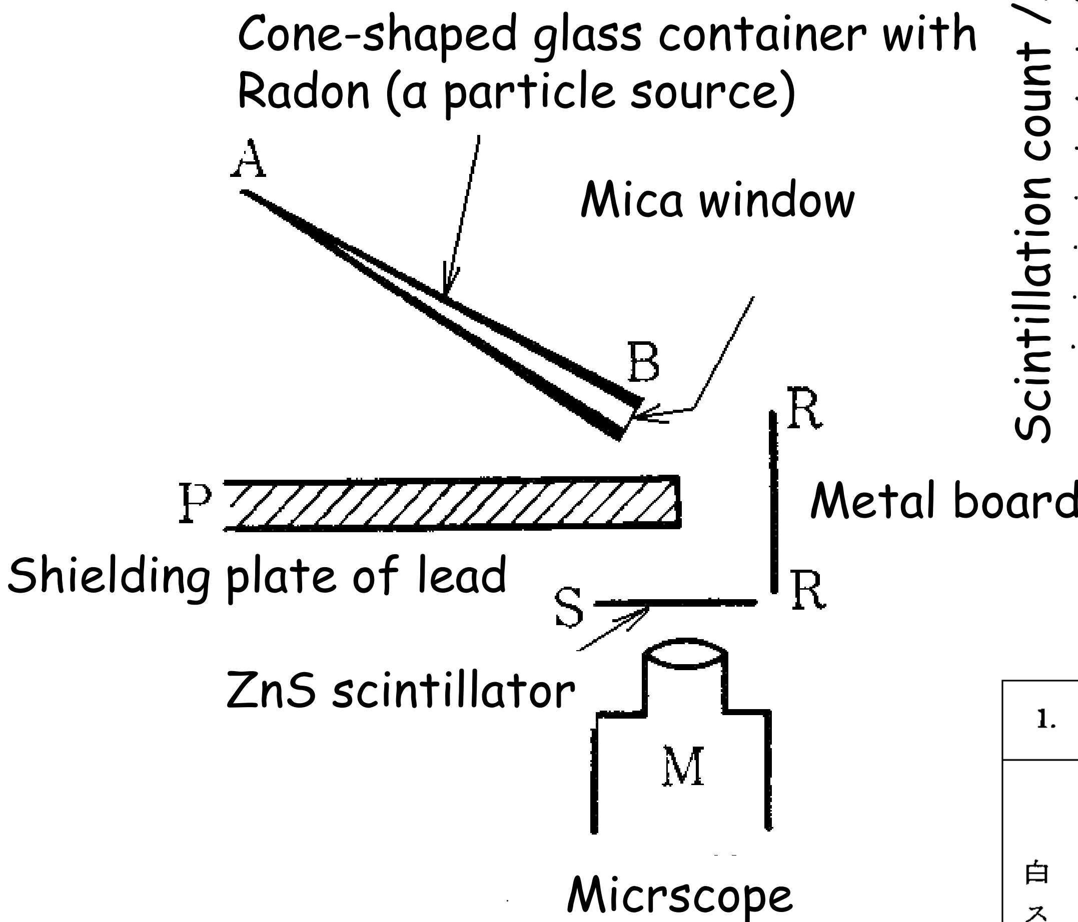
$$\frac{4}{3}\pi r^3 \times \rho = \left(\frac{4}{3}\pi R^3 \times \rho\right) \times \left(\frac{r}{R}\right)^3 = eZ\left(\frac{r}{R}\right)^3$$

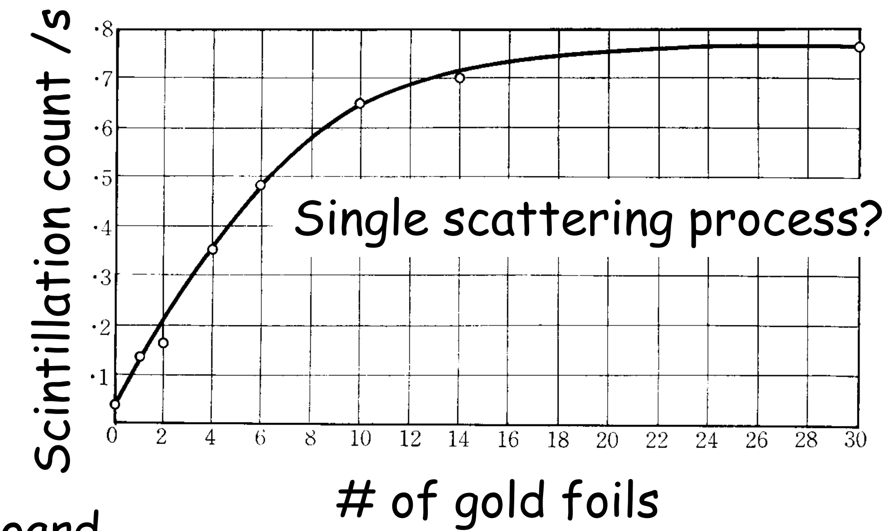
Centripetal force $F = -k_0 \dfrac{e \times \left\{eZ\left(\frac{r}{R}\right)^3\right\}}{r^2} = -\underbrace{\left(k_0 \dfrac{e^2 Z}{R^3}\right)}_{\parallel\ k} r$

$$\nu = \frac{3\times10^8\, m/s}{10^{-7}\, m} \approx 10^{16}\, Hz$$

$$R = \left(\frac{k_0 e^2 Z}{m \nu^2 4\pi^2}\right)^{\frac{1}{3}} \approx 10^{-10}\, m$$

Discovery of large scattering of a particles (1909)





A gold foil ~ 0.4mm

1. 金　属	2. 原子量 A	3. 1秒間のシンチレーションの数, Z	4. A/Z
鉛	207	62	30
金	197	67	34
白　金	195	63	33
ス　ズ	119	34	28
銀	108	27	25
銅	64	14. 5	28
鉄	56	10. 2	18. 5
アルミニウム	27	3. 4	12. 5

Thomson's model cannot explain the large scattering of a particles

1) Scattering angle by electrons in an atom

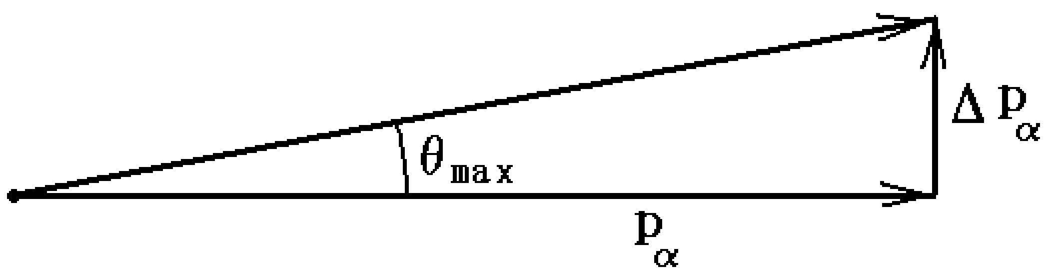
$$m_\alpha v_\alpha = m_\alpha v'_\alpha + m_e v'_e$$

$$\frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{2} m_\alpha v'^2_\alpha + \frac{1}{2} m_e v'^2_e$$

$$v'_e = \frac{2m_\alpha}{m_\alpha + m_e} v'_\alpha \approx 2v_\alpha$$

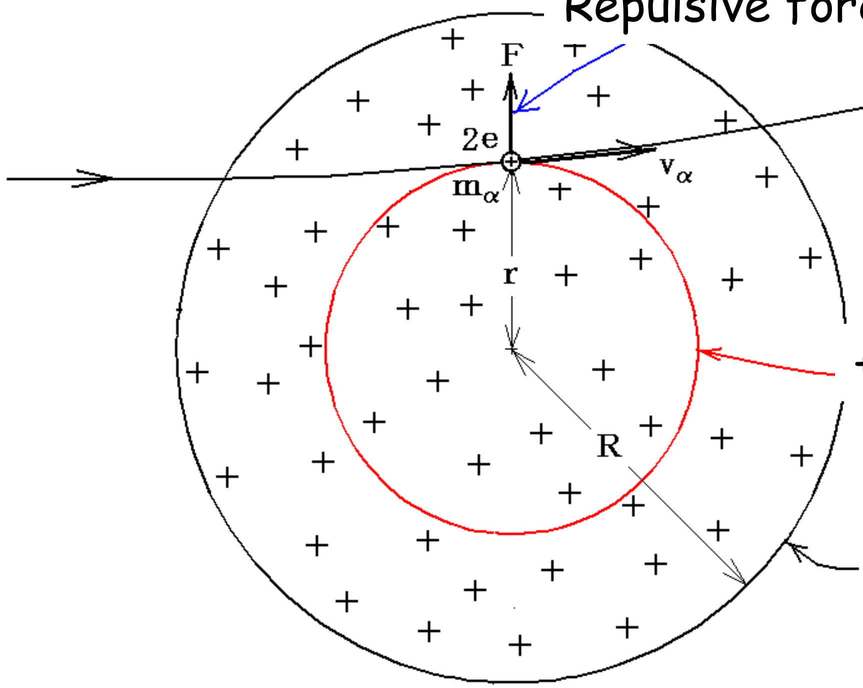
Change of momentum at maximum:

$$2m_e v_\alpha = \Delta p_\alpha$$



$$\theta_{max} \sim \frac{\Delta p_\alpha}{p_\alpha} \sim \frac{2 m_e v_\alpha}{4 \times 1836 \times m_e v_\alpha} \sim 10^{-4} \text{ rad}$$

2) Scattering angle by positive charge



Repulsive force $F = k_0 \dfrac{2e \times \left\{ eZ \left(\frac{r}{R}\right)^3 \right\}}{r^2} = \left(k_0 \dfrac{2e^2 Z}{R^3}\right) r$

Maximum repulsive force F_{max}

$$F_{\max} \sim k_0 \frac{2e^2 Z}{R^2}$$

+ charge within the radius r

$$\frac{4}{3}\pi r^3 \times \rho = \left(\frac{4}{3}\pi R^3 \times \rho\right) \times \left(\frac{r}{R}\right)^3 = eZ\left(\frac{r}{R}\right)^3$$

Total of + charge $\quad \frac{4}{3}\pi R^3 \times \rho = eZ$

ρ + density [C/m³]

Time of a particles passing through an atom at maximum: $\dfrac{2R}{v_\alpha}$

$$\Delta p_\alpha \sim F_{\max} \cdot \Delta t \sim k_0 \frac{2e^2 z}{R^2} \frac{2R}{v_\alpha} \sim k_0 \frac{4e^2 Z}{R v_\alpha}$$

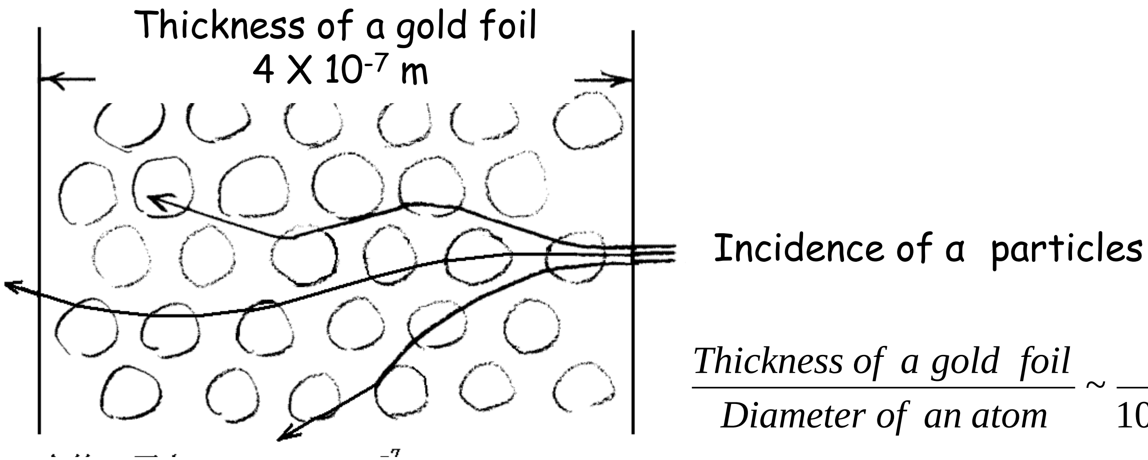
$k_0 \sim 9 \times 10^9 \, kgm^3/s^2C^2 \qquad R \sim 10^{-10} m$

$e \sim 1.6 \times 10^{-9} C \qquad v_\alpha \sim 1 \times 10^7 m/s$

$m_\alpha \sim 7 \times 10^{-27} kg \qquad Z \sim 80$

$$\theta_{\max} \sim \frac{\Delta p_\alpha}{p_\alpha} \sim \frac{k_0 \dfrac{4e^2 Z}{R v_\alpha}}{m_\alpha v_\alpha} \sim \frac{k_0 4e^2 Z}{m_\alpha R v_\alpha^2} \sim 10^{-3} rad$$

3) Estimate of angle of scattered particles



$$\frac{\textit{Thickness of a gold foil}}{\textit{Diameter of an atom}} \sim \frac{4\times10^{-7}\,m}{10^{-10}\,m/atom} \sim 4\times10^{3}\ atoms$$

Suppose 1000 atoms are lined up in the direction of thickness, how much degree particles can be scattered is expressed by the random walk model.

✔ Probability of the right and left direction per atom: ½ and ½ each

Straight: $\left(\frac{1}{2}\right)^{1000} \times \frac{1000!}{500!\,500!}$ 0.2°: $\left(\frac{1}{2}\right)^{1000} \times \frac{1000!}{501!\,499!}$ 0.4°: $\left(\frac{1}{2}\right)^{1000} \times \frac{1000!}{502!\,498!}$

100°: $\left(\frac{1}{2}\right)^{1000} = \frac{1}{2^{1000}} \approx \frac{1}{10^{300}}$

Geiger and Marsden's experiment:

Squared mean of scattering angle: ~ 1°

→ < 3° (99%)

Scattering: 10^{-4} rad, 1000~10000 times

→ Probability: ~10^{-3000}

Rutherford's assumption

1. Positive charge of atom: ~Ze

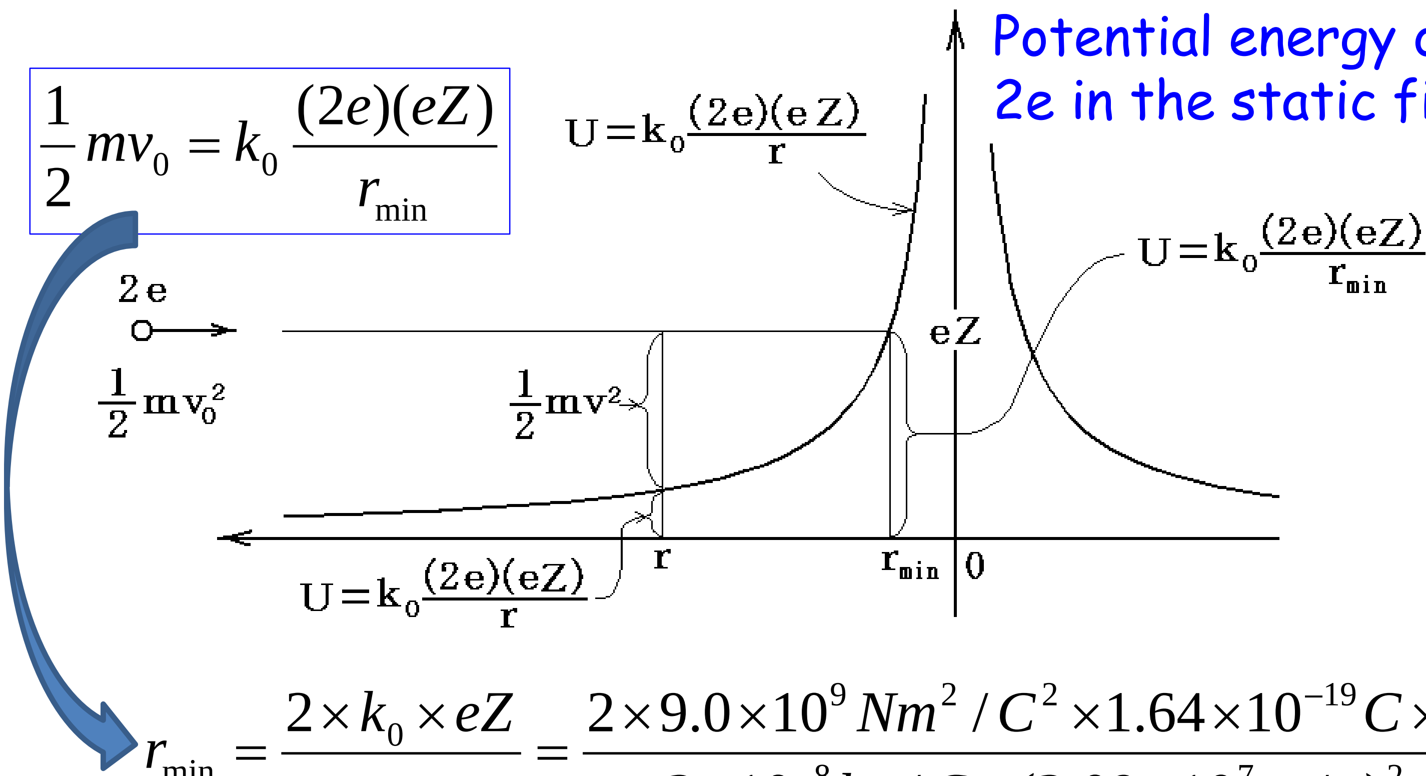
2. The mass of the scatter: the mass of α particle.
   (Momentum conservation)

3. Positive charge is concentrated in one point.
   (Strong electric field)

Rutherford's single scattering model (Nuclear atom model) (1911)

$$\frac{1}{2}mv_0 = k_0\frac{(2e)(eZ)}{r_{\min}}$$

Potential energy of the presence of 2e in the static field created by eZ



$$r_{\min} = \frac{2\times k_0\times eZ}{(\frac{m}{2e})\times v_0^2} = \frac{2\times 9.0\times 10^9\,Nm^2/C^2\times 1.64\times 10^{-19}C\times Z}{2\times 10^{-8}kg/C\times(2.09\times 10^7 m/s)^2} = 3\times Z\times 10^{-16}m$$

$$\sim 10^{-13}m \quad (cf.10^{-10}m)$$

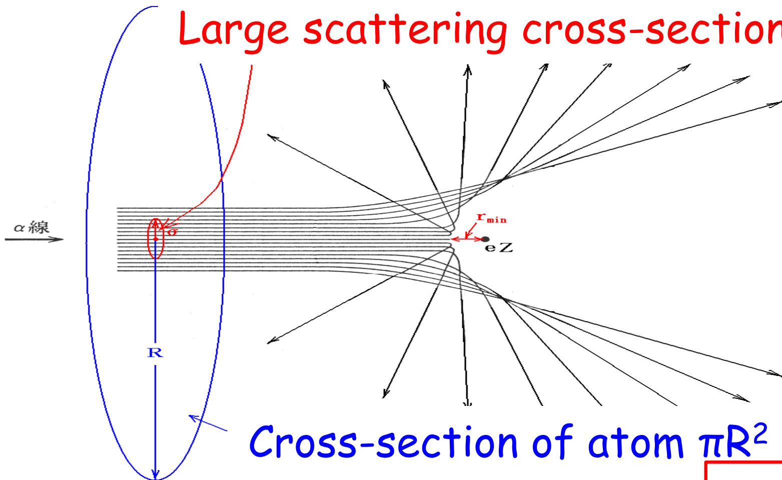
$$v_0 = 2.09\times 10^7 m/s$$

1000 times smaller than the atomic size

$$\alpha \ particle's\ mass-charge\ ratio\ \frac{m}{2e} = 2\times 10^{-8}kg/C$$

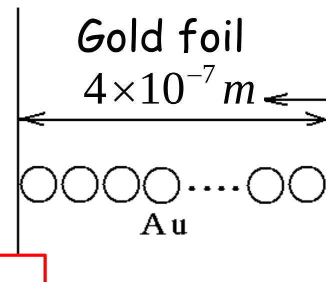
Probability of large angle scattering

Large scattering cross-section $\pi\sigma^2$



Cross-section of atom πR^2

$$\frac{\pi\sigma^2}{\pi R^2} \propto \frac{\text{\# of largely scattered particle}}{\text{\# of incident particle}}$$



Thickness

$$\frac{4\times10^{-7}\,m}{2.57\times10^{-10}\,\text{m}/\text{個}} = 1.56\times10^{3}\,\text{個}$$

\# of atoms lined up

$$\boxed{\frac{1}{20000}}$$: Probability of large scattering

Mass of Au

$197\,\text{g} \longleftrightarrow 6.0\times10^{23}\,\text{個}$

Avogadro number

Density of Au

$$1.93\times10^{7}\,\text{g} \longleftrightarrow 6.0\times10^{23}\,\frac{1.93\times10^{7}}{197}\,\text{個} \longleftrightarrow 1\,\text{m}^3$$

$$1\,\text{個} \longleftrightarrow 1\times\frac{1}{6.0\times10^{23}\,\dfrac{1.93\times10^{7}}{197}}\,\text{m}^3$$

$$= 17\times10^{-30}\,\text{m}^3$$

Volume /1 Au atom

Diameter of Au atom $\sqrt[3]{17\times10^{-30}\,\text{m}^3} = 2.57\times10^{-10}\,\text{m}$

Experimental probability of large scattering: 1/20000 with a gold foil that has 1.56×10^3 atomic layers in thickness.

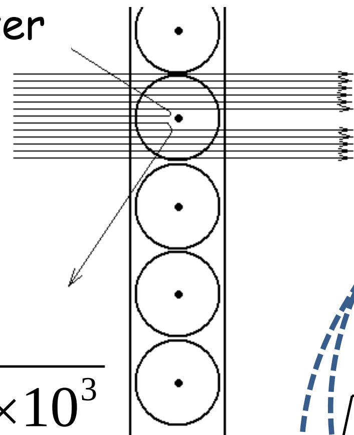
Assuming

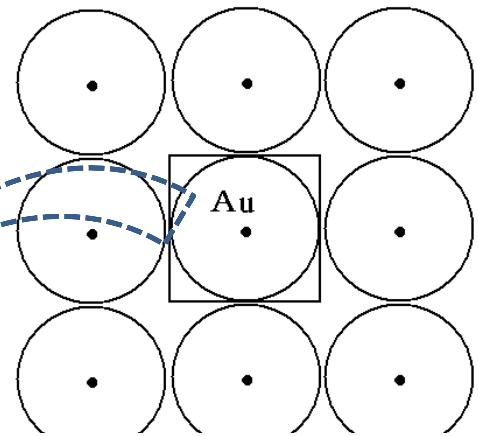
Single layer

Probability of large scattering

$$\frac{1}{20000 \times 1.56 \times 10^3}$$

Gold foil



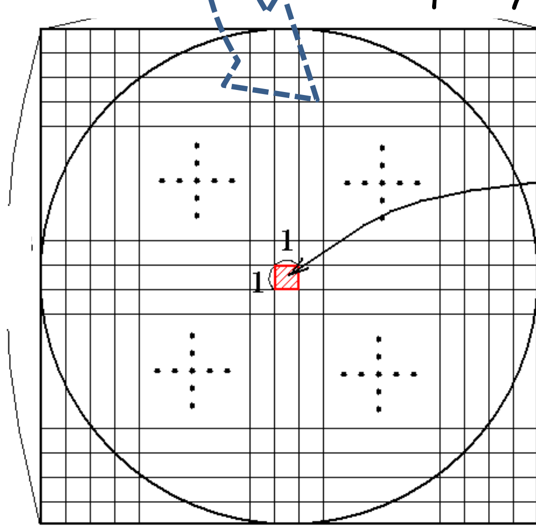


$$\sqrt{20000 \times 1.56 \times 10^3} = \sqrt{3.12 \times 10^7} = 5.6 \times 10^3$$

Single layer in two dimension

5.6×10^3 divided equally

5.6×10^3 divided equally



This area corresponds to $\pi\sigma^2 \sim \pi r_{min}^2$

$\sigma \sim r_{min}$: ~1/10000 of Au atom

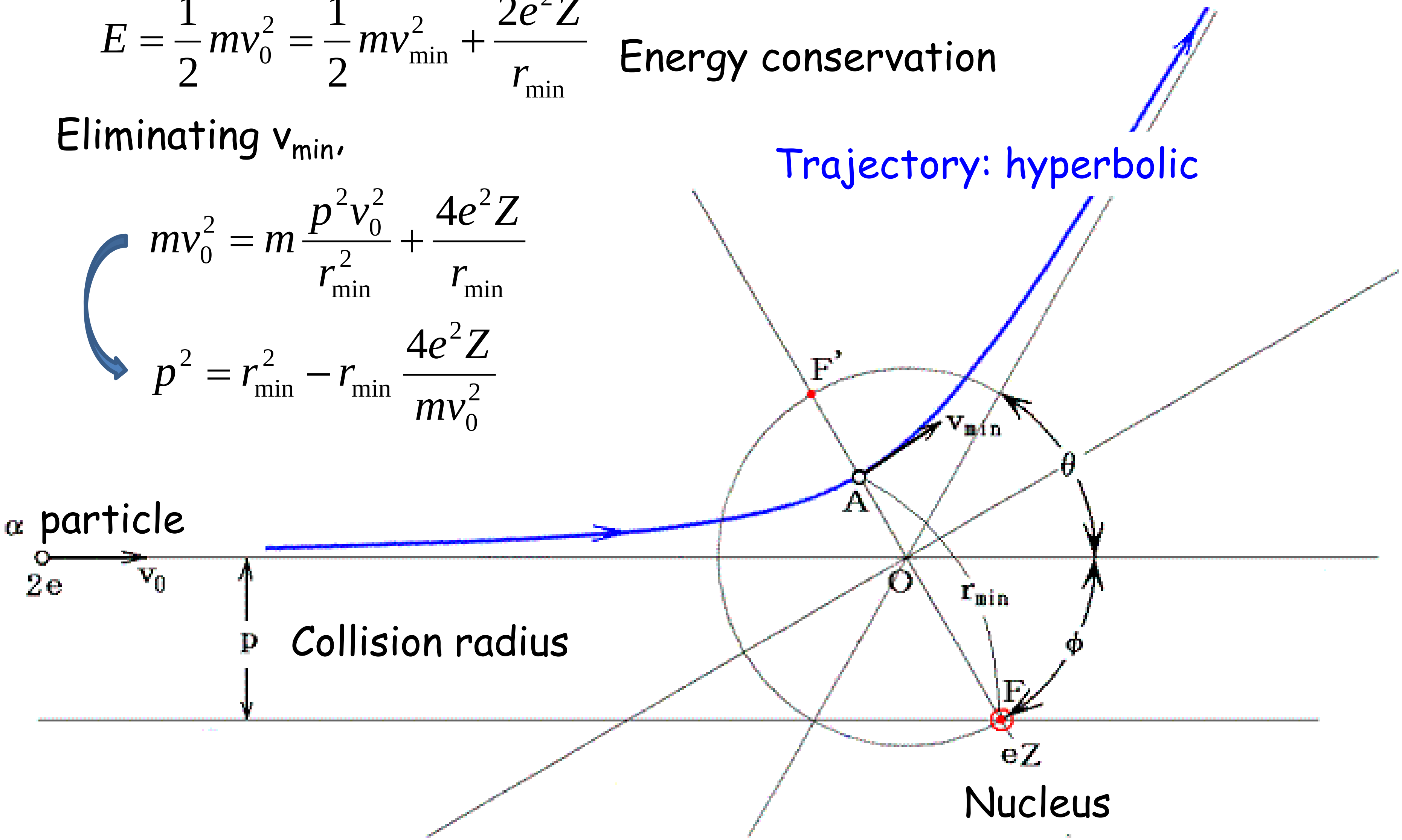
$$L = mpv_0 = mr_{\min} v_{\min}$$ Angular momentum conservation

$$E = \frac{1}{2} mv_0^2 = \frac{1}{2} mv_{\min}^2 + \frac{2e^2 Z}{r_{\min}}$$ Energy conservation

Eliminating $v_{\min}$,

$$mv_0^2 = m\frac{p^2 v_0^2}{r_{\min}^2} + \frac{4e^2 Z}{r_{\min}}$$

$$p^2 = r_{\min}^2 - r_{\min}\frac{4e^2 Z}{mv_0^2}$$



$\Delta OAB \equiv \Delta ONF \rightarrow b = p$

$$\tan\frac{\theta}{2} = \frac{a}{p}, \quad \cos\frac{\theta}{2} = \frac{p}{c}$$

$$r_{\min} = a + c = p\tan\frac{\theta}{2} + \frac{p}{\cos\frac{\theta}{2}} = p\,\frac{\sin\frac{\theta}{2}+1}{\cos\frac{\theta}{2}}$$

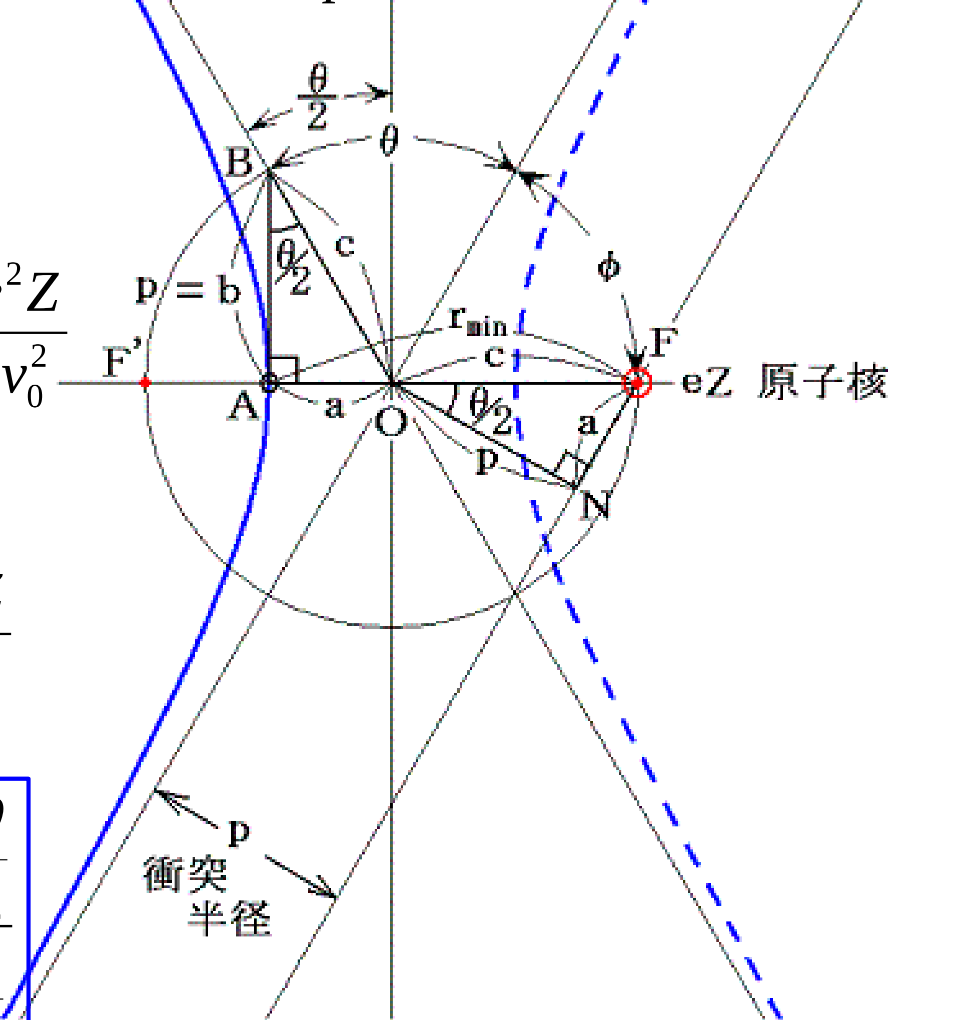
$$p^2 = r_{\min}^2 - r_{\min}\frac{4e^2Z}{mv_0^2}$$

$$p^2 = p^2\left(\frac{\sin\frac{\theta}{2}+1}{\cos\frac{\theta}{2}}\right)^2 - p\left(\frac{\sin\frac{\theta}{2}+1}{\cos\frac{\theta}{2}}\right)\frac{4e^2Z}{mv_0^2}$$

$$p\,\frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+1\right)}{\cos^2\frac{\theta}{2}} = p\,\frac{\sin\frac{\theta}{2}+1}{\cos\frac{\theta}{2}}\,\frac{4e^2Z}{mv_0^2}$$

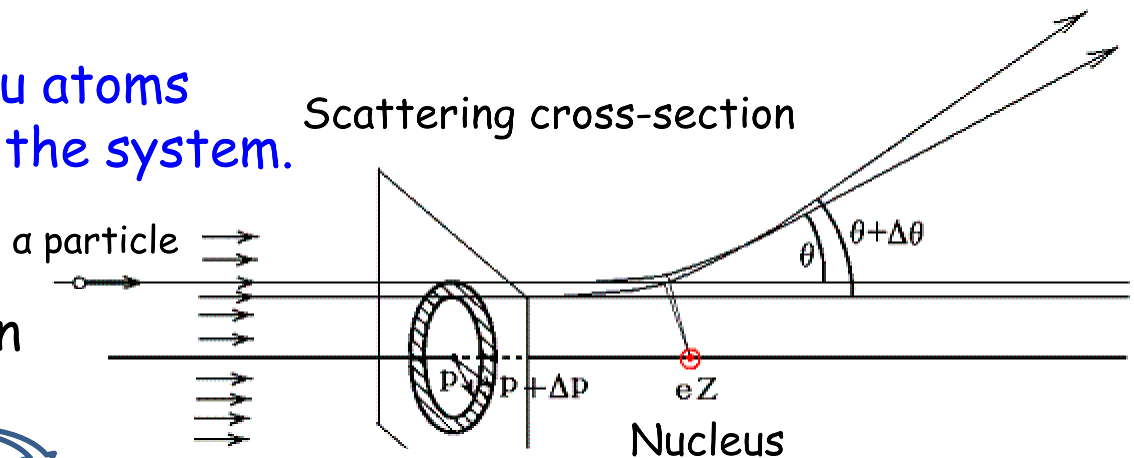
$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{4e^2Z}{2pmv_0^2}$$

$$p = \frac{2e^2Z}{mv_0^2}\,\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$



P & eZ → scattered in θ direction

Suppose there are N/m^3 Au atoms and n/m^2 α particles enter the system.



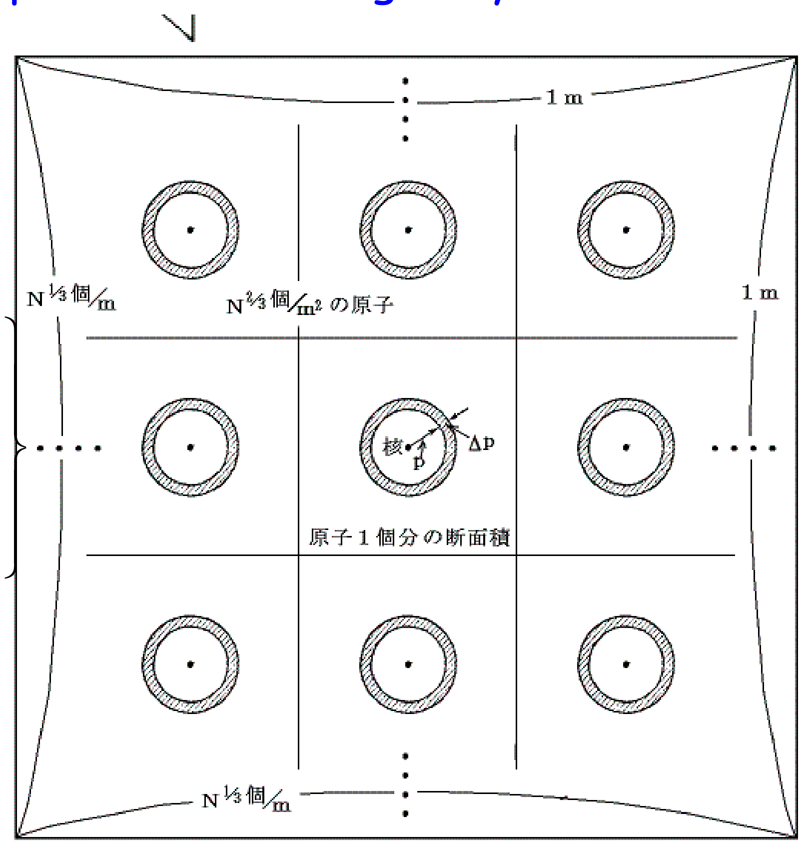
# of scattered particles in θ~θ+Δθ per unit time

# of scattered particles in a single layer

$$= n/s \times \frac{(Area\ of\ \ a\ ring) \times N^{2/3}}{1\,m^2} \times N^{1/3}$$

$$= n \times \left\{ \pi (p + \Delta p)^2 - \pi p^2 \right\} \times N$$

$$= n\,N \times \left\{ \pi \left(\frac{2e^2 Z}{m v_0^2} \frac{\cos\frac{\theta+\Delta\theta}{2}}{\sin\frac{\theta+\Delta\theta}{2}} \right)^2 - \pi \left(\frac{2e^2 Z}{m v_0^2} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \right)^2 \right\}$$



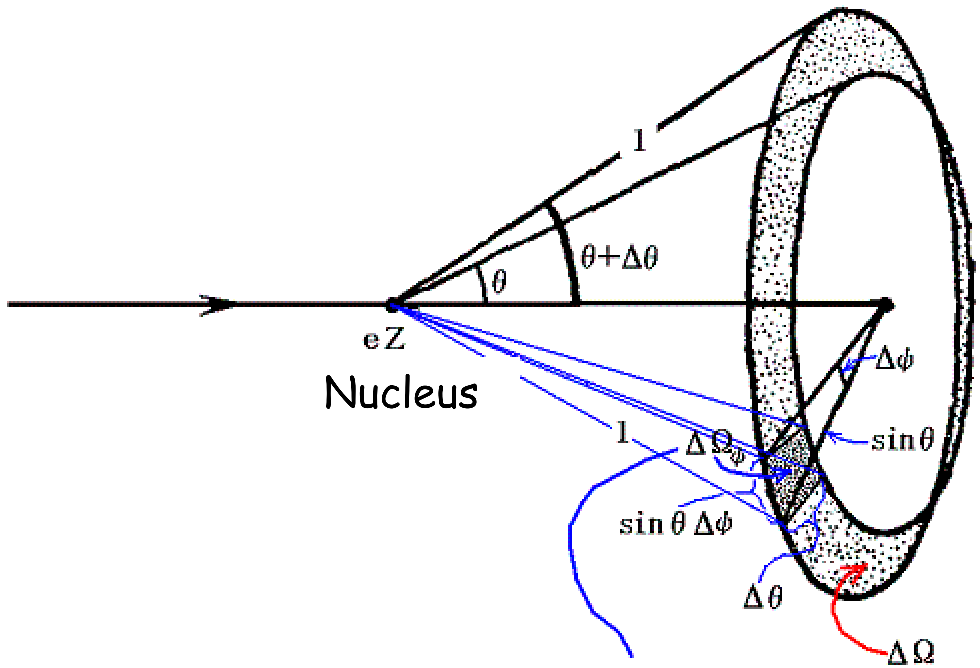
$$\cos\left(\frac{\theta}{2}+\frac{\Delta\theta}{2}\right)=\cos\frac{\theta}{2}\cos\frac{\Delta\theta}{2}-\sin\frac{\theta}{2}\sin\frac{\Delta\theta}{2}\approx\cos\frac{\theta}{2}-\frac{\Delta\theta}{2}\sin\frac{\theta}{2}$$

$$\sin\left(\frac{\theta}{2}+\frac{\Delta\theta}{2}\right)=\sin\frac{\theta}{2}\cos\frac{\Delta\theta}{2}+\cos\frac{\theta}{2}\sin\frac{\Delta\theta}{2}\approx\sin\frac{\theta}{2}+\frac{\Delta\theta}{2}\cos\frac{\theta}{2}$$

$$\approx n\,N\times\pi\left(\frac{2e^2Z}{mv_0^2}\right)^2\left\{\frac{\left(\cos\frac{\theta}{2}-\frac{\Delta\theta}{2}\sin\frac{\theta}{2}\right)^2}{\left(\sin\frac{\theta}{2}+\frac{\Delta\theta}{2}\cos\frac{\theta}{2}\right)^2}-\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right\}$$

$$\approx n\,N\times\pi\left(\frac{2e^2Z}{mv_0^2}\right)^2\left\{\frac{\left(\cos\frac{\theta}{2}-\frac{\Delta\theta}{2}\sin\frac{\theta}{2}\right)^2\sin^2\frac{\theta}{2}-\left(\sin\frac{\theta}{2}+\frac{\Delta\theta}{2}\cos\frac{\theta}{2}\right)^2\cos^2\frac{\theta}{2}}{\left(\sin\frac{\theta}{2}+\frac{\Delta\theta}{2}\cos\frac{\theta}{2}\right)^2\sin^2\frac{\theta}{2}}\right\}$$

$$\approx -n\,N\times\pi\left(\frac{2e^2Z}{mv_0^2}\right)^2\frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\Delta\theta$$



Solid angle $\Delta\Omega_\phi = \dfrac{(R\Delta\theta)(R\sin\theta\Delta\phi)}{R^2} = \sin\theta\,\Delta\theta\,\Delta\phi$

φ: Integrate 0~ 2π $\quad \Delta\Omega = 2\pi\Delta\Omega_\phi = 2\pi\sin\theta\,\Delta\theta = 4\pi\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\Delta\theta$

# of α particles (θ+Δθ, unit time) $= nN\left(\dfrac{e^2 Z}{m v_0^2}\right)^2 \dfrac{\Delta\Omega}{\sin^4\dfrac{\theta}{2}}$

Experimentally verified!!

Summary (What Rutherford did.)

✔ Rutherford discovered that atoms consist of small and heavy nucleus surrounded by electrons.

✔ Rutherford proposed a way to confirm his proposed atomic structure and verified the hypothesis.

✔ The magnitude of charge at the center can be determined.

✔ The possible maximum size of nucleus can be estimated from the ratio of the number of the largely scattered a particle with that of the incident a particle.