Group and phase velocity as a function of total relativistic energy and momentum of a particle

The formula of equivalence of mass and energy: $E = mc^2$

The formula of the relativistic momentum: $p = mv_g$

The formula of the relativistic mass: $m = \dfrac{m_0}{\sqrt{1-\dfrac{v_g^2}{c^2}}}$

E = total relativistic energy of a particle
m = relativistic mass of a particle
m_0 = rest mass of a particle
p = relativistic momentum of a particle
v_g = group velocity of a particle
c = speed of light in vacuum

$$\frac{E}{p} = \frac{c^2}{v_g}$$

The dimensions of the equation:
s = E/p is a velocity

Since the group velocity is always smaller than the speed of light, this velocity, s, must be greater than the speed of light, c.

$$v_f = \frac{E}{p}$$

Phase Velocity:

The phase velocity of any particle (massive or massless) is equal to its total relativistic energy divided by its momentum.

$$v_f v_g = c^2$$

The Product of the Phase Velocity and the Group Velocity:

The product between the phase velocity and the group velocity of any particle (massive or massless) equals the square of the speed of light in vacuum.

The Einstein's total relativistic energy formula:

$$E^2 = p^2c^2 + m_0^2c^4$$

$$\frac{d}{dp}\left(E^2\right) = \frac{d}{dp}p^2c^2 + m_0^2c^4 \quad \Rightarrow \quad 2E\frac{dE}{dp} = 2c^2p\frac{dp}{dp}$$

$$\frac{dE}{dp} = \frac{pc^2}{E} \quad \Rightarrow \quad \frac{dE}{dp} = \frac{pc^2}{mc^2} = \frac{p}{m}$$

$$\frac{dE}{dp} = v_g$$

Group Velocity:

The group velocity of any particle (massive or massless) is equal to the derivative of its total relativistic energy with respect to its relativistic momentum.

As a function of the deBroglie and the Compton wavelengths of a particle

$$v_f = \frac{E}{p}$$

$$p = \frac{h}{\lambda}$$ the de Broglie law

h = Planck's constant
λ = de Broglie wavelength

The Einstein's total relativistic energy formula:

$$E^2 = p^2c^2 + p_c^2c^2$$

$$p_c \equiv m_0 c$$ Compton momentum

The Compton wavelength of a particle of rest mass m_0

$$\lambda_c \equiv \frac{h}{p_c} = \frac{h}{m_0 c}$$

$$E^2 = c^2\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda_c^2}\right) = h^2c^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda_c^2}\right) = \frac{h^2c^2}{\lambda^2}\left(1 + \frac{\lambda^2}{\lambda_c^2}\right)$$

$$\Rightarrow \quad E = \frac{hc}{\lambda}\sqrt{1 + \frac{\lambda^2}{\lambda_c^2}}$$

$$\Rightarrow \quad v_f = \frac{E}{p} = \left(\frac{hc}{\lambda}\sqrt{1 + \frac{\lambda^2}{\lambda_c^2}}\right)\frac{\lambda}{h} = c\sqrt{1 + \frac{\lambda^2}{\lambda_c^2}}$$

Particular case of the phase velocity for photons:

For the particular case of photons, the rest mass is zero,

$$\lambda_c = \frac{h}{m_0 c} = \infty \qquad \Rightarrow \quad v_f = c\sqrt{1 + \frac{\lambda^2}{\lambda_c^2}} = c\sqrt{1+0} = c$$

The phase velocity for photons equals the speed of light.

$$v_f v_g = c^2 \quad \Rightarrow \quad v_g = \frac{c^2}{v_f}$$

$$\Rightarrow \quad v_g = \frac{c}{\sqrt{1+\dfrac{\lambda^2}{\lambda_c^2}}}$$

Particular case of the group velocity for photons:

For the particular case of photons, the rest mass is zero,

$$\Rightarrow \quad v_g = \frac{c}{\sqrt{1+0}} = 0$$

The group velocity for photons equals the speed of light.

As a function of the angular frequency and the wavenumber

$$v_f = \frac{E}{p} = \frac{\hbar}{\hbar}\frac{E}{p} = \frac{E}{\hbar}\frac{\hbar}{p} \qquad E = \hbar\omega, \quad p = \hbar k$$

$$\Rightarrow \quad v_f = \frac{\omega}{k}$$

Phase velocity:

The phase velocity is the ratio of the angular frequency to the wave number.

$$v_g = \frac{dE}{dp} = \frac{\hbar}{\hbar}\frac{dE}{dp} = \frac{d(E/\hbar)}{d(p/\hbar)} \qquad E = \hbar\omega, \quad p = \hbar k$$

$$\Rightarrow \quad v_g = \frac{d\omega}{dk}$$

Group Velocity:

The group velocity is the derivative of the angular frequency with respect to the wave number.