1. Free particle

Consider a case where particles of mass m are moving at a constant linear velocity along the x axis.

Potential energy: $V(x) = 0$

Time-independent Schrödinger eq.: $$-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

$$\Rightarrow \quad \psi^{\pm}(x) = N\exp\left(\pm\frac{i}{\hbar}\sqrt{2mE}\,x\right) = N\exp(\pm ikx) \qquad \left(k = \frac{\sqrt{2mE}}{\hbar} = \frac{p_x}{\hbar}\right)$$

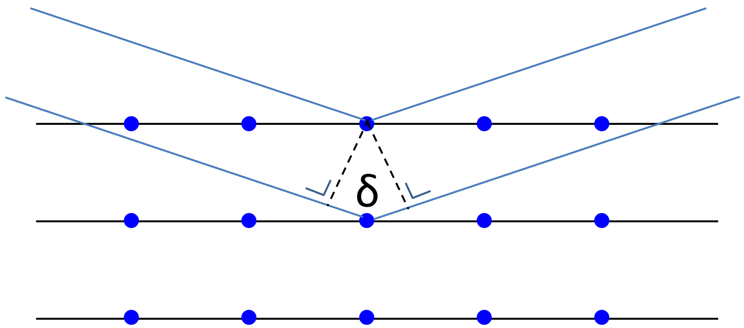
Momentum: $p_x = \sqrt{2mE}$ Observable: Real $\Rightarrow \quad E > 0$

Time-dependent Schrödinger eq.:

$$\Rightarrow \quad \Psi^{\pm}(x) = N\exp(\pm ikx \mp 2\pi\nu t) \qquad \left|\Psi^{\pm}(x)\right| = \left|\psi(x)\right|^2 = N^2$$

Diffraction

Why does a flow of electrons shows diffraction?



$$\left\{\Psi^+(x,t)+\Psi^+(x+\delta,t)\right\}^*\left\{\Psi^+(x,t)+\Psi^+(x+\delta,t)\right\}$$
$$=N^2\left\{2+\left(e^{-ik\delta}+e^{ik\delta}\right)\right\}=2N^2\left\{1+\cos(k\delta)\right\}$$

$$k\delta = 2n\pi \ (n = 0, 1, 2, \cdots)$$

$\delta = 2n\pi/k = n\lambda$: $4N^2$ max.

$\delta = (2n+1)(\lambda/2)$: 0 min.

Interference of probability distribution.

$\left|\Psi^{\pm}(x)\right|$: Probability of presence spreads out infinitely.

→ More localized picture?

$$\Psi = c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n + \cdots = \sum_i c_i\psi_i$$

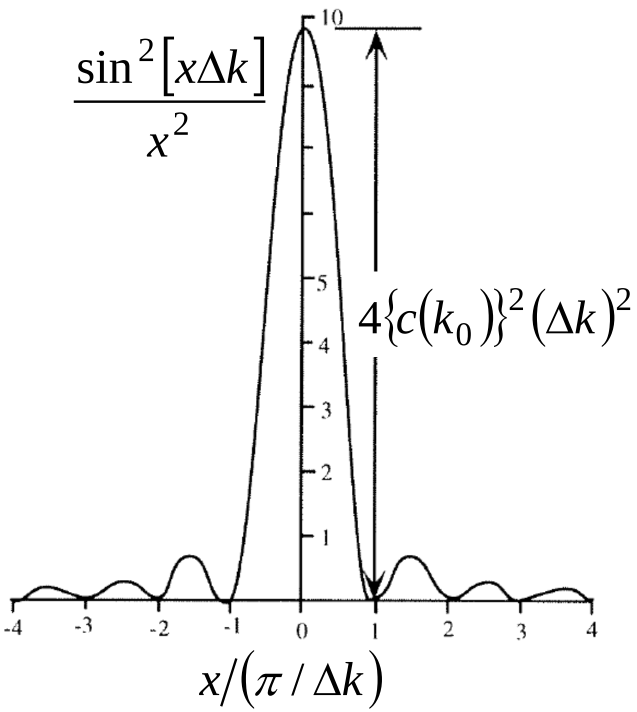
2. More localized picture

Consider a case of superposition, $k(=p_x/\hbar)$ $k_0-\Delta k \le k \le k_0+\Delta k$

$$\Psi(k,t)=\int_{k_0-\Delta k}^{k_0+\Delta k} c(k)\exp[i\{kx-\omega(k)t\}]dk \qquad \omega(k)=\omega_0+\left(\frac{d\omega}{dk}\right)_0 k$$

$c(k_0)$: const

$$\Rightarrow \quad \Psi(k,t)=2c(k_0)\frac{\sin[(x-(d\omega/dk)_0 t)\Delta k]}{x-(d\omega/dk)_0 t}\exp[i\{k_0x-\omega_0 t-(d\omega/dk)_0 k_0 t\}]$$



$$\Rightarrow \quad \Psi(k,0)=2c(k_0)\frac{\sin[x\Delta k]}{x}\exp[ik_0x]$$

$$\Rightarrow \quad |\Psi(k,0)|=4\{c(k_0)\}^2\frac{\sin^2[x\Delta k]}{x^2}$$

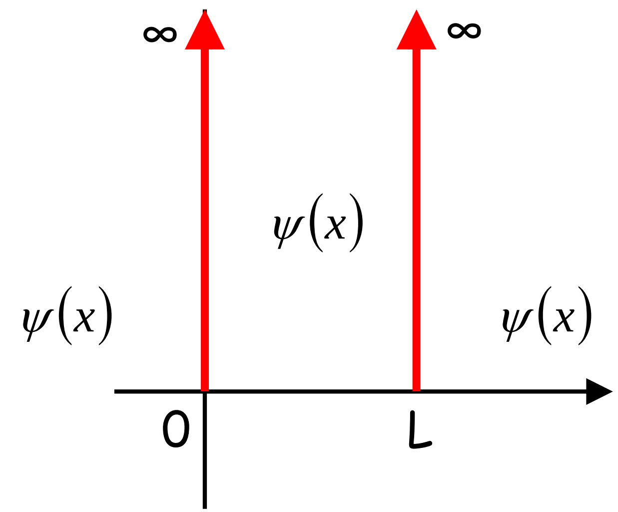
$$\Delta x=\frac{\pi}{\Delta k}-\frac{-\pi}{\Delta k}=\frac{2\pi}{\Delta k} \qquad \Delta P_x=\hbar\Delta k$$

$$\Rightarrow \quad \Delta x\cdot\Delta P_x=h$$

3. Infinite square well (Particle in a box)

Infinite square well potential

$$V = \begin{cases} 0 & if\ 0 \le x \le L \\ \infty & otherwise \end{cases}$$



Time-independent Schrödinger eq.

$$-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

Probability?
Energy?
Momentum?

Outside region? $\psi(x) = 0 \quad if\ \ x < 0,\ x > L$

Inside region? $-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$

Solution to the Schrödinger eq.

$$-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}=E\psi \quad\Rightarrow\quad \frac{\partial^2\psi}{\partial x^2}=-k^2\psi \qquad k=\frac{\sqrt{2mE}}{\hbar}$$

Harmonic oscillator

General solution:

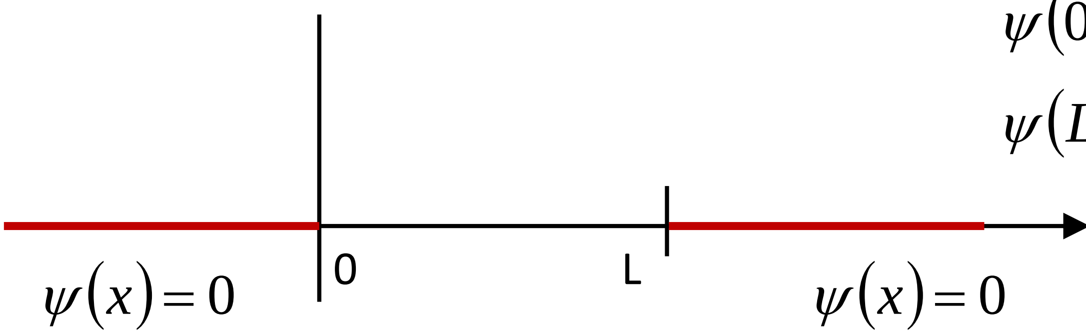
$$\psi(x)=A\sin(kx)+B\cos(kx) \qquad A, B:\text{constants}$$

What are A, B, and k?

Boundary conditions: form of solution

$$\psi(0)=0 \quad\Rightarrow\quad B=0$$

$$\psi(L)=0 \quad\Rightarrow\quad \psi(L)=A\sin(kL)=0$$



Boundary conditions: energy

$$\psi(L)=0 \quad \Rightarrow \quad \psi(L)=A\sin(kL)=0$$

$$kL = 0, \pm\pi, \pm 2\pi, \pm 3\pi \cdots$$

$\psi(x)=0 \qquad \sin(-x)=-\sin x$

A absorbs sign.

$$kL=\pi, 2\pi, 3\pi, \cdots = n\pi$$

$$\frac{\sqrt{2mE}}{\hbar}L=n\pi \quad \Rightarrow \quad E=\frac{n^2\pi^2\hbar^2}{2mL^2}$$

Discrete set of allowed energies

$$\psi(x)=A\sin\left(\frac{n\pi}{L}x\right)$$

Wave function

Normalization

$$\int_{-\infty}^{+\infty}\psi^*\psi dx=1 \qquad \int_0^L A^2\sin^2\left(\frac{n\pi}{L}x\right)dx=1$$

$$\sin^2 x=\frac{1-2\cos^2(2x)}{2}$$

$$A^2\cdot\frac{1}{2}L=1 \quad \Rightarrow \quad A=\sqrt{\frac{2}{L}}$$

$$\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

Final form of wavefunction

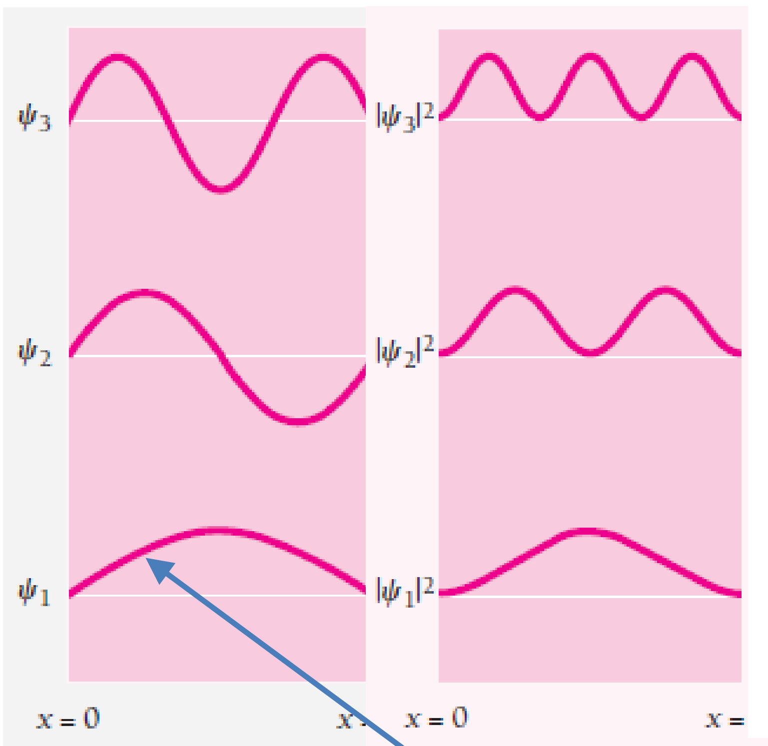
Solutions and energies

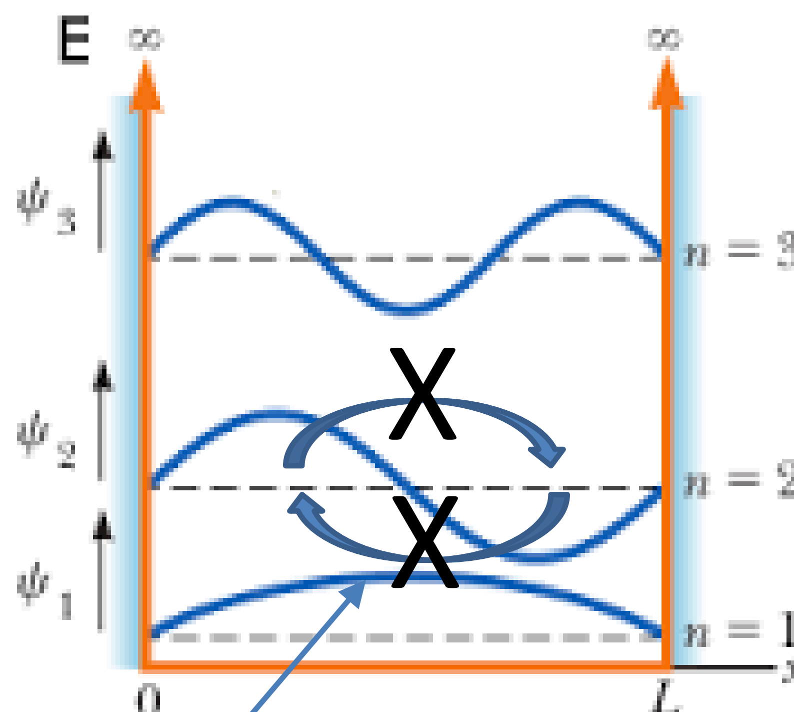
Wave function:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$$

Energy:

$$E = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$





Not amplitude of wave

<x> of a particle in a box

$$\langle x\rangle = \int_{-\infty}^{\infty}\psi^* x\,\psi\,dx = \int_{-\infty}^{\infty} x|\psi|^2 dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\int_0^L x\frac{1-\cos\left(\frac{2n\pi}{L}x\right)}{2}dx = \frac{2}{L}\int_0^L\left\{\frac{x}{2}-\frac{x}{2}\cos\left(\frac{2n\pi}{L}x\right)\right\}dx$$

$$= \frac{2}{L}\left\{\int_0^L\frac{x}{2}dx - \frac{x\sin(2n\pi x/L)}{4n\pi/L} + \int_0^L\frac{\sin(2n\pi x/L)}{4n\pi/L}dx\right\}$$

$$= \frac{2}{L}\left[\frac{x^2}{4} - \frac{x\sin(2n\pi x/L)}{4n\pi/L} - \frac{\cos(2n\pi x/L)}{8(n\pi/L)^2}\right]_0^L = \frac{L}{2}$$

In all quantum states!

Average ≠ Probability

<p> of a particle in a box

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi dx = \int_{-\infty}^{\infty} \psi^* \left(\frac{\hbar}{i} \frac{d}{dx} \right) \psi dx = \frac{\hbar}{i} \frac{2}{L} \frac{n\pi}{L} \int_0^L \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx = 0$$??

$$\int \sin \alpha x \cos \alpha x dx = \frac{1}{2\alpha} \sin^2 \alpha x$$

A particle in a box should have eigenvalues:

$$p_n = \pm\sqrt{2mE_n} = \pm\frac{n\pi\hbar}{L}$$??

A particle is moving back and forth:

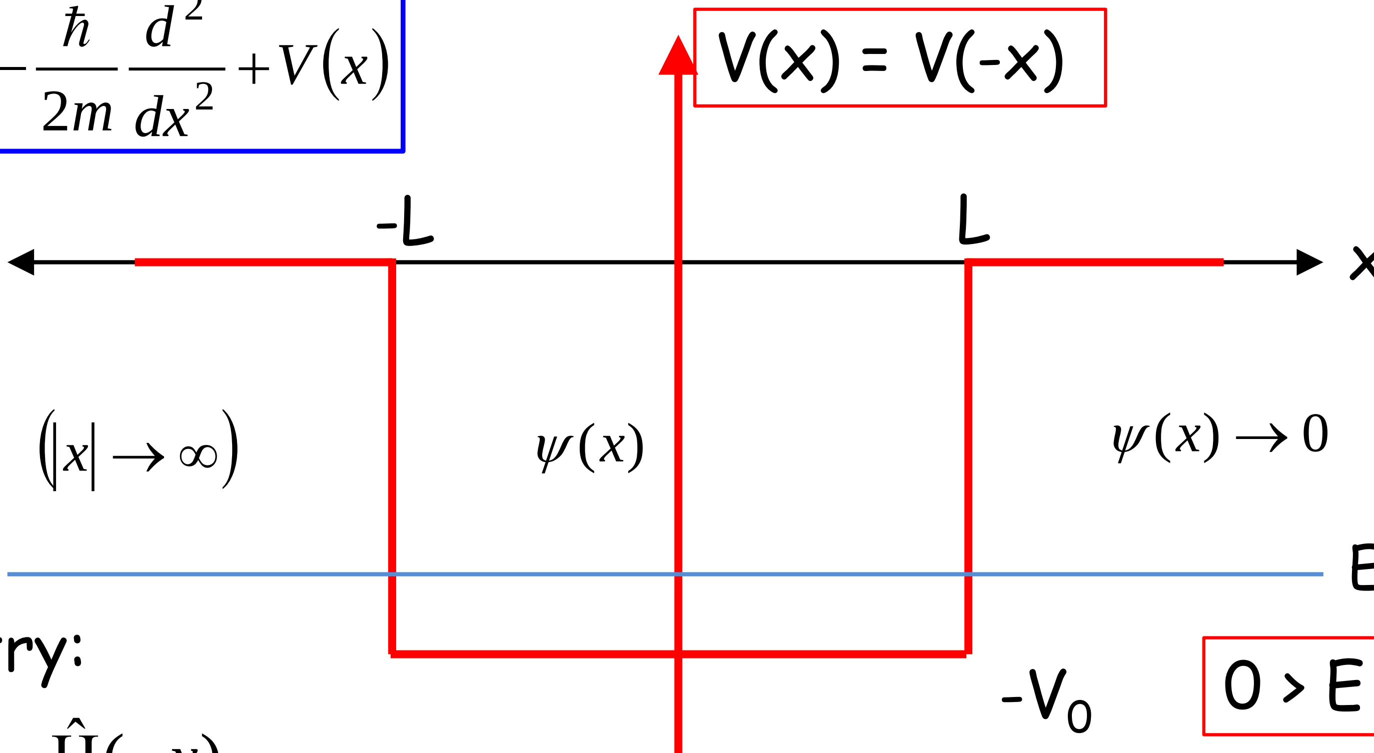
$$p_{ave} = \frac{1}{2}\left(\frac{n\pi\hbar}{L} - \frac{n\pi\hbar}{L} \right) = 0$$

→ Order estimate

4. Finite square well potential

Hamiltonian:

$$\hat{H} = -\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)$$



$\psi(x) \to 0 \quad (|x| \to \infty)$

$\psi(x) \to 0 \quad (|x| \to \infty)$

$0 > E > -V_0$

Symmetry:

$$\hat{\mathrm{H}}(x) = \hat{\mathrm{H}}(-x)$$

$$\Rightarrow \qquad \psi(x) = \psi(-x) \qquad \left.\frac{d\psi}{dx}\right|_{x=0} = 0$$

$$\psi(-x) = -\psi(x) \qquad \psi(0) = 0$$

General solutions

$$\hat{H}\psi = E\psi \quad \Rightarrow \quad -\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

x < -L

E < V; V = 0

$$\frac{\partial^2\psi}{\partial^2 x} = +k^2\psi$$

E < 0

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = Ae^{-kx} + Be^{kx}$$

x → -∞

$$\psi(x) = Be^{kx}$$

-L < x < L

E > V

$$\frac{\partial^2\psi}{\partial^2 x} = -\ell^2\psi$$

$$\ell^2 = (E + V_0)\frac{2m}{\hbar^2}$$

$$\psi(x) = C\sin(\ell x) + D\cos(\ell x)$$

x < L

E < V; V = 0

$$\frac{\partial^2\psi}{\partial^2 x} = +k^2\psi$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = Fe^{-kx} + Ge^{kx}$$

x → +∞

$$\psi(x) = Fe^{-kx}$$

Even solution boundary conditions

x < -L	-L < x < L	x < L
E < V; V = 0	E > V	E < V; V = 0
$\psi(x) = Be^{kx}$	$\psi(x) = C\sin(\ell x) + D\cos(\ell x)$	$\psi(x) = Fe^{-kx}$

$$C = 0,\ B = F$$

Ψ continuous: $Fe^{-kL} = D\cos(\ell L)$

$\partial_x\Psi$ continuous: $-kFe^{-kL} = -\ell D\sin(\ell L)$

$\div \Rightarrow$ $\boxed{-k = -\ell\tan(\ell L)}$

$$k = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$\ell = \sqrt{(E + V_0)\frac{2m}{\hbar^2}}$$

The eigen values of energy are determined by this equation.

Check your understandings: Odd solutions

- ✔ Write forms of Ψ(x) in the three domains for odd Ψ(x).
- ✔ Write a boundary condition for continuity of Ψ.
- ✔ Write a boundary condition for continuity of ∂Ψ.
- ✔ Show that you get k = -ℓcot(ℓL).

Summary of solutions

$$\frac{k}{\ell} = \tan(\ell L)$$

$$-\frac{k}{\ell} = \cot(\ell L)$$

$$\psi(x) = \begin{cases} Be^{kx} & (x < -L) \\ D\cos(\ell x) & (-L < x < L) \\ Be^{-kx} & (-L < x) \end{cases}$$

$$\psi(x) = \begin{cases} Be^{kx} & (x < -L) \\ C\sin(\ell x) & (-L < x < L) \\ -Be^{-kx} & (-L < x) \end{cases}$$

Define

$$\xi = \ell L = \frac{\sqrt{2m(E+V_0)}}{\hbar} L \qquad \eta = kL = \frac{\sqrt{-2mE}}{\hbar} L \qquad \xi, \eta \geq 0$$

Energy quantized

Cannot be solved analytically. → See graphically.

By eliminating E,

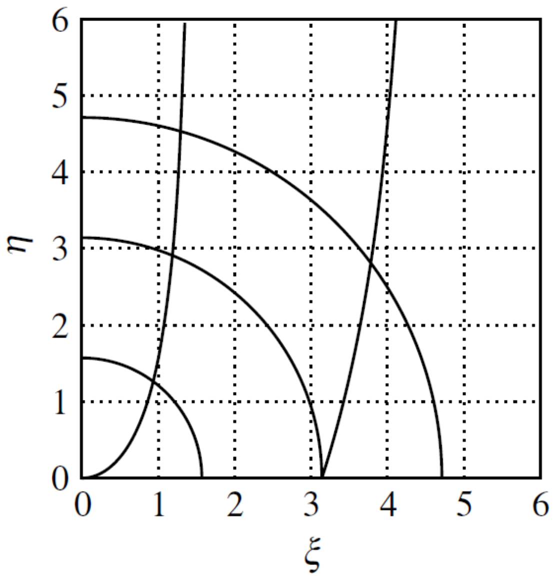
How energy is quantized !

$$\xi^2 + \eta^2 = \frac{2mV_0L^2}{\hbar^2}$$

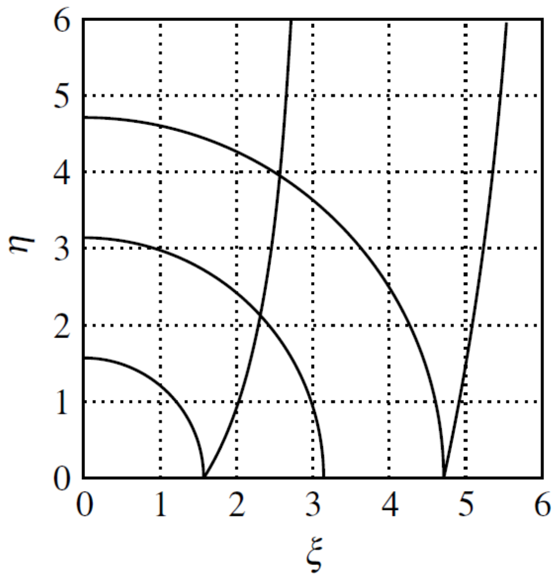
depends on the potential depth V_0.

From the boundary conditions,

Even $\eta = \xi \tan(\xi)$

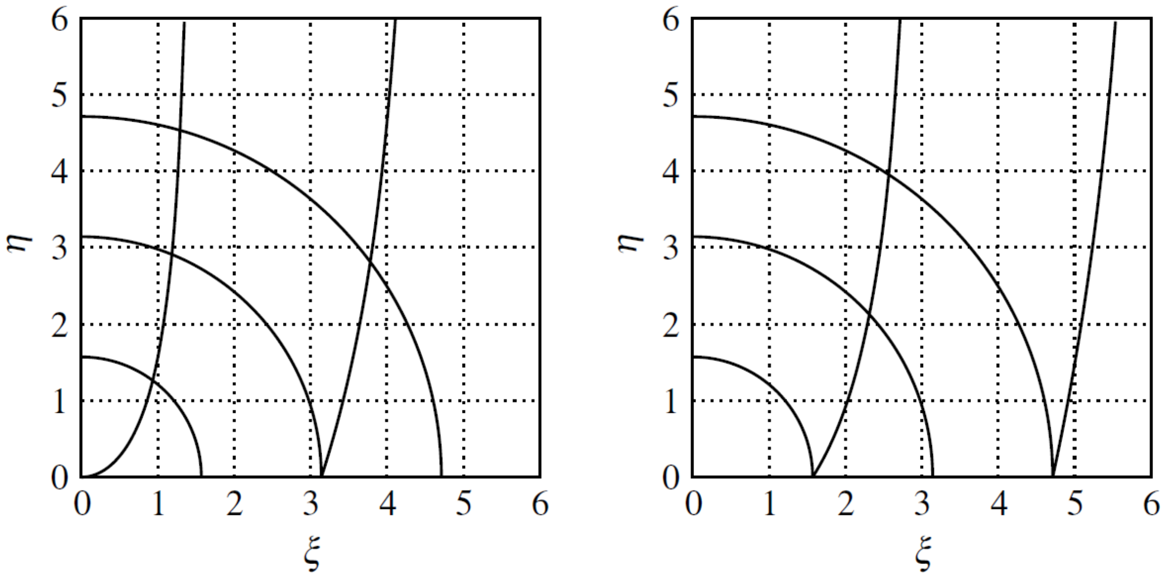


Odd $\eta = -\xi \cot(\xi)$



$$V_0L^2 = \frac{n^2\pi^2\hbar^2}{8m}$$

$$\xi^2 + \eta^2 = \frac{n^2\pi^2}{4} \quad (n = 1, 2, 3)$$



of solutions depends on V_0L^2

Even: $\xi = n\pi \Rightarrow \eta = 0; \quad \xi = \left(n + \frac{1}{2}\right)\pi \Rightarrow \eta = \infty$

Odd: $\xi = (n+1)\pi \Rightarrow \eta = 0; \quad \xi = \left(n + \frac{3}{2}\right)\pi \Rightarrow \eta = \infty$

$$\frac{n^2\pi^2\hbar^2}{8m} < V_0L^2 < \frac{(n+1)^2\pi^2\hbar^2}{8m} \qquad n = 0, 1, 2, \cdots$$

# of solutions $\Rightarrow n+1$

Another way

Even function: $\frac{k}{\ell} = \tan(\ell L)$ $\quad k^2 = \frac{-2mE}{\hbar^2}$ $\quad \ell^2 = \frac{2m}{\hbar^2}(E + V_0)$

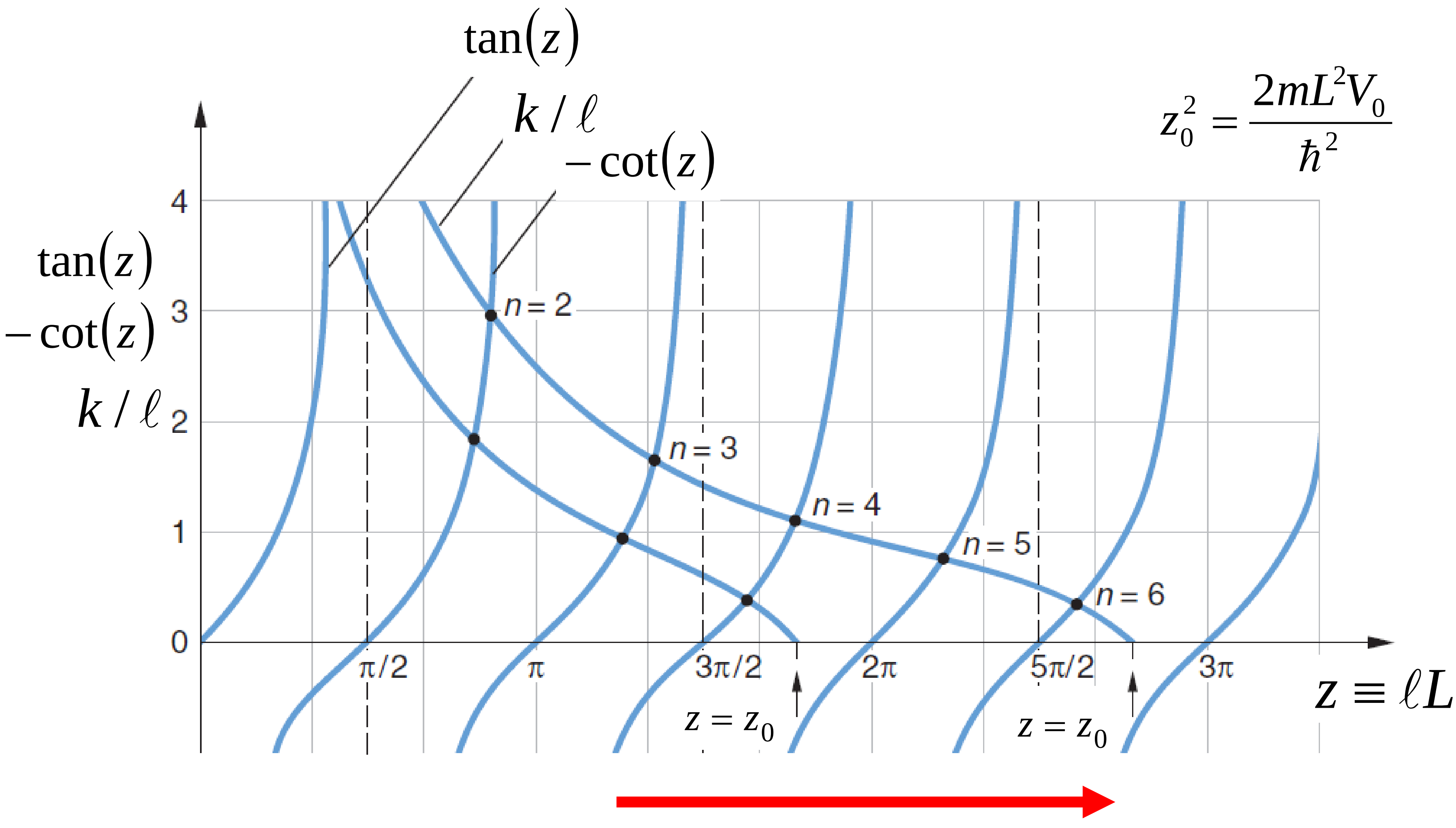
Odd function: $-\frac{k}{\ell} = \cot(\ell L)$

Substituting $z \equiv \ell L$ $\quad z^2 = \frac{2mL^2}{\hbar^2}(E + V_0)$

$$\frac{k}{\ell} = \sqrt{\frac{-2mE/\hbar^2}{z^2/L^2}} = \cdots\cdots\cdots = \sqrt{\frac{z_0^2}{z^2} - 1} \qquad z_0^2 = \frac{2mL^2V_0}{\hbar^2}$$

Even: $\boxed{\tan(z) = \sqrt{\frac{z_0^2}{z^2} - 1}}$ **Odd:** $\boxed{-\cot(z) = \sqrt{\frac{z_0^2}{z^2} - 1}}$

Another graphical solutions



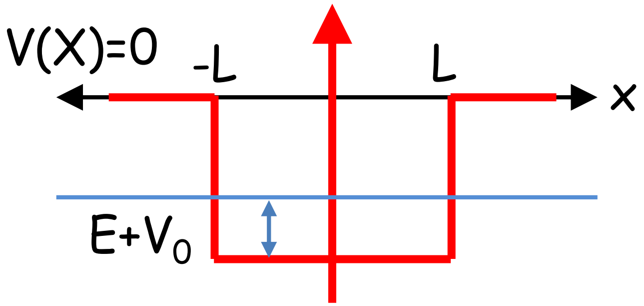
Deeper potential

Limiting cases

Wide deep well: **Large z_0**

cf. Infinite sqaure well
Width: 2L

$$z = \frac{n\pi}{2} \quad (n = 1, 2, 3, \quad \cdots) \quad \Rightarrow \quad E + V_0 = \frac{\hbar^2 n^2 \pi^2}{4 \cdot 2mL^2}$$



Shallow narrow well:

will always have one even bound state.

Wave functions

Even functions:

$$Ae^{-kx} = Ae^{-k(L+x-L)} = Ae^{-kL}e^{-k(x-L)}$$

At x=L:

$$Ae^{-kL} = D\cos(\ell L)$$

$$\psi(x) = \begin{cases} D\cos(\ell L) & (|x| \leq L) \\ D\cos(\ell L)Ae^{-k(|x|-L)} & (|x| > L) \end{cases}$$

Normalization:

$$1 = 2\left[\int_0^a D^2\cos^2(\ell x)dx + \int_a^\infty D^2\cos^2(\ell L)e^{-2k(x-L)}dx\right]$$

$$= D^2\left[L + \frac{1}{2\ell}\sin 2\ell a + \frac{1}{k}\cos^2(\ell L)\right]$$

$$1 = D^2\left(L + \frac{1}{k}\right)$$

$$\sin 2\ell L = 2\sin\ell L\cos\ell L = 2\sin\ell L\frac{\ell}{k}\sin\ell L = \frac{2\ell}{k}\sin^2\ell L$$

$$|D| = \frac{1}{\sqrt{L + \frac{1}{k}}}$$

Odd functions:

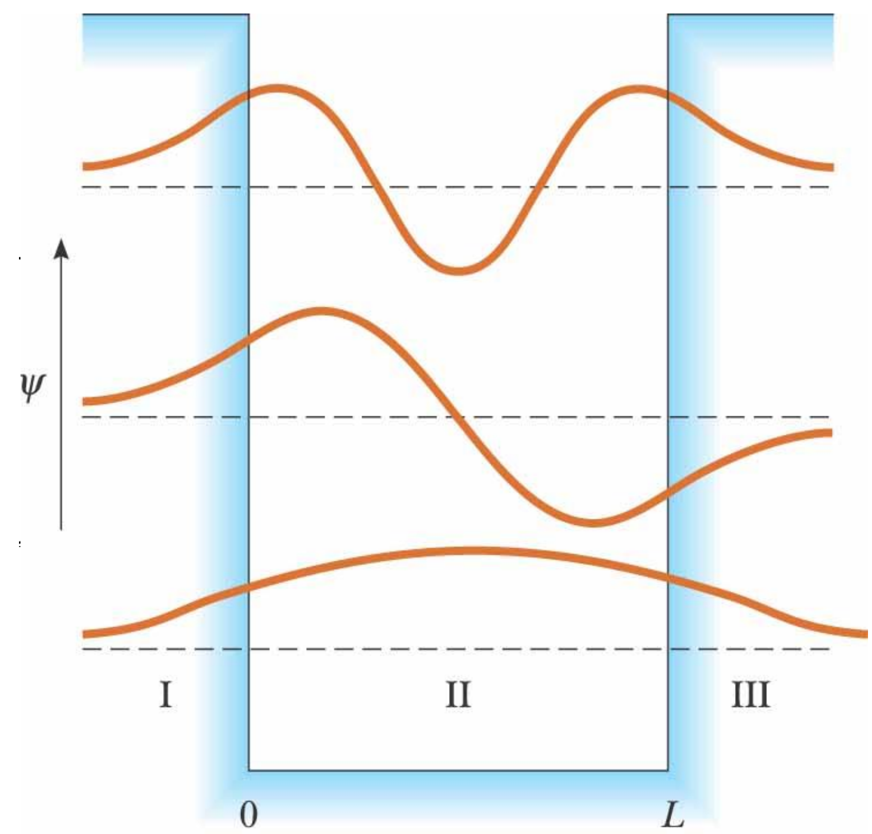
$$\psi(x) = \begin{cases} C\sin(\ell x) & (|x| \le L) \\ \dfrac{x}{|x|} C\sin(\ell L) A e^{-k(|x|-L)} & (|x| > L) \end{cases}$$

Normalization:

$$|C| = \frac{1}{\sqrt{L + \dfrac{1}{k}}}$$

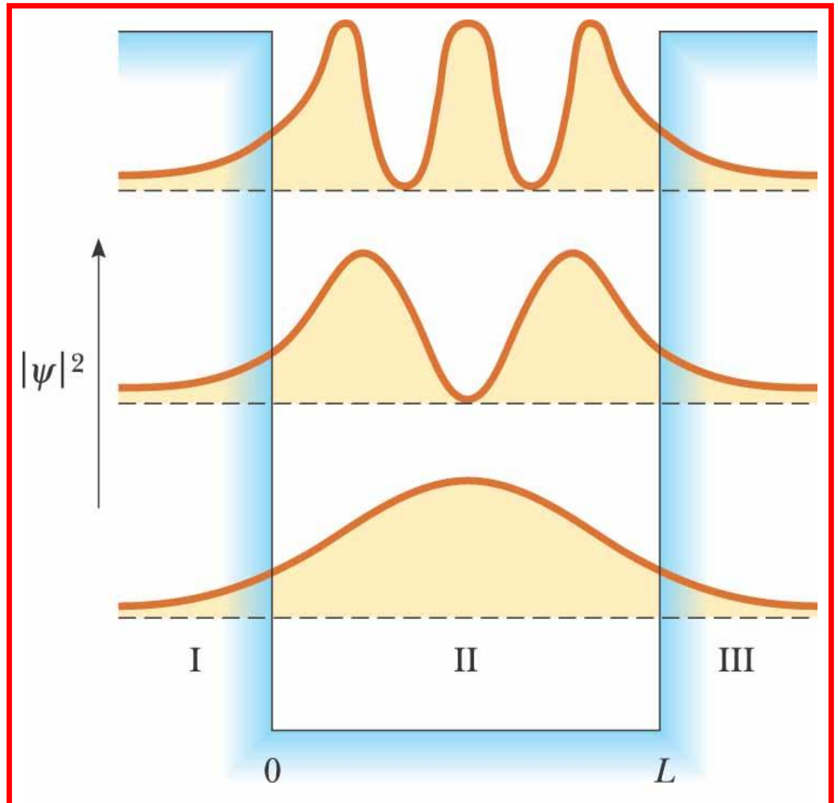
✔ Outside the potential well, classical physics forbids the presence of the particle

✔ Quantum mechanics shows the wave function decays exponentially to approach zero.



Graphical Results for Probability Density, $|\psi(x)|^2$

- The probability densities for the lowest three states are shown
- The functions are smooth at the boundaries
- Outside the box, the probability to find the particle decreases exponentially, **but it is not zero!**



5. Harmonic oscillator

- ✔ Oscillators under restoring force -kx

  potential energy $\frac{1}{2}kx^2$
- ✔ Vibrations of molecules and lattice vibrations can be regarded as harmonic oscillators.

$$-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \qquad \alpha\equiv\frac{m\omega}{\hbar},\quad \lambda\equiv\frac{2E}{\hbar\omega},\quad \xi\equiv\sqrt{\alpha}x$$

See supplemental #1.

Asymptotic solution: $\xi\to$ **large**

$$\Rightarrow\quad \frac{d^2\psi(\xi)}{d\xi^2}-\xi^2\psi(\xi)=-\lambda\psi(\xi) \qquad \Rightarrow\quad \frac{d^2\psi}{d\xi^2}=\xi^2\psi$$

$$\psi(\xi)=Nf(\xi)\exp\left(-\frac{\xi^2}{2}\right) \quad \Leftarrow \quad \psi(\xi)=N\exp\left(-\frac{\xi^2}{2}\right)$$

See supplemental #2. **Remove asymptotic solution:**

$$\Rightarrow\quad \frac{d^2f}{d\xi^2}-2\xi\frac{df}{d\xi}+(\lambda-1)f=0$$

$$\lambda-1=2n$$

Hermite differential eq.

Supplemental #1

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x) \qquad x=\sqrt{\frac{\hbar}{m\omega}}\xi$$

$$-\frac{\hbar}{2m}\frac{\partial^2\psi\left(\sqrt{\frac{\hbar}{m\omega}}\xi\right)}{\partial x^2}+\frac{1}{2}m\omega^2\frac{\hbar}{m\omega}\xi^2\psi(x)=E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x}\right)=-\frac{\hbar^2}{2m}\sqrt{\frac{m\omega}{\hbar}}\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial\xi}{\partial x}=-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{\partial^2\psi}{\partial\xi^2}$$

$$-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{\partial^2\psi}{\partial\xi^2}+\frac{1}{2}m\omega^2\frac{\hbar}{m\omega}\xi^2\psi(x)=E\psi(x)$$

$$\Rightarrow\quad\frac{\partial^2\psi}{\partial\xi^2}=\left(\xi^2-\lambda\right)\psi \qquad \lambda\equiv\frac{2E}{\hbar\omega}$$

Supplemental #2

$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial \xi^2} = \left(\xi^2 - \lambda\right)\psi \qquad\qquad \psi(\xi) = Nf(\xi)\exp\left(-\frac{\xi^2}{2}\right)$$

$$\frac{d\psi}{d\xi} = \frac{df}{d\xi} e^{-\xi^2/2} - f\xi e^{-\xi^2/2}$$

$$\frac{d^2\psi}{d\xi^2} = \frac{d^2 f}{d\xi^2} e^{-\xi^2/2} - 2\frac{df}{d\xi}\xi e^{-\xi^2/2} - fe^{-\xi^2/2} + f\xi^2 e^{-\xi^2/2}$$

$$= \left(\frac{d^2 f}{d\xi^2} - 2\frac{df}{d\xi}\xi + \left(\xi^2 - 1\right)\right)e^{-\xi^2/2}$$

$$\Rightarrow \qquad \frac{d^2 f}{d\xi^2} - 2\xi\frac{df}{d\xi} + (\lambda - 1)f = 0$$

Solution by power series

$$\frac{d^2 f}{d\xi^2} - 2\xi \frac{df}{d\xi} + (\lambda - 1) f = 0$$

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + \cdots = \sum_{\ell=0}^{\infty} c_\ell \xi^\ell \qquad f'(\xi) = \sum_{\ell=0}^{\infty} \ell c_\ell \xi^{\ell-1}$$

$$f''(\xi) = 1\cdot 2c_2 + 2\cdot 3c_3\xi + 3\cdot 4c_4\xi^2 + \cdots + (\ell+1)(\ell+2)c_{\ell+2}\xi^\ell + \cdots$$

$$= \sum_{\ell=0}^{\infty} (\ell+1)(\ell+2)c_{\ell+2}\xi^\ell$$

$$\Rightarrow \qquad \sum_{\ell=0}^{\infty} \{(\ell+1)(\ell+2)c_{\ell+2} - 2\ell c_\ell + (\lambda-1)c_\ell\}\xi^\ell = 0$$

This equation holds for any ξ

$$\Rightarrow \qquad (\ell+1)(\ell+2)c_{\ell+2} = (2\ell - \lambda + 1)c_\ell$$

Recurrence relation

$$\Rightarrow \qquad c_{\ell+2} = \frac{(2\ell - \lambda + 1)}{(\ell+1)(\ell+2)} c_\ell$$

Infinitely??

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \cdots$$

Termination of power series

If the power series does not terminate,

the infinite expansion of the terms may occur..

→ Inconsistency of the prerequisite that the wave functions are normalizable.

To avoid infinite expansion of the terms,

→ Terminate the highest power.

$$\boxed{\lambda = 2\ell + 1} \quad \Rightarrow \quad E = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$\lambda \equiv \frac{2E}{\hbar\omega} \qquad \ell \to n$$

Energy

Quantized.

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \cdots$$

Either evens or odds. Not both.

The first few

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \cdots$$

$$c_0 \neq 0,\ n = 0,\ c_1 = 0$$

$$f_0 = c_0 \qquad \psi_0(\xi) = c_0 e^{-\xi^2/2}$$

$$c_0 = 0,\ n = 1,\ c_1 \neq 0$$

$$f_1 = c_1\xi \qquad \psi_1(\xi) = c_0 \xi e^{-\xi^2/2}$$

$$c_0 \neq 0,\ n = 2,\ c_1 = 0$$

$$c_{j+2} = \frac{2j - (2n+1) + 1}{(j+2)(j+1)} c_j = \frac{-2(n-j)}{(j+2)(j+1)} c_j$$

$$f_2 = c_0 + c_0 \frac{-2(2-0)}{2 \cdot 1} \xi^2 = c_0 (1 - 2\xi^2)$$

$$\psi_2(\xi) = c_0 (1 - 2\xi^2) e^{-\xi^2/2}$$

Hermite polynomials

Wave functions:

$$\psi_n(\xi) = N_n H_n(\xi)\exp\left(-\frac{\xi^2}{2}\right), \qquad \xi = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x$$

$H_n(\xi)$: **Hermite polynomial**

$$H_0(\xi) = 1$$
$$H_1(\xi) = 2\xi$$
$$H_2(\xi) = 4\xi^2 - 2$$
$$H_3(\xi) = 8\xi^3 - 12\xi$$
$$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$$

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi)\exp\left(-\xi^2\right)d\xi = 2^n n!\sqrt{n} \qquad (n = m)$$

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi)\exp\left(-\xi^2\right)d\xi = 0 \qquad (n \neq m)$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \frac{N_n^2}{\sqrt{\alpha}} \int_{-\infty}^{\infty} |H_n(\xi)|^2 d\xi = \frac{N_n^2}{\sqrt{\alpha}} 2^n n!\sqrt{n} = 1$$

$$\Rightarrow \qquad N_n = \left(\frac{1}{2^n n!}\sqrt{\frac{\alpha}{\pi}}\right)^{\frac{1}{2}}$$

Harmonic oscillators

$$\psi_n(\xi) = \left(\frac{1}{2^n n!} \sqrt{\frac{2m\omega}{h}} \right)^{\frac{1}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp\left(-\frac{1}{2} \frac{m\omega}{\hbar} x^2 \right)$$

$$E_{n+1} - E_n = \left((n+1) + \frac{1}{2} \right)\hbar\omega - \left(n + \frac{1}{2} \right)\hbar\omega = \hbar\omega \qquad E_0 = \frac{1}{2}\hbar\omega$$