Session 4 ~ 7 Problems with classical physics
- Duality of photons and electrons -

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Goal

1. To understand the limitation of classical physics in the microscopic world of atoms and molecules, electrons and nuclei.
2. Some experimental observation that led to the concept of Particle properties of waves of wave-like properties of particles
3. what is light? what is electron?

Scenario

- Introduction,

✓ The concept of particle and wave are clear in classical physics.

Stones : particles dropped in a lake

ripples : waves carry energy and momentum

✓ Classical physics mirrors "physical reality" of our sense impressions

However,

✓ The size of matters becomes extremely small (atoms, molecules, electrons ⋯)

⇒ These are neither particles nor waves.

✓ We see electrons as particles because they possess charge and mass, and behave according to the laws of particle mechanics.

✓ We will see, however, that it is just as correct to interprete a moving electron as a wave manifestation as it is to interprete it as a particle manifestation.

✓ We regard electromagnetic waves as waves, because under suitable circumstances, they exhibit diffraction, interference, and polarization

✓ Similarly, we see that under other circumstances, electromagnetic waves behave as if they consist of streams of particles

✓ Together with special relativity, the wave-particle duality is central to understandings of mordern physics.

- Electromagnetic wave

Maxwell's eqs.

$$\left.\begin{aligned} \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \\ \nabla \cdot \vec{E} &= 4\pi\rho \\ \nabla \cdot \vec{B} &= 0 \end{aligned}\right\}$$

$$\left[\nabla \times (\nabla \times) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E}(\vec{r},t) = -\frac{4\pi}{c}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t)$$

$$\Rightarrow \nabla \times (\nabla \times \vec{E}) - \frac{\omega^2}{c^2}\varepsilon\vec{E} = 0$$

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{wave eq.}$$

$$\Rightarrow E = A\cos(kz - \omega t + \phi)$$

for waves propagating in atmosphere.

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 2.998 \times 10^8 \text{ m/s}$$

Coupled electric and magnetic oscillations that move with the speed of light exhibit typical wave behaviors.

Light = Electromagnetic wave

- Blackbody radiation

✓ Hertz's experiments : light consisted of em waves that obeyed Maxwell's eqs.
This lasted only a dozen years.

✓ The first sign that something was seriously amiss came from attempts to understand the origin of radiation emitted by bodies of matter.

✓ Glow of a hot piece of metal, which gives off visible light.
: it's color varies with the temperature of the metal, going from red to yellow to white as it is hotter and hotter.

✓ All objects radiate such energy continuously.

✓ The ability of a body to radiate $\propto$ The ability to absorb radiation.
A body at a constant temperature is in thermal equilibrium with its surroundings.

✓ An ideal body : the one that absorbs all radiation incident upon it.
= A blackbody.

✓ Around the late 19th century, it was one of the issues to measure temperature in a blast furnance.

✓ A hole in the wall of a hollow object is an excellent approximation of a blackbody : light is trapped by reflection back and forth until it is absorbed.

- Rayleigh - Jean's formula

Light in a hollow object : standing waves (Harmonic oscillators)

① Average energy of a single harmonic oscillator.

② # of oscillators per unit frequency

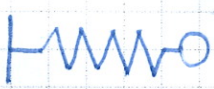
③ Total energy density = ① × ②

①

$$\langle y \rangle = \int y(x) P(x)\, dx$$

$y(x)$: Physical quantity as a function x

$P(x)$: Probability distribution function.

 $\{q, p\}$ q: position, p: momentum

Energy of a spring oscillator = K.E + P.E.

K.E. $E_m = a p^2$ (a: constant)

According to statistical mechanics,

$$P(E_m) \propto \exp\left[-\frac{E_m}{kT}\right] = \exp\left[-\frac{ap^2}{kT}\right]$$

$P(E_m)$: Probability of a state E_m in thermally equallibrium.

By normalization,

$$P(E_m) = \frac{\exp\left[-\frac{ap^2}{kT}\right]}{\int \exp\left[-\frac{ap^2}{kT}\right] dp}$$

$$\Rightarrow \quad \langle E_m \rangle = \frac{\int ap^2 \exp\left[-\frac{ap^2}{kT}\right] dp}{\int \exp\left[-\frac{ap^2}{kT}\right] dp} \qquad (\star)$$

$$\int ap^2 \exp\left[-\frac{ap^2}{kT}\right] dp = -\frac{kT}{2} \int p \frac{d}{dp} \exp\left[-\frac{ap^2}{kT}\right] dp$$

$$= -\frac{kT}{2} \left[p \exp\left[-\frac{ap^2}{kT}\right]\right]_0^\infty + \frac{kT}{2} \int \exp\left[-\frac{ap^2}{kT}\right] dp$$

$$= \frac{kT}{2} \int \exp\left[-\frac{ap^2}{kT}\right] dp$$

$$(\star) \Rightarrow \quad \langle E_m \rangle = \frac{kT}{2}$$

P.E. $E_p = b q^2$ (q: position, b: constant)

By the same way,

$$\Rightarrow \langle E_p \rangle = \frac{kT}{2}$$

$\therefore$ Energy of a single oscillator

$$= \frac{kT}{2} + \frac{kT}{2} = kT \;/\!/$$

② # of oscillators per unit frequency

Consider a cube with a side of L.

$$\lambda_s = \frac{2L}{s} \quad (s: 1, 2, 3, \cdots). \qquad \nu_s = s \frac{c}{2L}$$

# of modes = # of oscillators

In three dimensions,

$$\nu = \sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2} = \frac{c}{2L}\sqrt{s_x^2 + s_y^2 + s_z^2}$$

$$\Rightarrow \quad \frac{2\nu L}{c} = \sqrt{s_x^2 + s_y^2 + s_z^2}$$

This eq. is equivalent to a sphere with the radius $\frac{2\nu L}{c}$
in a three dimension space (s_x, s_y, s_z)

# of modes = # of grids in 1/8 part of the sphere.

If the grids are dense enough, # of modes g $(0 \sim \nu)$

$$g = \frac{4}{3}\pi\left(\frac{2\nu L}{c}\right)^3 \times \frac{1}{8} \times 2 = \frac{8\pi\nu^3 L^3}{3c^3}$$

$g(\nu \to \nu + d\nu)$

$$\frac{dg}{d\nu} = \frac{8\pi\nu^2 L^3}{c^3} \longrightarrow dg = \frac{8\pi\nu^2 L^3}{c^3} \cdot d\nu$$

$\Rightarrow$ # density (# modes per unit volume) $g_\nu = \frac{8\pi\nu^2}{c^3}$

③ Energy density in a hollow

Energy density = Average energy of a single harmonic oscillator

$\times$ Density modes

$$= kT \times \frac{8\pi\nu^2}{c^3} = \frac{8\pi\nu^2 kT}{c^3}$$

- Planck's quantum hypothesis (From continuous to discrete energy)

Instead of the average energy kT.

and. $\langle E_m + E_p \rangle = \langle ap^2 + bq^2 \rangle$

$$\propto \iint (ap^2 + bq^2) \exp\left[-\frac{ap^2+bq^2}{kT}\right] dp\,dq$$

dealing with p and q being continuous values,

$$E = ap^2 + bq^2 = n\varepsilon \quad (n = 0, 1, 2, \cdots)$$

$$\langle E \rangle = \frac{\sum_n^\infty n\varepsilon \exp\left[-\frac{n\varepsilon}{kT}\right]}{\sum_n^\infty \exp\left[-\frac{n\varepsilon}{kT}\right]}$$

Here, for the numerator

$$\sum_n^\infty n\varepsilon e^{-n\varepsilon/kT} = -\frac{\partial}{\partial\left(\frac{1}{kT}\right)} \sum_n^\infty e^{-n\varepsilon/kT} = -\frac{\partial}{\partial\left(\frac{1}{kT}\right)} \frac{1}{1 - e^{-\varepsilon/kT}}$$

$$= \frac{\varepsilon e^{-\varepsilon/kT}}{(1 - e^{-\varepsilon/kT})^2}$$

for the denominator,

$$\sum_n^\infty e^{-n\varepsilon/kT} = \frac{1}{1 - e^{-\varepsilon/kT}}$$

$$\Rightarrow \langle E \rangle = \frac{\varepsilon e^{-\varepsilon/kT}}{1 - e^{-\varepsilon/kT}} = \frac{\varepsilon}{e^{\varepsilon/kT} - 1}$$

✓ Planck's law

$$\text{Energy density} = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1}$$

If $h\nu \ll kT$.

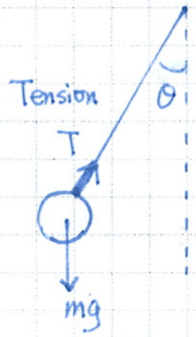
$$\approx \frac{8\pi\nu^2}{c^3} \frac{h\nu}{1 + \frac{h\nu}{kT} - 1} = \frac{8\pi\nu^2}{c^3} kT.$$

(Rayleigh-Jean's formula)

- About ε.

Consider ε in a more detail $\Rightarrow$ $\varepsilon \propto \nu$

Here, we consider a swing harmonic oscillator



Eq of motion

$$mL\ddot{\theta} = -mg\sin\theta$$

$$\theta \ll 1 \Rightarrow \theta = a\cos(2\pi\nu t + \phi)$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Pulling the string to change L (How ν changes by L)

$$\frac{d\nu}{dL} = -\frac{1}{4\pi L}\sqrt{\frac{g}{L}} = -\frac{\nu}{2L} \longrightarrow \delta\nu = -\frac{\nu}{2L}\delta L \quad ①$$

Consider the tension T. (How work is done by T).

$$T = \underset{\text{Gravitational}}{mg\cos\theta} + \underset{\text{Centrifugal}}{mL\dot{\theta}^2} \simeq mg\left(1 - \frac{1}{2}\theta^2\right) + mL\dot{\theta}^2$$

$$\theta = a\cos(2\pi\nu t + \phi)$$

$$\Rightarrow T = mg - mga^2\left\{\frac{1}{2}\cos^2(2\pi\nu t + \phi) - \sin^2(2\pi\nu t + \phi)\right\}$$

Pulling the string slowly by δL

$$\Rightarrow (\text{work by } T) = \langle T \rangle(-\delta L)$$

$$\langle T \rangle(-\delta L) = \left\langle mg - mga^2\left\{\frac{1}{2}\cos^2(2\pi\nu t + \phi) - \sin^2(2\pi\nu t + \phi)\right\}\right\rangle(-\delta L)$$

$$= \underbrace{-mg\delta L}_{\text{Potential energy}} - \underbrace{\frac{mga^2}{4}\delta L}_{\text{Oscillation energy}} \Rightarrow \delta E = -\frac{mga^2}{4}\delta L \quad ②$$

On the other hand, Oscillation energy = P.E. + K.E.

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta) \simeq \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mgL\theta^2$$

$$= \frac{1}{2} \times \underbrace{4\pi^2\nu^2}_{\frac{g}{L}} a^2 mL^2\sin^2(2\pi\nu t + \phi) + \frac{1}{2}mgLa^2\cos^2(2\pi\nu t + \phi)$$

$$= \frac{1}{2}mga^2L \quad ③$$

With ② and ③

$$\delta E = -\frac{E}{2L}\delta L$$

$$\delta E = -\frac{E}{2L}\delta L \quad + \quad ① \quad \Rightarrow \quad \frac{\delta\nu}{\delta E} = \frac{\nu}{E}$$

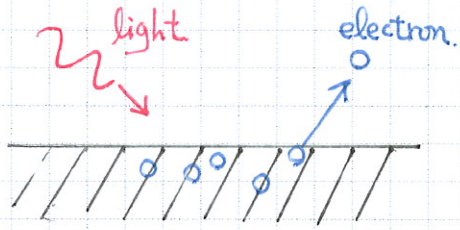
$$\delta\left(\frac{E}{\nu}\right) = \frac{\delta E}{\nu} - \frac{E}{\nu^2}\delta\nu$$

$$= \frac{E}{\nu}\left(\frac{\delta E}{E} - \frac{\delta\nu}{\nu}\right) = 0$$

$$\Rightarrow \quad \frac{E}{\nu} = \text{const.}$$

- Photoelectric effect

The phenomenon in which electrons jumps out from a metal surface illuminated by light.



1. Threshold frequency ν_0

   $\nu < \nu_0$: no electrons come out regardless of high intensity of light.

2. Energy of electron $E_{el} \propto \nu$ ✕ $E_{el} \propto$ Fluence of light I

3. # of ejected electrons $\propto I$

4. Even if I is low, $\nu > \nu_0 \Rightarrow$ Electrons are ejected.

Millikan's experiments.

max K.E.

$K_m = E - W_0$

$= h\nu - h\nu_0$

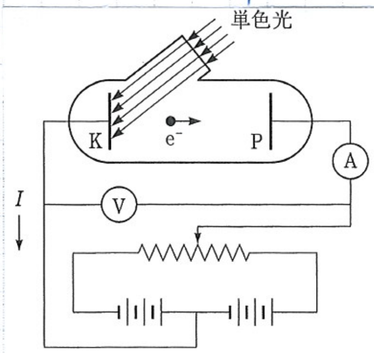


図 1.2　光電効果の実験の概念図

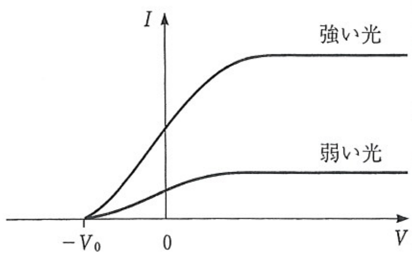


図 1.3　正極電圧 V と電流 I の関係（概念図

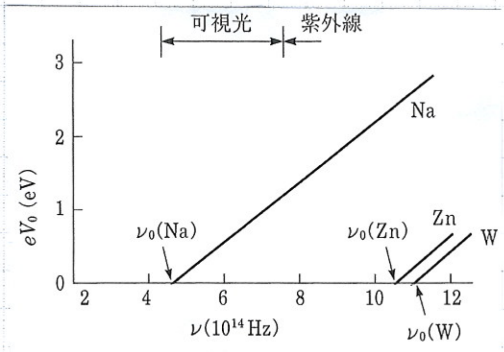


図 1.4　単色光の振動数 ν と阻止電圧 V_0 の関係
縦軸の単位は $1\,\mathrm{eV} = 1.6 \times 10^{-19}$ J

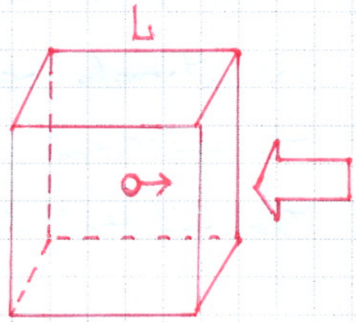
- Momentum of photon

Provided that light has a particle-like property, it can have "momentum". Here, we discuss the momentum of a photon.

Our discussion here uses the cavity radiation in a cube.

✓ Suppose that the length of one side of a cube L is slowly changed.

✓ Suppose that one photon propagates in the direction of the changing side, which pushes the surface when bumping against it.



Energy conservation :

(Change of the photon energy)

= (Work of moving the surface by the external force)

= (Pressure by the photon) × (moved length of the surface)

Change of the photon energy

L and ν

$$\nu = s\frac{c}{2L} \longrightarrow \frac{d\nu}{dL} = -s\frac{c}{2L^2} \longrightarrow \delta\nu = -s\frac{c}{2L}\cdot\frac{\delta L}{L}$$

$$= -\frac{\nu}{L}\delta L$$

Photon energy $E = h\nu$.

$$\frac{dE}{d\nu} = h \longrightarrow \delta E = h\delta\nu = -\frac{h\nu}{L}\delta L$$

Pressure by the photon

The change of the momentum when bumped back at the surface

$\Rightarrow$ $2p$

of the bumping per unit time : $\frac{c}{2L}$

$\Rightarrow$ Total pressure : $P = 2p \times \frac{c}{2L} = \frac{pc}{L}$

$$-\frac{h\nu}{L}\delta L = -\frac{pc}{L}\delta L \Rightarrow p = \frac{h\nu}{c} \quad \text{(Momentum of a photon)}$$

$$\delta E = -P\delta L \quad (L \to \text{small, External workload} \to \text{positive})$$

- Compton effect

When X-ray is scattered by matters, the wavelength (the frequency) of the X-ray shifts.

- If X-ray is considered to be waves,

① When X-ray impinges on matters, the electons in the matters are strongly oscillated.

② The oscillated electrons give off spherical electromagnetic radiation

③ The radiated fields are observed as scattered light.

$\Rightarrow$ (Freq. of scattered light = Freq. of electron oscillation
= Freq. of input X-ray   This is not true!)

Consider a collision b/w a photon and an electron,

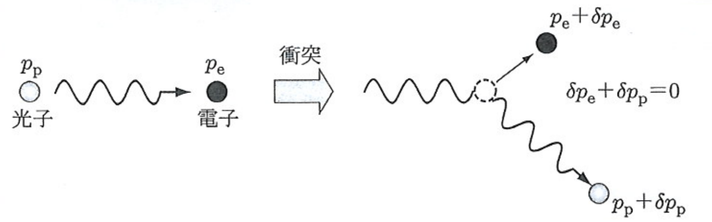


図 1.14 衝突散乱

Light : particle-like

$P = \frac{h\nu}{c}$ $P \propto \nu$

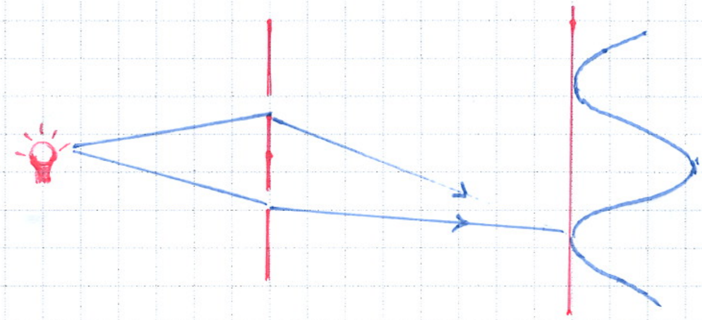
- Wave-like and particle-like properties.

✓ Light : Wave-like & particle-like properties.

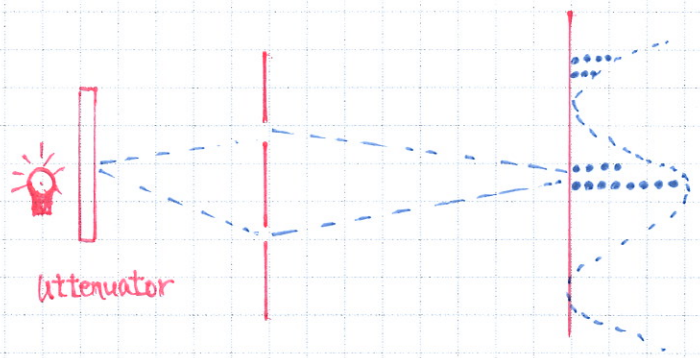
How these two properties, which look completely different, are consistant?

The answer is that the energy of light has a minimum unit whose behaviors follow wave-like properties.

(Wave)



(Photon)



How the wave-like and particle-like properties are consistent?

✓ The state of light is expressed by the probability amplitude of photons : a

✓ The probability amplitude behaves like wave (i.e. it has a phase) : $a = |a| e^{i\theta}$

✓ When we observe light, its energy has a minimum unit and discrete : $h\nu$

✓ The probability of observing a photon is given by the absolute square of the probability amplitude : $|a|^2$

✓ When some probability amplitudes are overlapped, there occurs interference : $|a+b|^2 = |a|^2 + |b|^2 + 2\mathrm{Re}[ab^*]$

Note : "Particle-like property" ≠ Light particles fly over a space
= Light energy has a minimum unit, nothing else.

- Wave properties of particle

(Logic)

- ✓ Waves behave like particles (Photon).
- ✓ Particles behave like waves?
- ✓ Louis de Broglie proposed that moving objects have waves as well as particle characteristics.
- ✓ This revolutionary concept was suggested in 1924. without a strong experimental mandate.
- ✓ The existence of de Broglie waves was experimentally demonstrated in 1927. : Electron scattering/diffraction experiments.

- de Broglie waves

A moving object behaves in certain ways as though it has a wave nature.

Momentum of a photon:

$$p = \frac{h\nu}{c} = \frac{h}{\lambda} \quad \text{with } c = \lambda\nu$$

Photon wavelength:

$$\lambda = \frac{h}{p}$$

de Broglie's hypothesis: There is a symmetry in nature b/w particles and waves.

Particles with a momentum p have an associated wavelength λ.

$$\lambda = \frac{h}{\gamma m_0 v} = \frac{h}{mv} \quad \text{with } m = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

m: Relativistic mass

m_0: Rest mass

ex.).

1. A 46g golf ball with a velocity of 30 m/s.

$v \ll c \Rightarrow \gamma = 1.$

$$\lambda = \frac{h}{m_0 v} = 4.8 \times 10^{-34}\ m$$

2. An electron with a velocity of 10^7 m/s

$v \ll c \Rightarrow m_0 = m = 9.1 \times 10^{-31}\ kg.$

$$\lambda = \frac{h}{mv} = 7.3 \times 10^{-11}\ m$$

cf. Bohr radius of hydrogen atom

: $0.529\,\text{Å} = 5.29 \times 10^{-11}\ m.$

- What is wave? Waves of what?

Water waves : Height of water surface, Sound : Pressure

Light : EM fields. Matter waves : ? Ψ What is Ψ?

✓ Wave function : $\Psi(\vec{r}, t) = A(\vec{r}, t) + iB(\vec{r}, t)$

no direct physical significance!

Ψ : $-1 \sim +1$ negative values : meaningless

✓ The probability of experimentally finding a particle described by the wave function Ψ at the point (x, y, z) at the time t is proportional to the value of $|\Psi|^2$

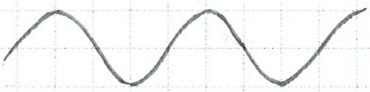
$$\rho = |\Psi(\vec{r}, t)|^2 = \Psi^* \Psi$$

✓ The probability P to find a particle in an experiment in a volume element $dv = dx\,dy\,dz$ is

$$P = \rho\, dv \qquad \int \rho\, dv = \int |\Psi|^2 dv = 1.$$

- Discribing de Broglie waves

✓ Infinite wave



No starting and ending points just continue infinitly

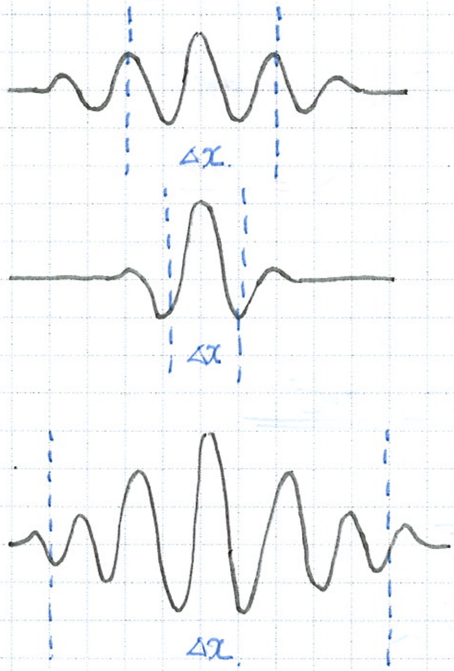
A general form

$$\psi(\vec{r},t) = A e^{i(k\vec{r}-\omega t)}$$

$$\omega = 2\pi\nu, \quad k = \frac{\lambda}{2\pi}$$

→ No information and no meaning as light signals.

✓ Wave packet



Probability density.

$$P = |\psi(\vec{r},t)|^2$$

Only ψ: no physical meaning. Can be negative.

$$\Rightarrow \psi^*\psi = |\psi|^2$$

If. $\psi = A e^{i(k\vec{r}-\omega t)}$

$$|\psi|^2 = A^2 = \text{const}$$

Not true for de Broglie waves.

Spacially and temporally waves are localized carrying energy and information

Wave packets are created by the superposition of two waves with different freqs.

One important quantity describing waves is their propagating velocity v_p.

What is this velocity for de Broglie matter waves?

$$v_p = \lambda\nu = \frac{c^2}{v} > c \;?$$

$$E = h\nu, \quad E = \gamma m_0 c^2 \quad ; \quad \nu = \frac{\gamma m_0 c^2}{h} \qquad \lambda = \frac{h}{\gamma m_0 v}$$

⇒ Phase velocity and Group velocity

Consider two waves : E_1 and E_2 with the freq. components of $\omega_c+\Delta\omega$, $\omega_c-\Delta\omega$ and the wave vectors of $k_0+\Delta k$, $k_0-\Delta k$.

$$E(z,t) = E_0\cos[(k_0+\Delta k)z-(\omega_c+\Delta\omega)t] + E_0\cos[(k_0-\Delta k)z-(\omega_c-\Delta\omega)t]$$

$$= \underbrace{2E_0\cos(\Delta k z-\Delta\omega t)}_{\text{Envelope func.}}\cos(k_0 z-\omega_c t)$$

$$2E_0\cos\left[\Delta k\left(z-\frac{\Delta\omega}{\Delta k}t\right)\right]$$

Phase velocity :

$$v_p = \frac{\omega_0}{k_0}$$

Group velocity :

$$v_g = \frac{\Delta\omega}{\Delta k}.$$

Angular freq. $\omega = 2\pi\nu = 2\pi\frac{mc^2}{h} = \frac{2\pi}{h}\frac{m_0c^2}{\sqrt{\quad}}$

Wave vector $k = \frac{2\pi}{\lambda} = \frac{2\pi}{h}\frac{m_0v}{\sqrt{\quad}}$

ω, k : functions of an object's velocity v.

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{dv}\cdot\frac{dv}{dk}.$$

$$\frac{d\omega}{dv} = \frac{2\pi mv}{h\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$\frac{dk}{dv} = \frac{2\pi m}{h\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$\Rightarrow$ $v_g = v$.

The group velocity of a wave packet describing a particle

$\Rightarrow$ corresponds to its velocity