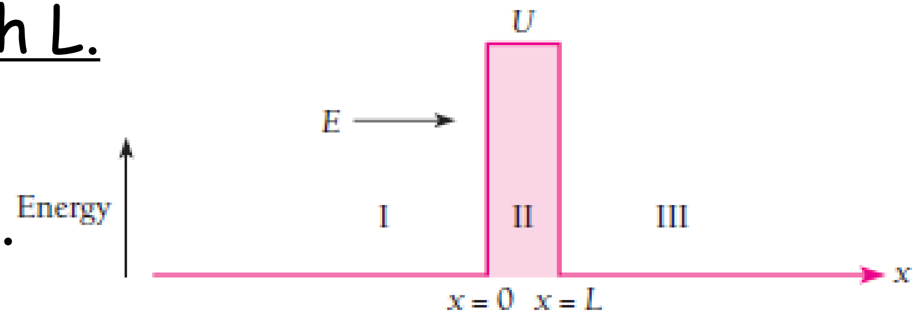
Tunnel effect

✔ Suppose a particle with kinetic energy E strikes a barrier with the height U and the width L.

✔ Classically the particle cannot overcome the barrier.



✔ Quantum mechanically **the particle can penetrated the barrier and appear on the other side.**

✔ It is said to have tunneled through the barrier.

- Emission of alpha particles from radioactive nuclei by tunneling through the binding potential barrier.
- Tunneling of electrons from one metal to another through an oxide film.
- Tunneling in a more complex systems described by a generalized coordinate varying in some potential.

Approximation

✔ The transmission coefficient T is the probability of a particle incident from the left (region I) to be tunneling through the barrier (region II) and continue to travel to the right (region III).

$$\boxed{T = e^{-2k_2 L}} \qquad k_2 = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

✔ Depends exponentially on the width of barrier L and the difference between the particle kinetic energy and the barrier height $(U_0\text{-}E)^{1/2}$ and mass of the particle $m^{1/2}$.

Ex.) An electron with kinetic energy E = 1 eV tunnels through a barrier with U_0 = 10 eV and width L = 0.5 nm. What is the transmission probability? $\Rightarrow \quad T = 1.1\times 10^{-7}$

✔ The probability is small, even for a light particle and a thin barrier - but it can be experimentally observed and used in devices.

Schrödinger eq.

Outside of barrier (regions I and III)

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi_{I,III}}{\partial^2 x}+E\psi_{I,III}=0$$

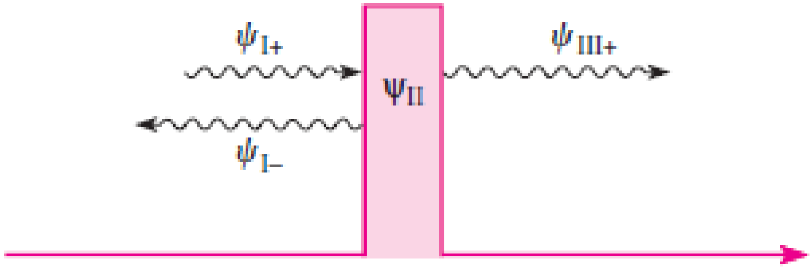
$$\psi_I(x)=Ae^{ik_1x}+Be^{-ik_1x}$$

$$\psi_{III}(x)=Fe^{ik_1x}+Ge^{-ik_1x}$$

$$k_1=\frac{\sqrt{2mE}}{\hbar}=\frac{p}{\hbar}=\frac{2\pi}{\lambda}$$

Incoming wave: $\psi_{I+}(x)=Ae^{ik_1x}$ Reflected wave: $\psi_{I-}(x)=Be^{-ik_1x}$

Transmitted wave: $\psi_{III+}(x)=Fe^{ik_1x}$



Incoming flux of particles with the group velocity v_{I+} :

$S=|\psi_{I+}|^2 v_{I+}$ # of particles per second that arrive there

Transmission

Probability: $$T = \frac{|\psi_{III+}|^2 v_{III+}}{|\psi_{I+}|^2 v_{I+}}$$ Ratio of flux of transmitted particles to incident particles

Inside of barrier (regions II)

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_{II}}{\partial^2 x} + (U-E)\psi_{II} = 0 \qquad U > E$$

$$\psi_{II}(x) = Ce^{-k_2 x} + De^{k_2 x} \qquad k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

Real quantity

- ✔ Exponentially decaying or increasing wave (no oscillations)
- ✔ Does not describe a moving particle
- ✔ But probability in barrier region is non-zero

A particle may emerge into region III,
or may return to region I.

Boundary conditions

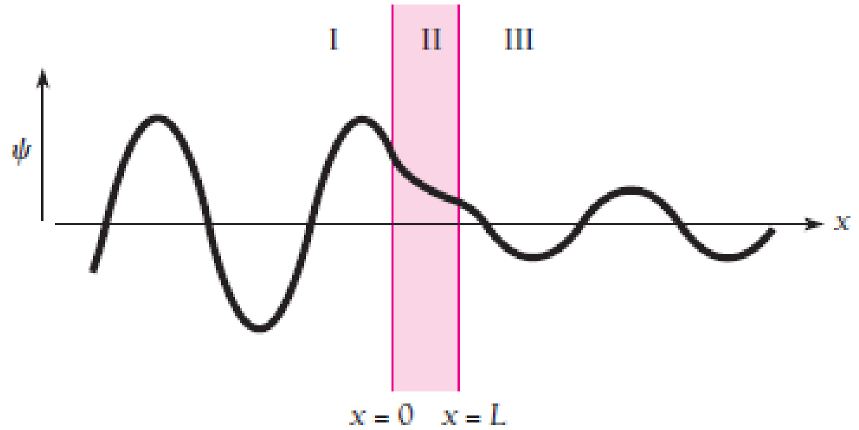
At the left edge of the well,

$$\psi_I(0)=\psi_{II}(0) \quad ; \quad \left.\frac{\partial \psi_I}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=0}$$

At the right edge of the well,

$$\psi_{II}(L)=\psi_{III}(L) \quad ; \quad \left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=L} = \left.\frac{\partial \psi_{III}}{\partial x}\right|_{x=L}$$

✔ Solve the four equations for the four coefficients and express them relative to A ($|A|^2$ is proportional to incoming flux)



From the boundary conditions

$$\begin{cases} A+B=C+D \\ ik_1(A-B)=-k_2(C-D) \end{cases}$$

$$\left.\begin{aligned} Ce^{-k_2L}+De^{k_2L} &= Fe^{ik_1L} \\ -k_2\left(Ce^{-k_2L}-De^{k_2L}\right) &= ik_1Fe^{ik_1L} \end{aligned}\right\} \Rightarrow$$

$$2Ce^{-k_2L}=\left(1-\frac{ik_1}{k_2}\right)Fe^{ik_1L}$$

$$C=\frac{k_2-ik_1}{2k_2}e^{ik_1L}e^{k_2L}F$$

$$2De^{k_2L}=\left(1+\frac{ik_1}{k_2}\right)Fe^{ik_1L}$$

$$D=\frac{k_2+ik_1}{2k_2}e^{ik_1L}e^{-k_2L}F$$

$$\Rightarrow \quad A+B=e^{ik_1L}\left(\cosh k_2L-i\frac{k_1}{k_2}\sinh k_2L\right)F$$

$$A-B=e^{ik_1L}\left(\cosh k_2L-i\frac{k_2}{k_1}\sinh k_2L\right)F$$

$$A=\frac{1}{2}e^{ik_1L}\left(2\cosh k_2L-i\frac{k_1^2+k_2^2}{k_1k_2}\sinh k_2L\right)F$$

$$B=e^{ik_1L}\frac{k_2^2-k_1^2}{2k_1k_2}\sinh k_2L\times F$$

$$=\frac{e^{ik_1L}}{2k_1k_2}\left(2k_1k_2\cosh k_2L-i\left(k_1^2+k_2^2\right)\sinh k_2L\right)F$$

Transmission coefficient

Find A/F from set of boundary condition equations.

Another form:

$$\frac{A}{F}=\left[\frac{1}{2}+\frac{i}{4}\left(\frac{k_2}{k_1}-\frac{k_1}{k_2}\right)\right]e^{(ik_1+k_2)L}+\left[\frac{1}{2}-\frac{i}{4}\left(\frac{k_2}{k_1}-\frac{k_1}{k_2}\right)\right]e^{(ik_1-k_2)L}$$

Simplify: Assume barrier U to be high relative to particle energy E

$$\frac{k_2}{k_1}-\frac{k_1}{k_2}\approx\frac{k_2}{k_1}$$

Simplify: Assume barrier to be wide ($k_2L>1$)

$$e^{k_2L}>>e^{-k_2L}$$

$$\Rightarrow\qquad\frac{A}{F}=\left[\frac{1}{2}+\frac{i}{4}\frac{k_2}{k_1}\right]e^{(ik_1+k_2)L}$$

Transmission probability

$$\frac{AA^*}{FF^*} = \left[\frac{1}{4} + \frac{k_2^2}{16k_1^2}\right] e^{2k_2 L}$$

with $\left(\frac{k_2}{k_1}\right)^2 = \frac{U-E}{E}$

$$\Rightarrow \qquad T = \frac{FF^* v_{III+}}{AA^* v_{I+}} = \left(\frac{AA^*}{FF^*}\right)^{-1}$$

$$k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

$$= \left[\frac{16}{4 + (k_2/k_1)^2}\right] e^{-2k_2 L} \boxed{\approx e^{-2k_2 L}}$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$

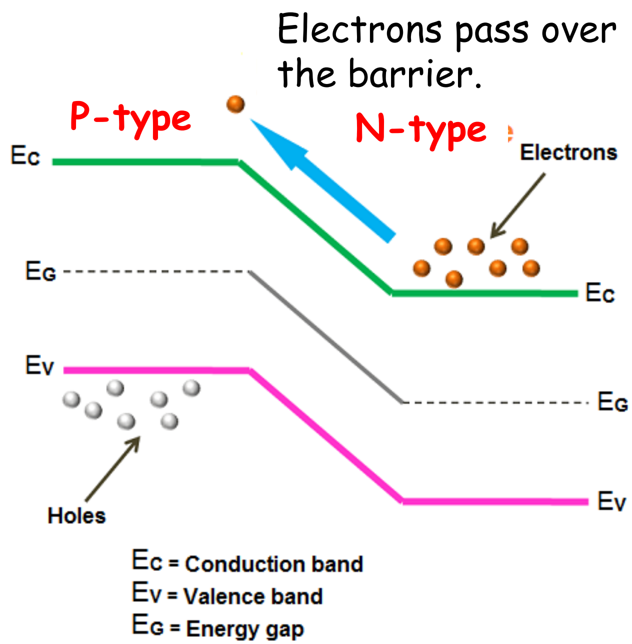
✔ T is exponentially sensitive to width of barrier.
✔ T can be measured in terms of a particle flow (e.g. an electrical current) through a tunnel barrier.
✔ Makes this effect a great tool for measuring barrier thicknesses or distances for example in microscopy applications

Rigorous transmission probability

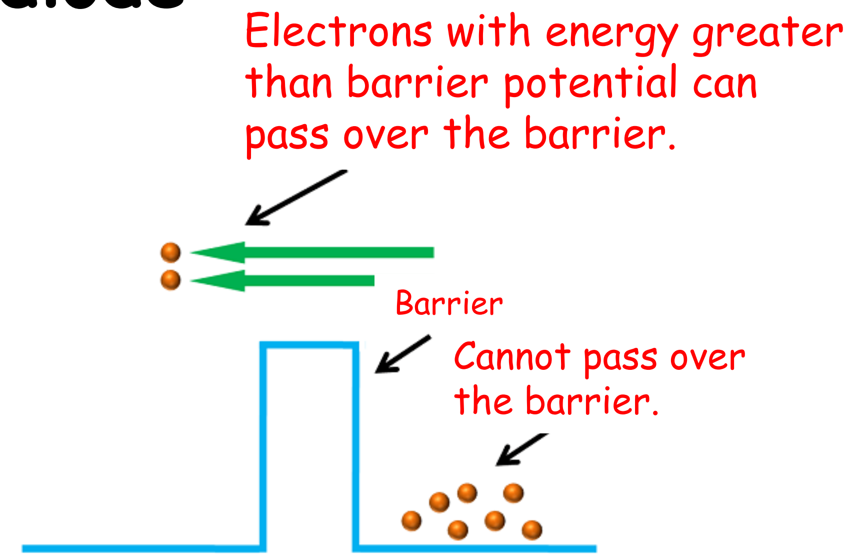
$$T = \frac{FF^* v_{III+}}{AA^* v_{I+}} = \left(\frac{AA^*}{FF^*} \right)^{-1}$$

$$= \frac{1}{1 + \frac{U^2}{4E(U-E)} \sinh^2 \left(\frac{\sqrt{2m(U-E)}}{\hbar} L \right)}$$

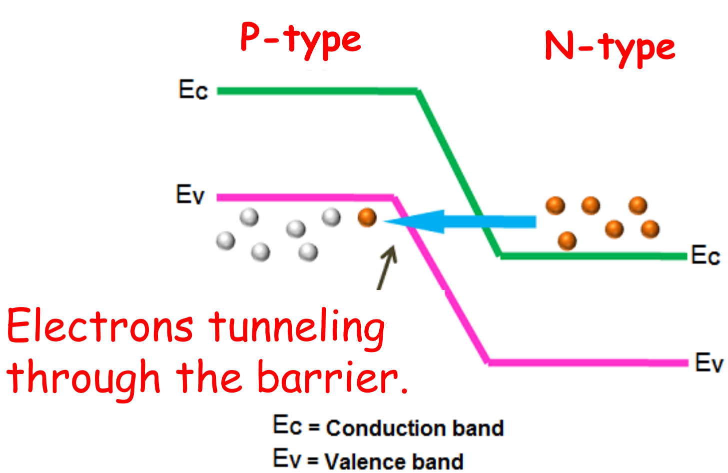
Tunnel diode

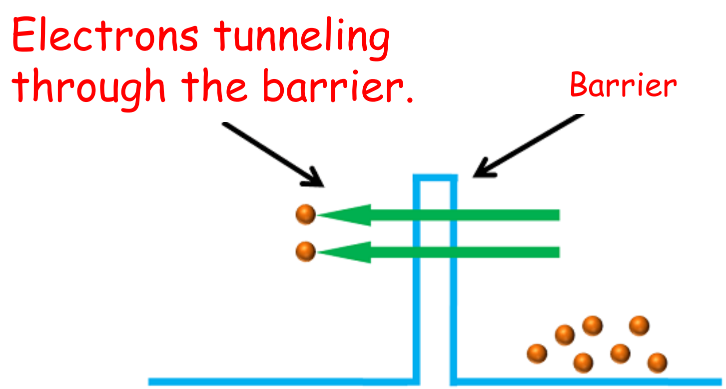


Ordinary P-N junction diode

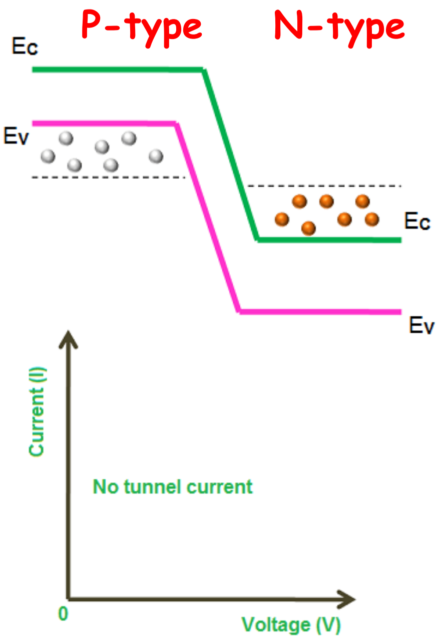


Ordinary p-n junction diodes

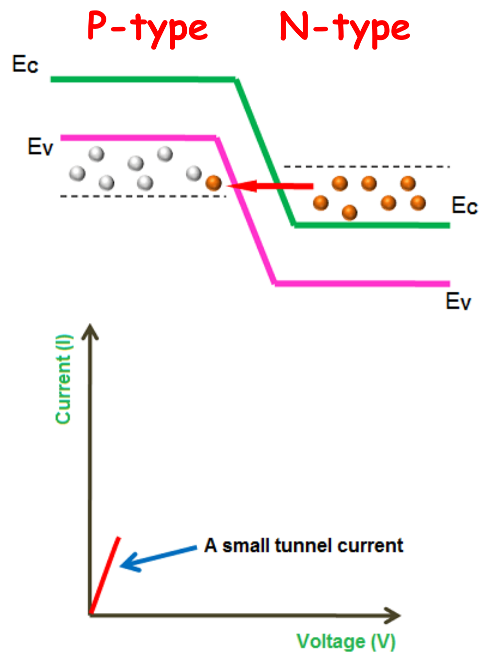




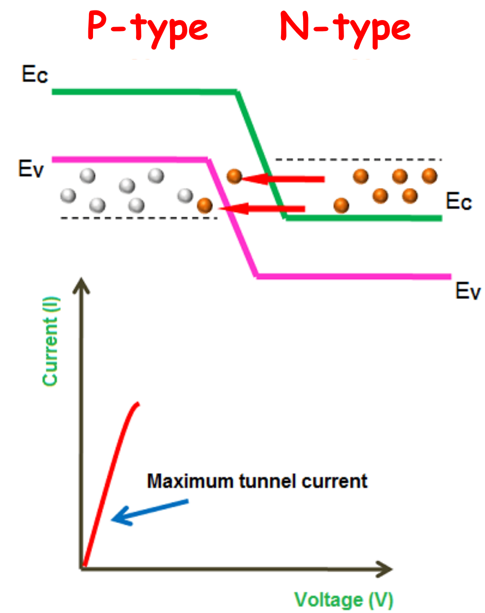
Tunnel diodes



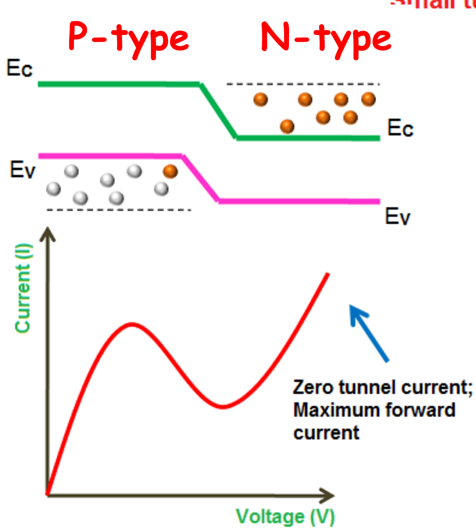
Unbiased tunnel diode



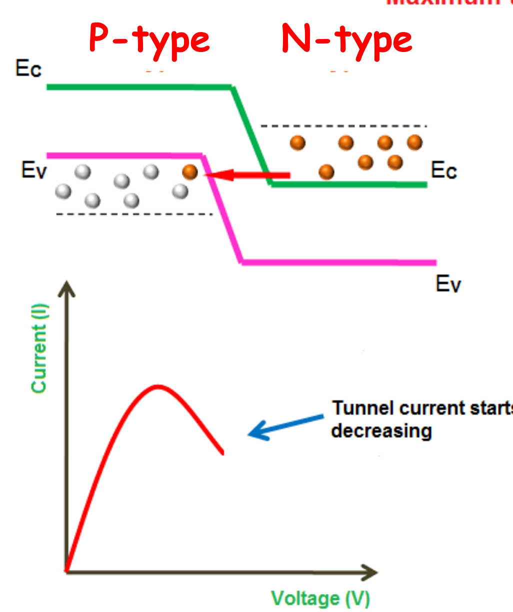
Small tunnel current



Maximum tunnel current



Zero tunnel current; maximum forward current



Tunnel current starts decreasing