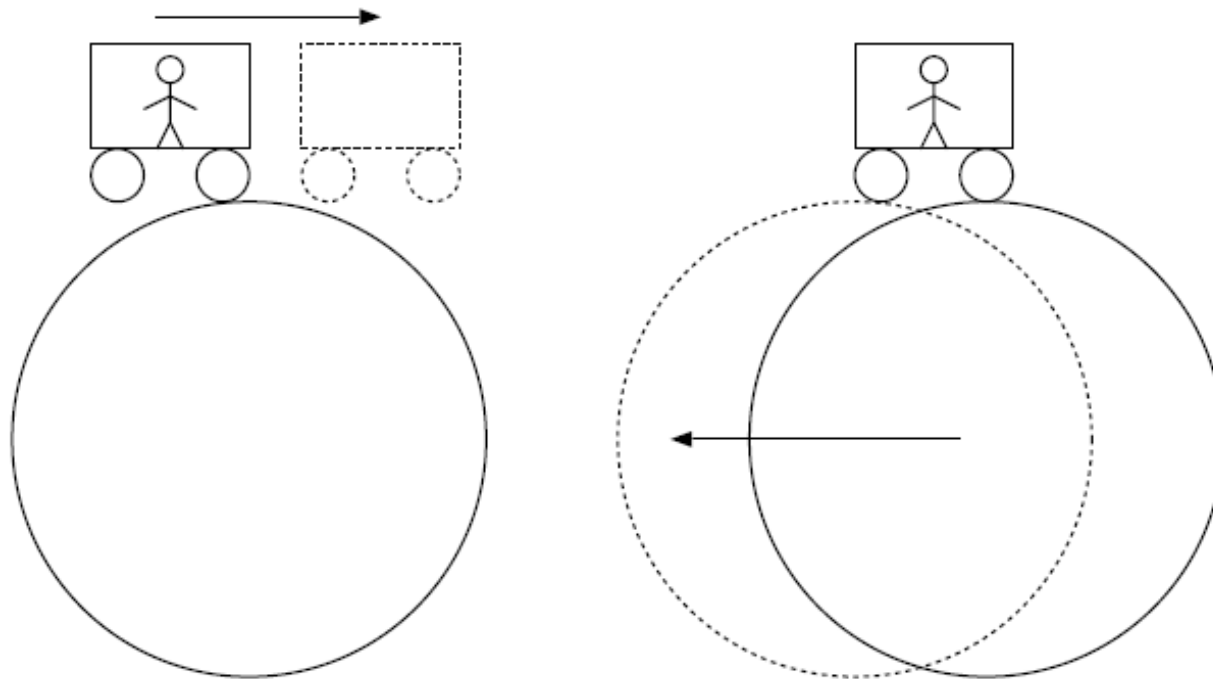


What is "special relativity"?

There is nothing in the world that can claim that "I am absolutely stationary".



"Absolute space" in electromagnetism

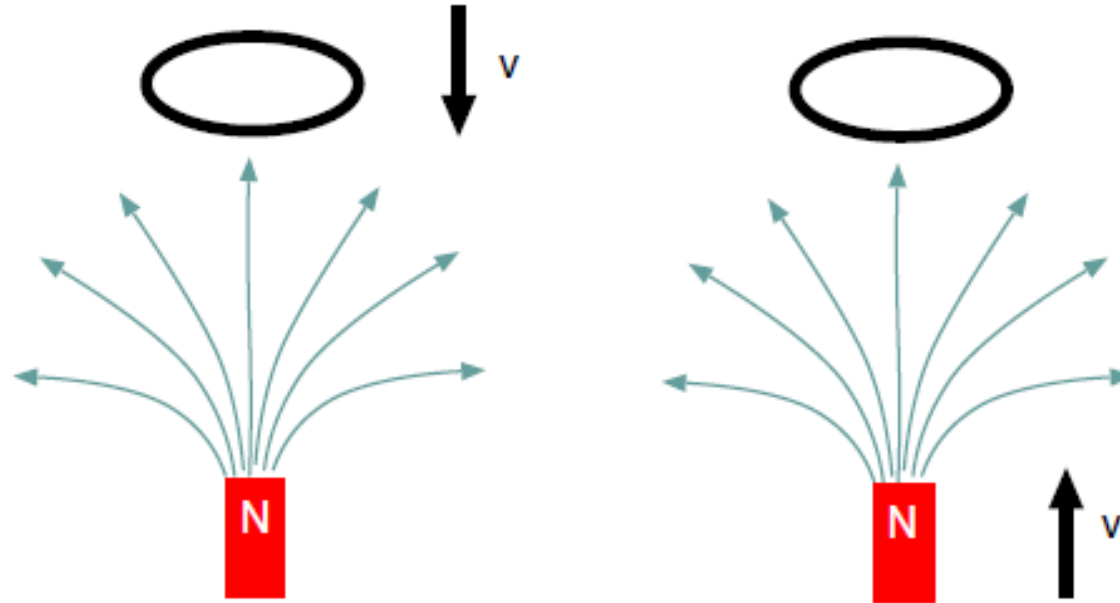
About the end of the 19th century, physicists were aware that there was no "absolute space" in mechanics, but in electromagnetism people thought that there might be "absolute space". An indicator of the absolute space was "ether" which was a medium of light. It was discovered by Maxwell that light is electric and magnetic waves called electromagnetic waves. In the case of waves, there should be a medium, and it is natural to think that the speed of the wave would change if the medium moves.

However, attempts to detect "ether's wind" failed, and electromagnetism also found that there is no "absolute space" (or even if there is it) it can not be detected.

Make a theory that absolutely has no absolute space.

From any frame of reference
physical laws are the same??

Example



As the electrons move downward in the magnetic field, the electrons are moved by the Lorentz force.

An induced electric field that swirls due to the change in magnetic flux density is generated.

What is the theory that absolute space does not exist mechanically and electromagnetically?

Framework of Relativity

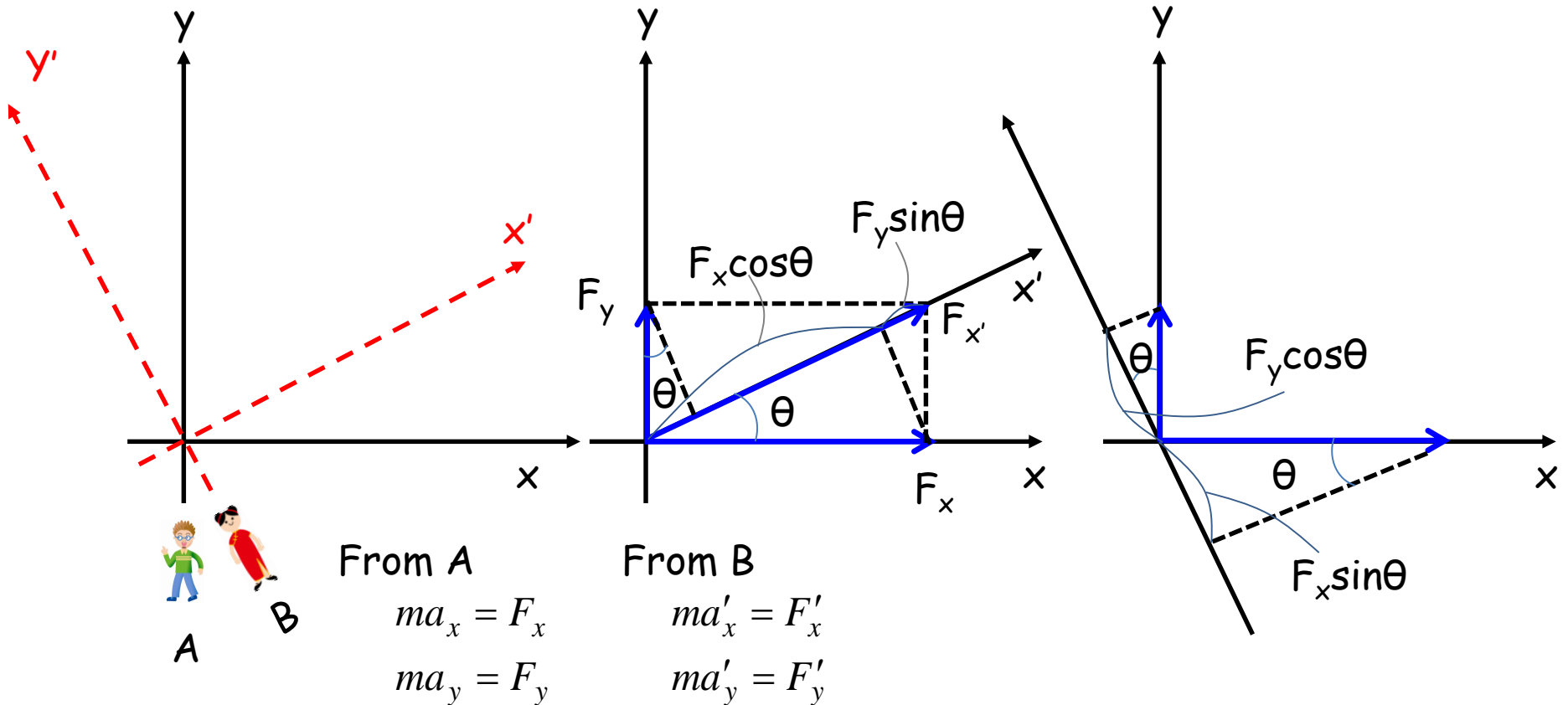
The theory should be the one that electric and magnetic fields look like mixed, which makes electric fields seen as magnetic fields while you are moving, and vice versa. But the final result is more than that.

If we modify the theory so that there is no absolute space from electromagnetics, mechanics will also be modified as a result. On the contrary, it turns out that the scale measuring the length of the objects has to change depending on the states of the observer.

Specifically, "When looking while moving (or when the object moves) the object shrinks". Furthermore, the theory of relativity will deny not only "absolute space" but also "absolute time". It turns out that even if the position is different, time is not the same thing. There is also a result that "time goes late when moving", and that "happening at the same time for one person is not simultaneous for another person" occurs.

Covariance of equations

Equations representing physical laws are expressed in the same form for the observers.



Inertial frame of reference

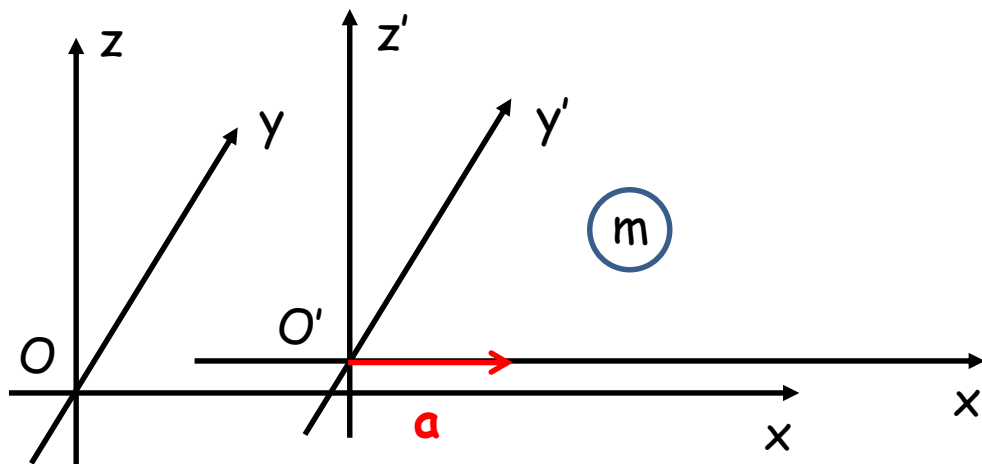
Newton's law of inertia:

An object not subjected to external force is stationary or moves at a constant linear motion.

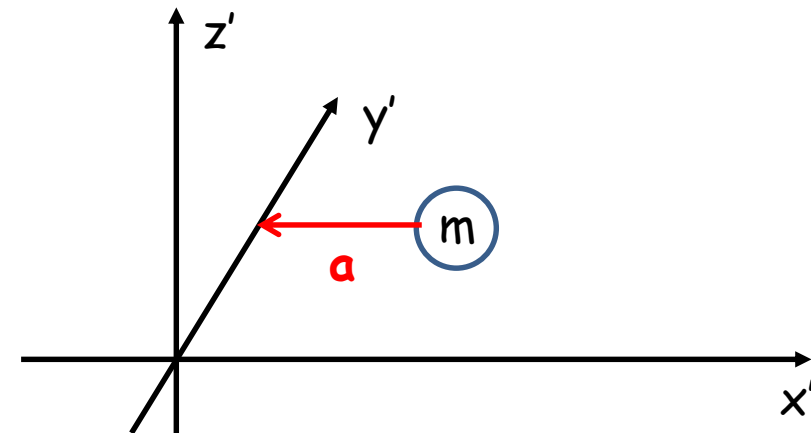
Where is the inertial system?

The inertial system is the product of thought, not reality.

The reason to come up with an inertial system is that there are cases where it can be regarded as an inertial system in a limited range.

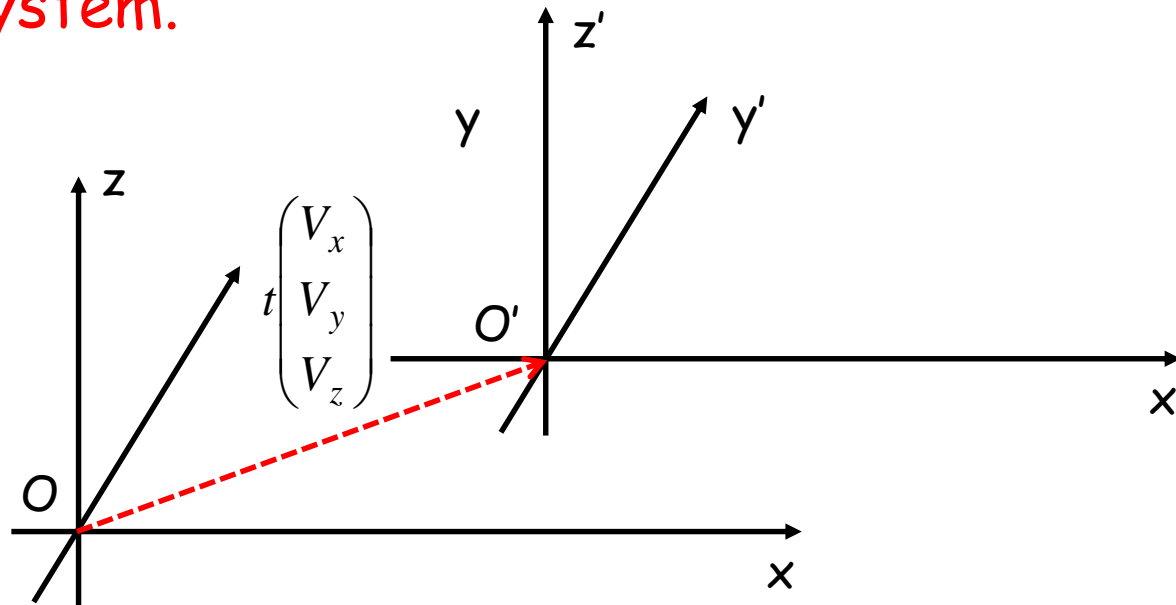


Acceleration frame



Galilean transform

The coordinate system that is moving at a constant linear velocity with respect to the inertial system is an inertial system.



$$\mathbf{x}' = \mathbf{x} - t\mathbf{V}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - t \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Covariance to Galilean Transform

$$\frac{dx'}{dt} = \frac{dx}{dt} - V_x, \quad \frac{dy'}{dt} = \frac{dy}{dt} - V_y, \quad \frac{dz'}{dt} = \frac{dz}{dt} - V_z$$

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}, \quad \frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}, \quad \frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2}$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x} \frac{\partial}{\partial z'} = \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial z'^2} = \Delta'$$

✓ Newton's equation of motion

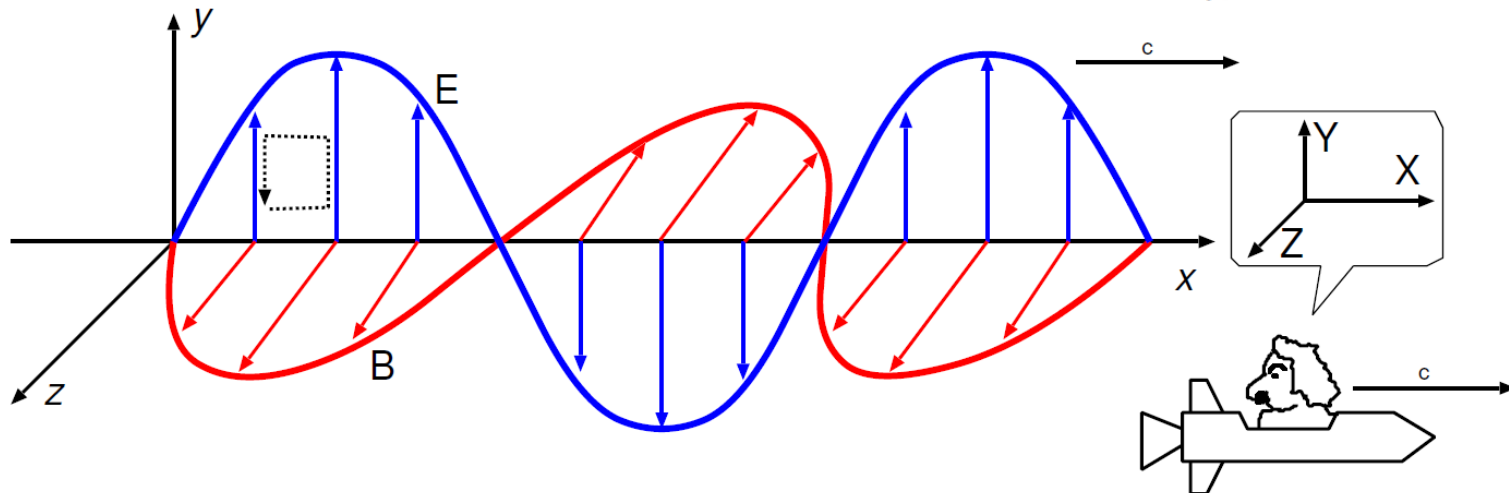
✓ Newton's gravity equation

$$\Delta\phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}) \quad \rightarrow \quad \Delta'\phi(\mathbf{x}') = 4\pi G\rho(\mathbf{x}')$$

Relativity in electromagnetics

Can you see electrostatic fields and static magnetic fields shaped like waves when flying at the speed of light?

$$E_x = E_z = 0, E_y = E_0 \sin k(x - ct), B_x = B_y = 0, B_z = \frac{E_0}{c} \sin k(x - ct)$$



$$X = x - ct, T = t$$

$$E_X = E_Z = 0, E_Y = E_0 \sin kX, B_X = B_Y = 0, B_Z = \frac{E_0}{c} \sin kX$$

$$\text{rot} \vec{E} \quad Z \quad \partial_X E_Y = k E_0 \cos kX \quad \frac{\partial \vec{B}}{\partial T} = 0$$

$$\text{rot} \mathbf{E} \neq -\frac{\partial \mathbf{B}}{\partial t}$$

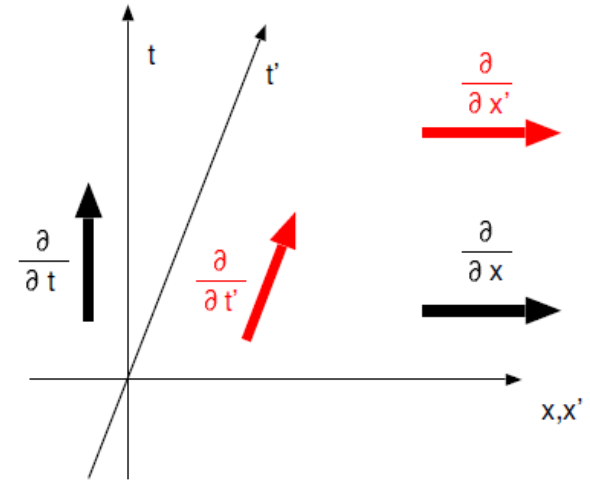
How about Maxwell's wave equation?

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t} \frac{\partial}{\partial z'} = \frac{\partial}{\partial t'} - V_x \frac{\partial}{\partial x'} - V_y \frac{\partial}{\partial y'} - V_z \frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x} \frac{\partial}{\partial z'} = \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

d'Alembertian (Wave operator): \square

$$\begin{aligned} \square &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \\ &= \left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 + \left(\frac{\partial}{\partial z} \right)^2 - \frac{1}{c^2} \left(\frac{\partial}{\partial t} \right)^2 \\ &= \left(\frac{\partial}{\partial x'} \right)^2 + \left(\frac{\partial}{\partial y'} \right)^2 + \left(\frac{\partial}{\partial z'} \right)^2 - \frac{1}{c^2} \left(\frac{\partial}{\partial t'} - V_x \frac{\partial}{\partial x'} - V_y \frac{\partial}{\partial y'} - V_z \frac{\partial}{\partial z'} \right)^2 \\ &\neq \square' \end{aligned}$$



Galilean Transform of Maxwell eqs.

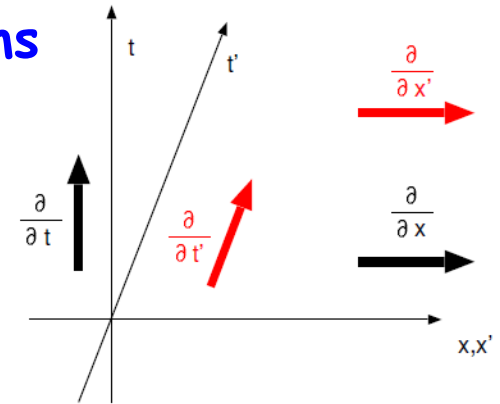
$\text{div} \vec{B} = 0 \quad \text{div} \vec{E} = 0$: No change in x and x' systems

$\text{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Maxwell eqs. in the x' system

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = -\frac{\partial B_z}{\partial t'}$$

$$\begin{aligned} \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} - v_x \frac{\partial B_z}{\partial x} - v_y \frac{\partial B_z}{\partial y} - v_z \frac{\partial B_z}{\partial z} \\ &= -\frac{\partial B_z}{\partial t} - v_x \frac{\partial B_z}{\partial x} - v_y \frac{\partial B_z}{\partial y} + v_z \frac{\partial B_x}{\partial x} + v_z \frac{\partial B_y}{\partial y} \\ &= -\frac{\partial B_z}{\partial t} - v_x \frac{\partial B_z}{\partial x} + v_z \frac{\partial B_x}{\partial x} - v_y \frac{\partial B_z}{\partial y} + v_z \frac{\partial B_y}{\partial y} \end{aligned}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x} (\vec{v} \times \vec{B})_y - \frac{\partial}{\partial y} (\vec{v} \times \vec{B})_x \quad \text{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \text{rot}(\vec{v} \times \vec{B})$$



$$\begin{aligned} \text{div} \vec{B} = 0 & \quad \text{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \text{rot}(\vec{v} \times \vec{B}) \\ \text{div} \vec{D} = \rho & \quad \text{rot} \vec{H} = \frac{\partial \vec{D}}{\partial t} - \text{rot}(\vec{v} \times \vec{D}) + \vec{j} + \rho \vec{v} \end{aligned}$$

Hertz eqs.

In the universe there is a special coordinate system in which the Maxwell equation holds, and the Hertz equation is established in the coordinate system that is moving with respect to the special coordinate system??

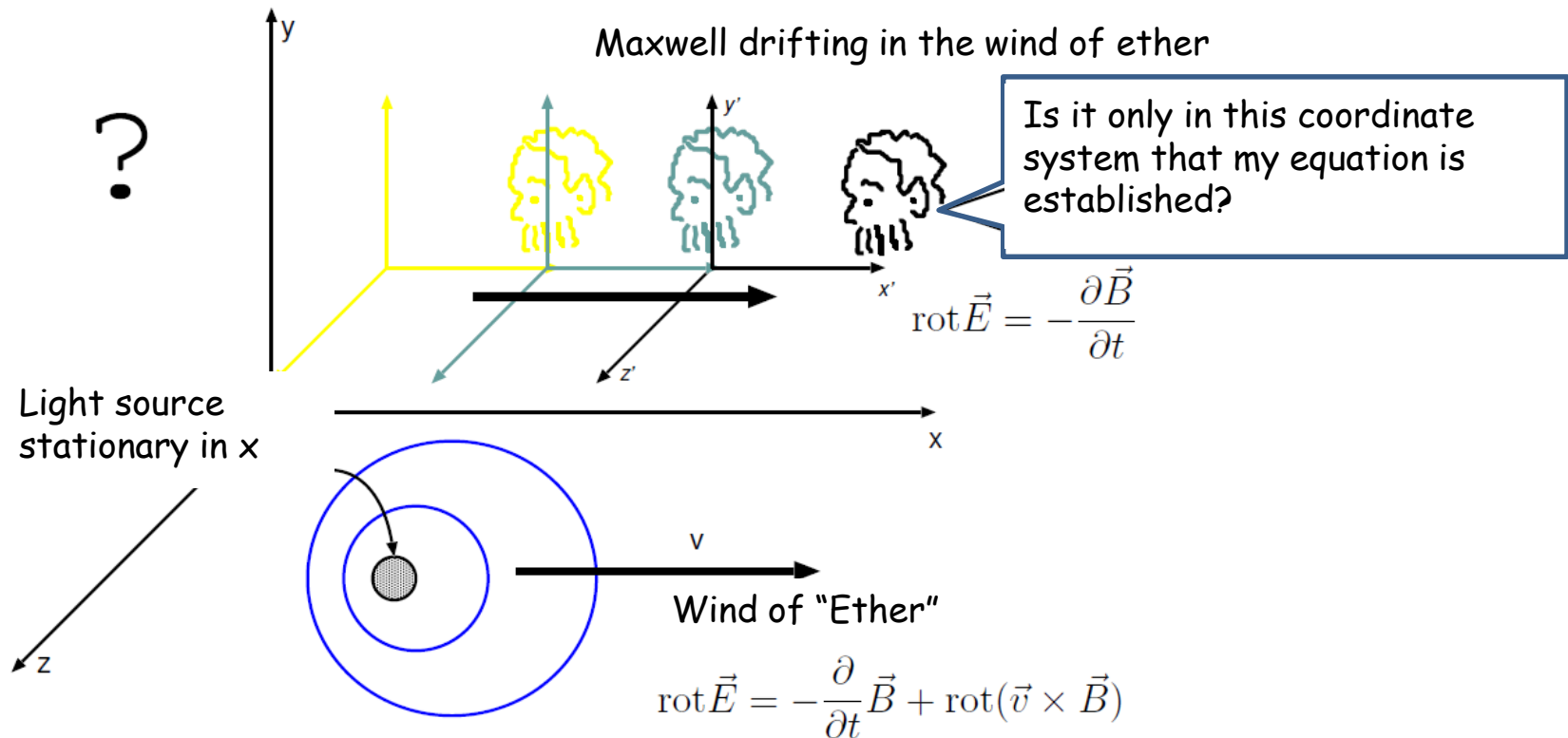
Ether: presence of absolute frame of reference?

Two coordinate systems that can be converted with Galilean transform to each other.

$$x' = x - vt$$

x' : Maxwell's equations hold.

x : Hertz's equations

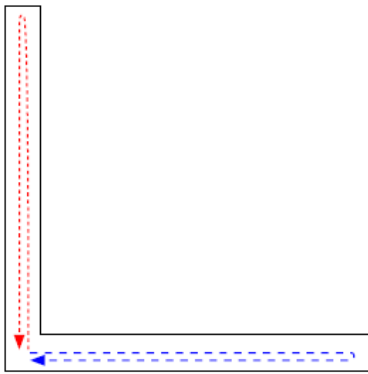


The electromagnetism based on Maxwell's equation has weak point that it can not be applied by Galilean transform, in other words, "it can only be applied with special coordinate system (ether stationary system)".

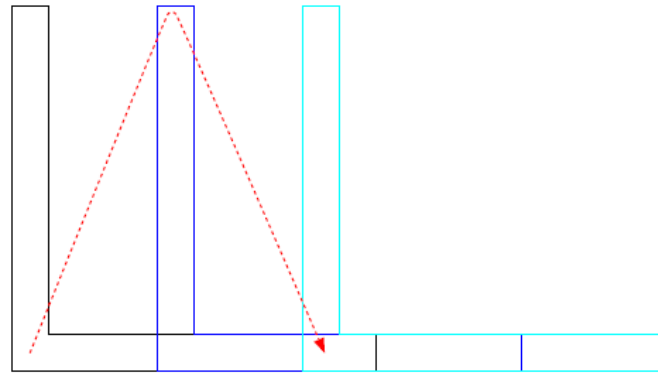
Then, is the Earth moving against ether?

Michelson-Morley experiment

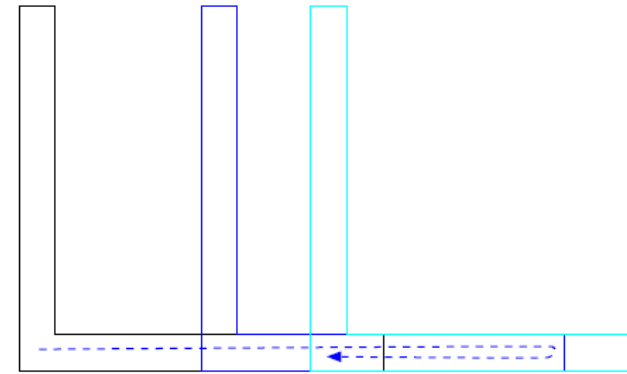
The simplest way would be to "fire light at point A and receive at point B. Divide by the time it took the distance between point A and point B". In modern times where you can precisely measure time using atomic clocks, you can do exactly this experiment. However, we can not do such measurements at the time. Let's detect speed change by using interference



Experimental equipment is stationary against ether.



North-South light with experimental equipment moving relative to ether.



East-West light with experimental equipment moving relative to ether.

Considering that the Earth will move greatly in the east-west direction from the north-south direction (assuming the sun is stationary and thinking that we are watching the movement of the Earth from the sun, this is plausible) It seems that the difference seems to come out.

Michelson-Morley experiment

From the viewpoint where ether is stationary,

$$(ct)^2 = (vt)^2 + L^2$$

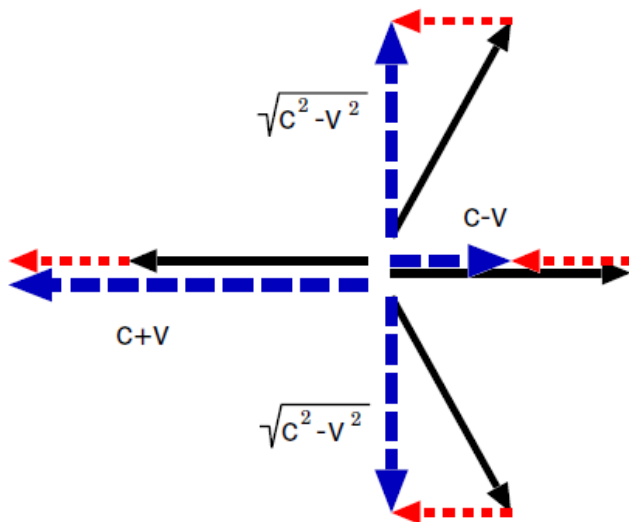
$$t_{\text{南北}} = \frac{2L}{\sqrt{c^2 - v^2}}$$

$$L + vt_1 = ct_1$$

$$t_{\text{東西}} = \frac{L}{c - v} + \frac{L}{c + v} = \frac{2cL}{c^2 - v^2}$$

$$L - vt_2 = ct_2$$

From the viewpoint where the experimental apparatus is stationary,

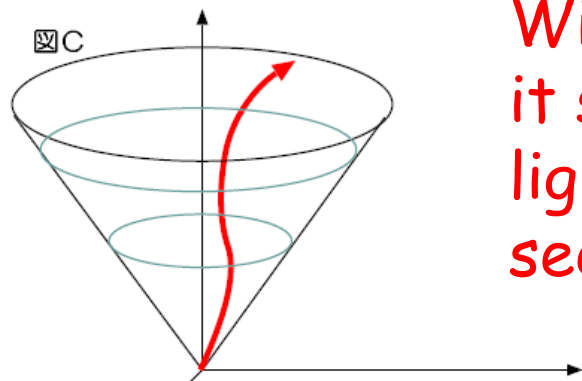
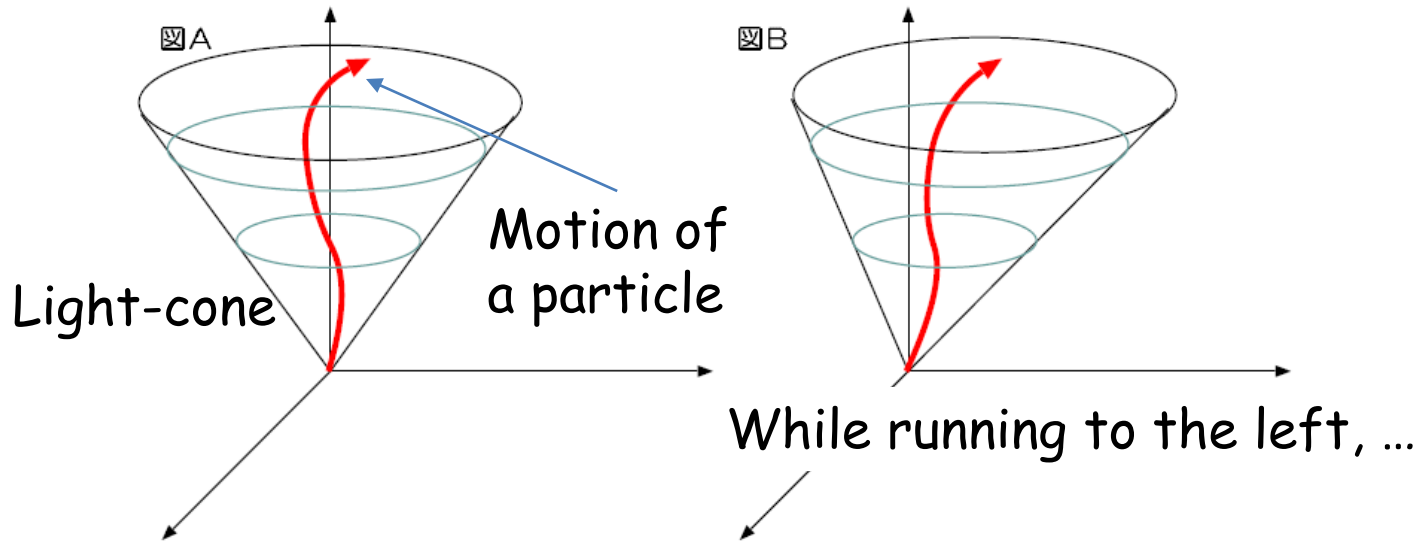


→
The speed of light
when there was no ether wind.

←- - -
Wind of ether

- - - →
The speed of light in
the wind of ether

Light propagation and Galilean transformation



With the idea of Galilean transformation, it seems unlikely to explain the fact that light is the same speed no matter who sees it.

The speed of light does not change upon looking while moving, even though the movement of the substance is changing.

So far, we have learned that the Maxwell equation is not invariant with Galilean transformation.

- ✓ Maxwell's equations can be thought of as equations that can be established only in a specific coordinate system.
- ✓ It can be considered that the Galilean transformation is incorrect.

However, the former was denied by experiment, so we need to think about the latter.

As a result of Michelson Morley and other experiments, there is the fact that "the light velocity is c , regardless of how it is moved." In other words, Maxwell's equation should be considered to be established in all inertial systems. That is, we must make a theory to match it. Therefore, it is necessary to correct the Galilean transformation. Einstein called it **special relativity** that "every law of physics is the same in all inertial systems". If the Maxwell equation is also included in this physical law, this should include the principle that the light velocity is invariance. And in order for this principle to be established, coordinate transform which is not Galilean transformation must be made.

Special relativity

Principle of invariant light speed

The speed of light is absolute and does not change.

The value of the light speed is actually derived from the law of electromagnetism. Therefore, assuming the special relativity principle, the principle of the light speed invariance is conversely required. Since the speed of light is a value automatically determined from the law of electromagnetics, if the speed of light changes depending on the observers, the law of electromagnetism will also be changed by the observers. If the law of electromagnetism is changed depending on the observers, it contradicts the special relativity principle that "every physical law does not depend on the observers".

Lorentz transformation

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(-\beta(ct) + x)$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

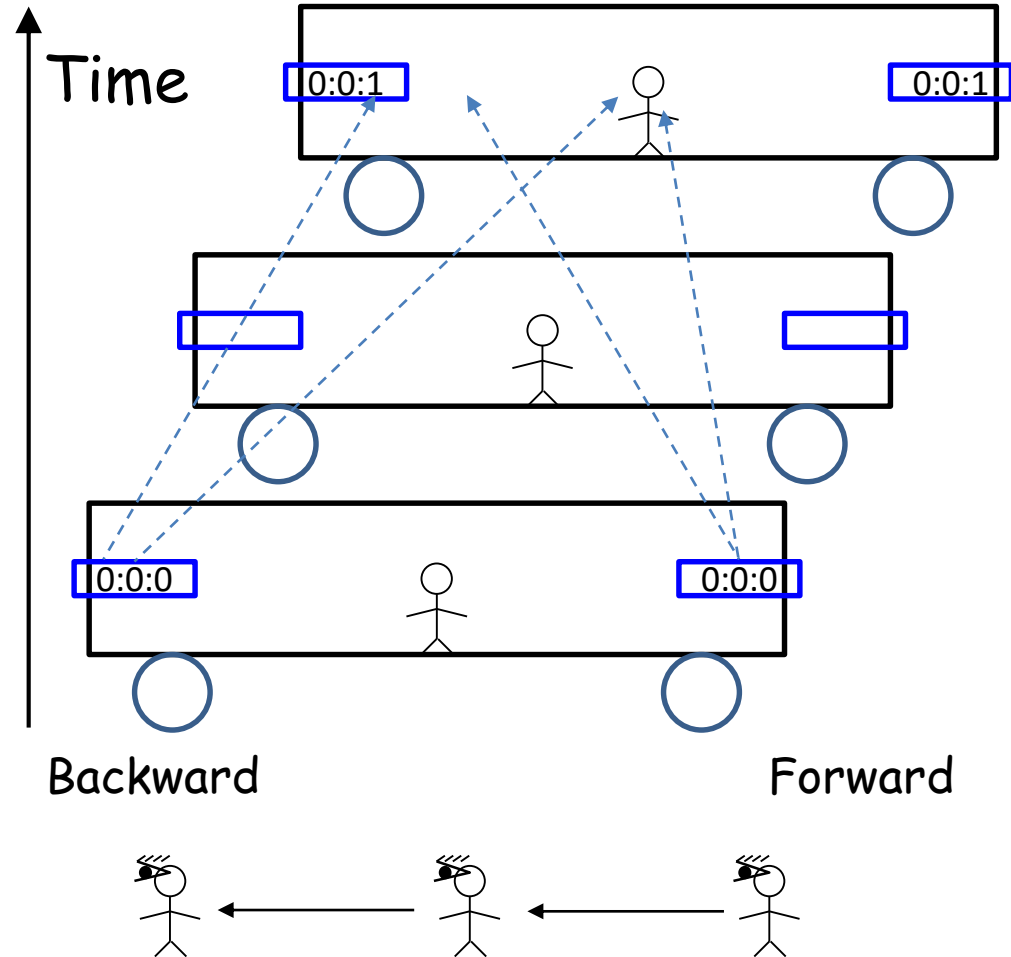
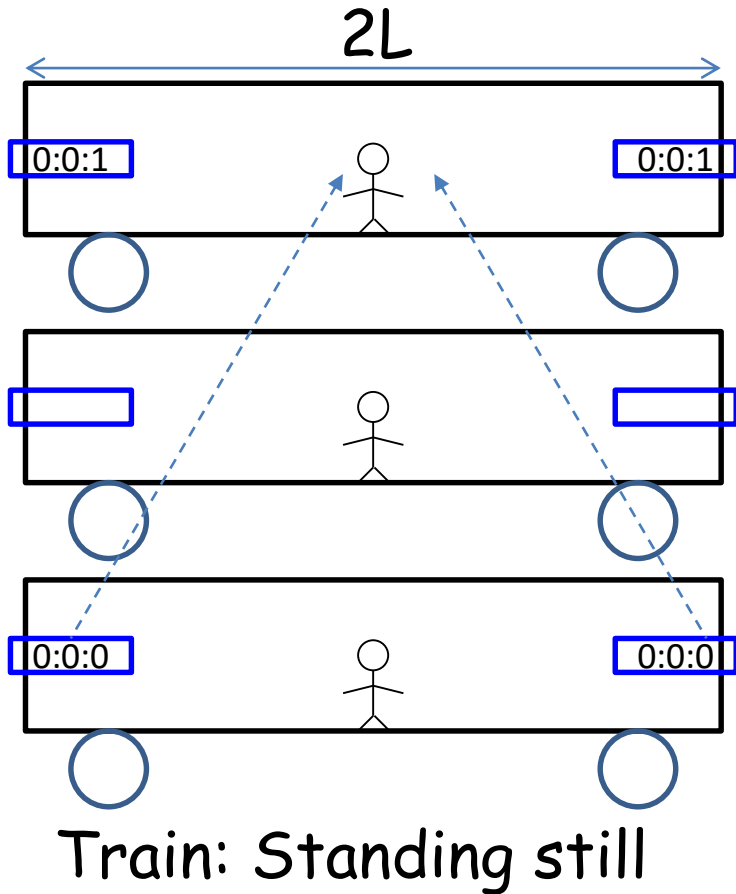
$$x' = x - Vt$$

$$y' = y$$

$$z' = z$$

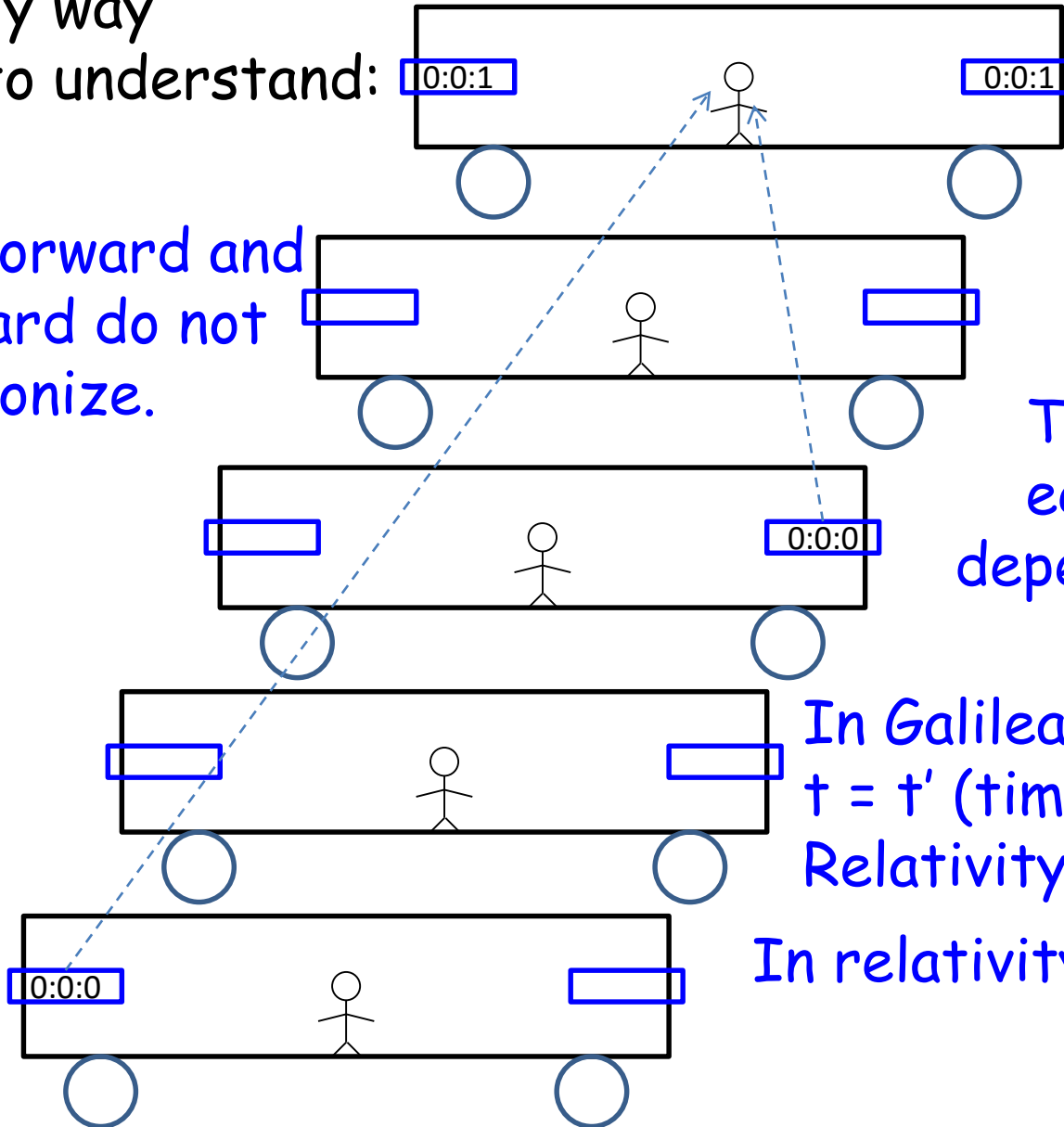
$$\left(\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \beta = \frac{V}{c} \right)$$

What can be deduced from the invariance of the speed of light. - Lorentz transform -



Relative simultaneity

The only way
to understand:



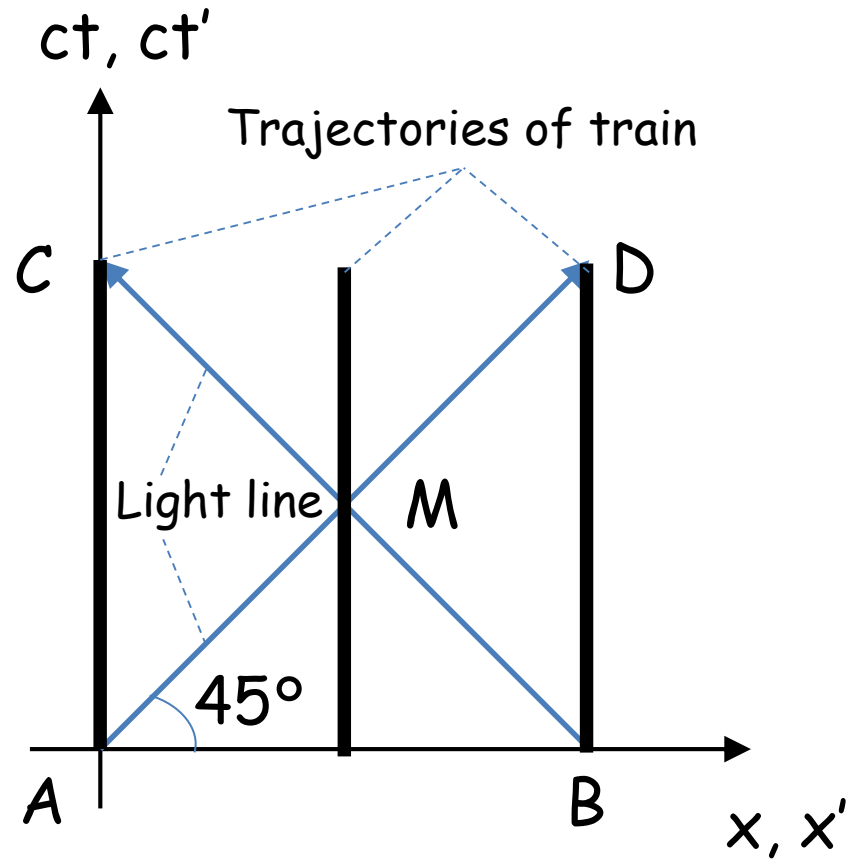
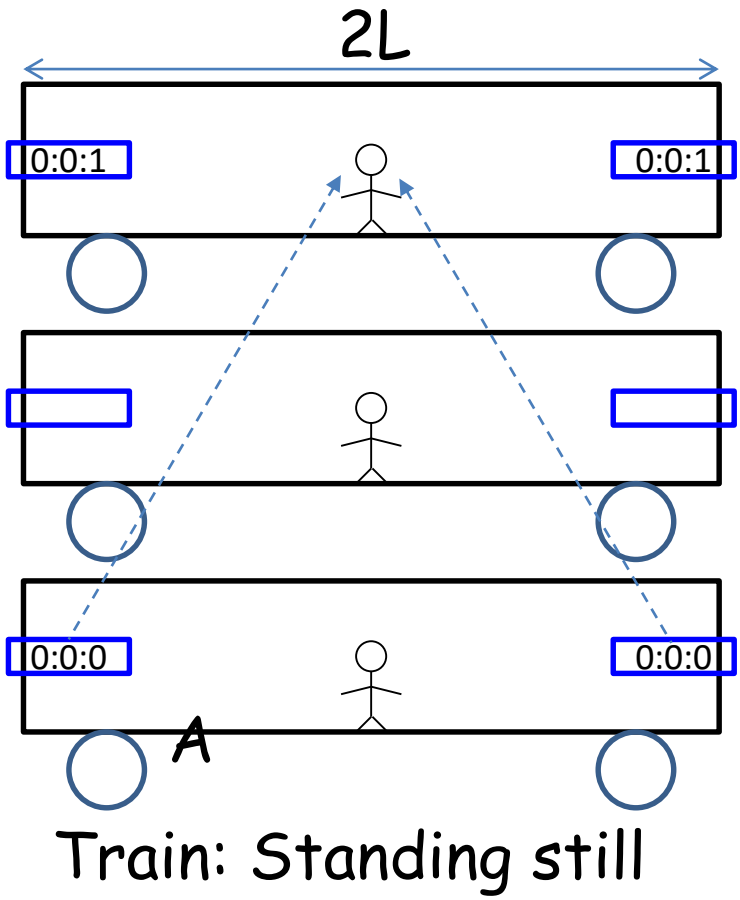
Time forward and
backward do not
synchronize.

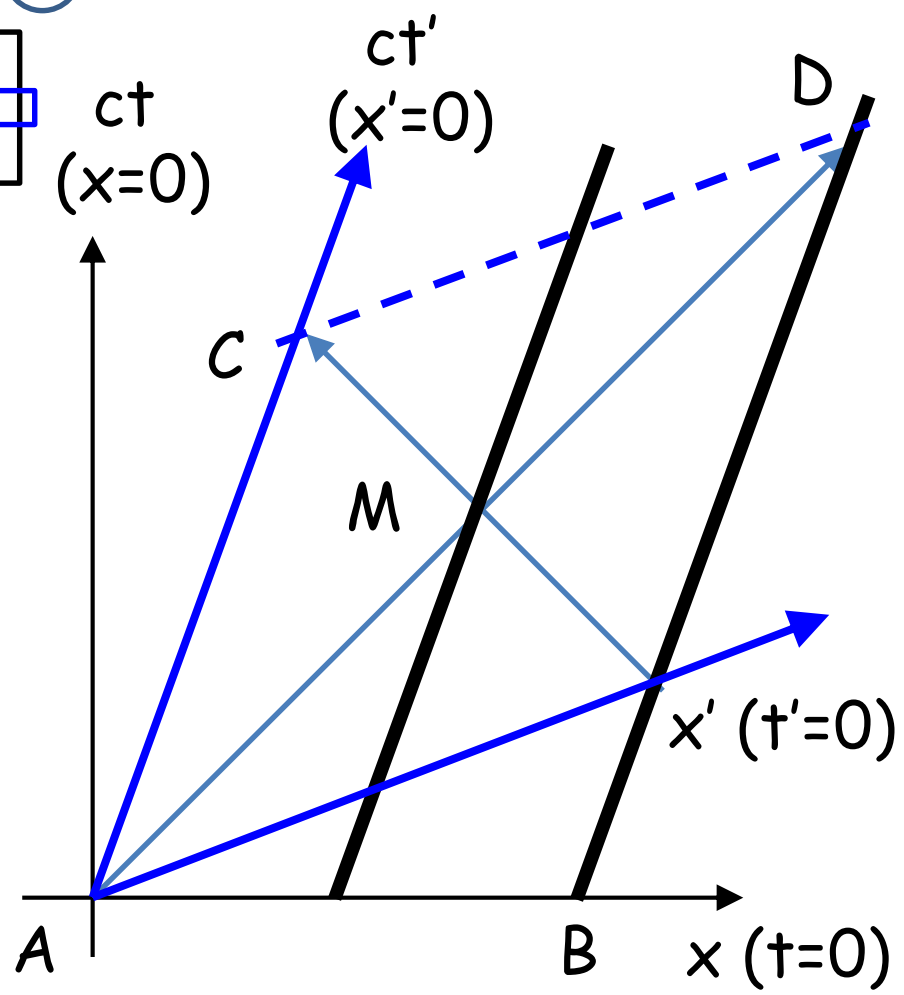
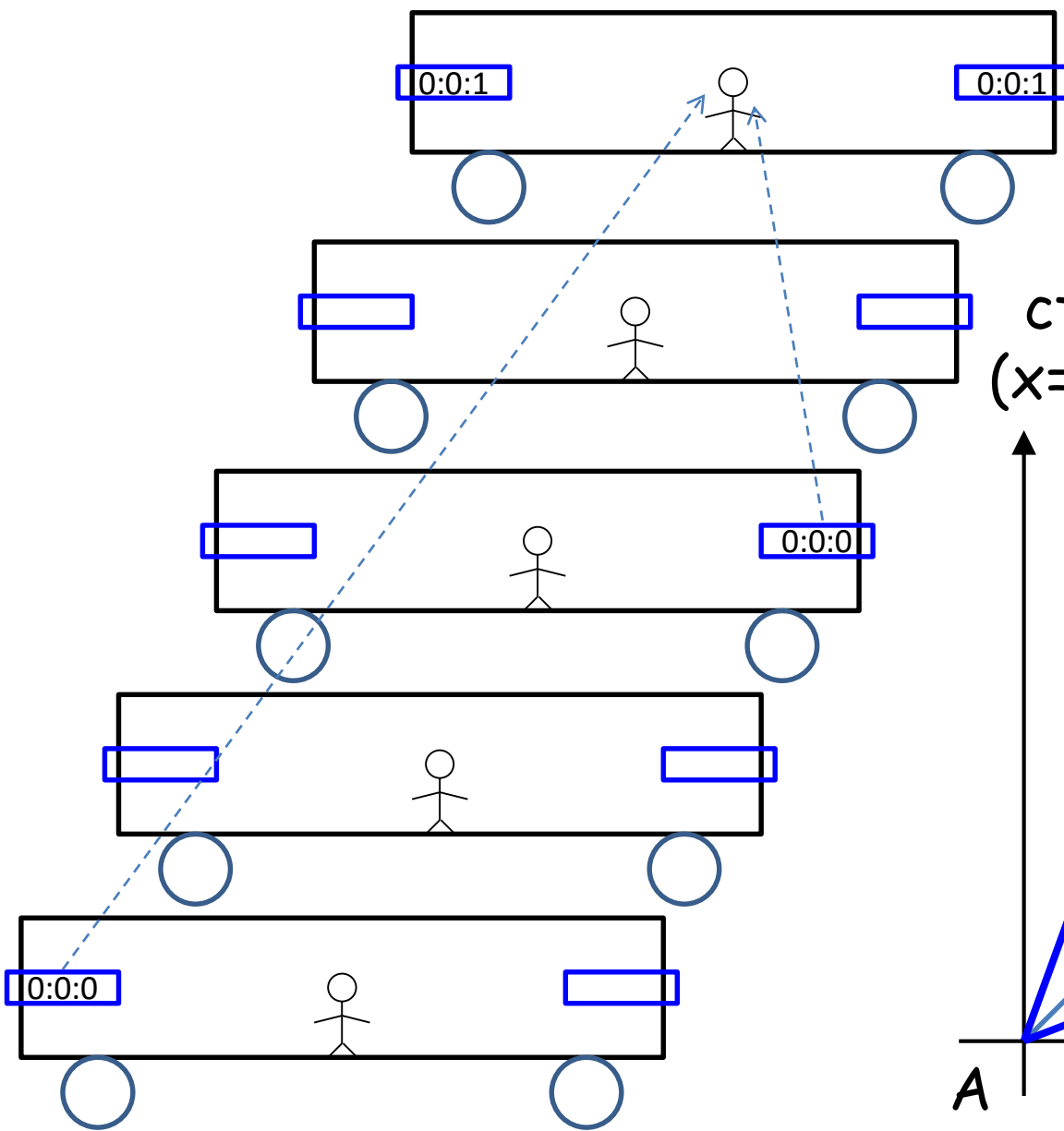
The concept of
equal-time does
depend on observers.

In Galilean transformation,
 $t = t'$ (time: absolute). But,
Relativity does not allow this.

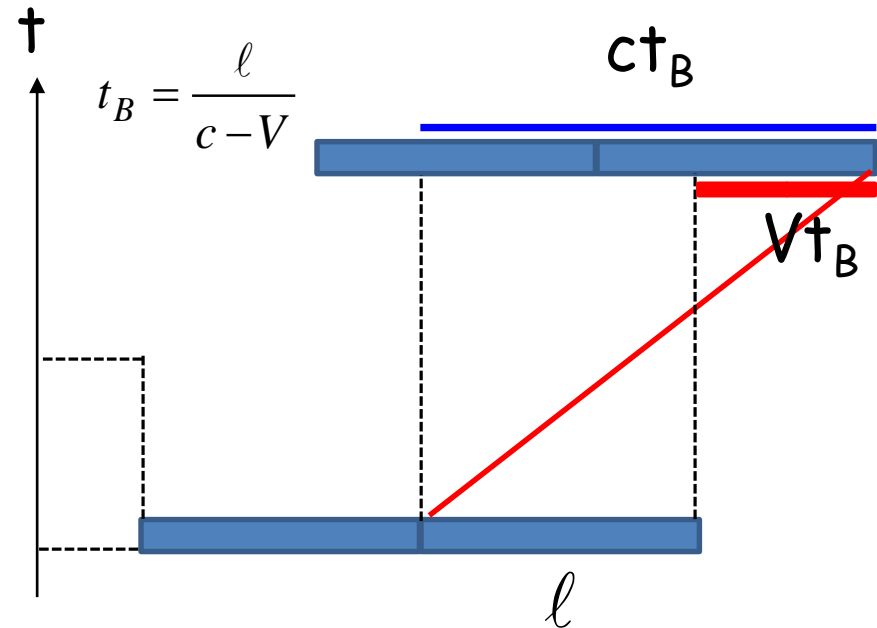
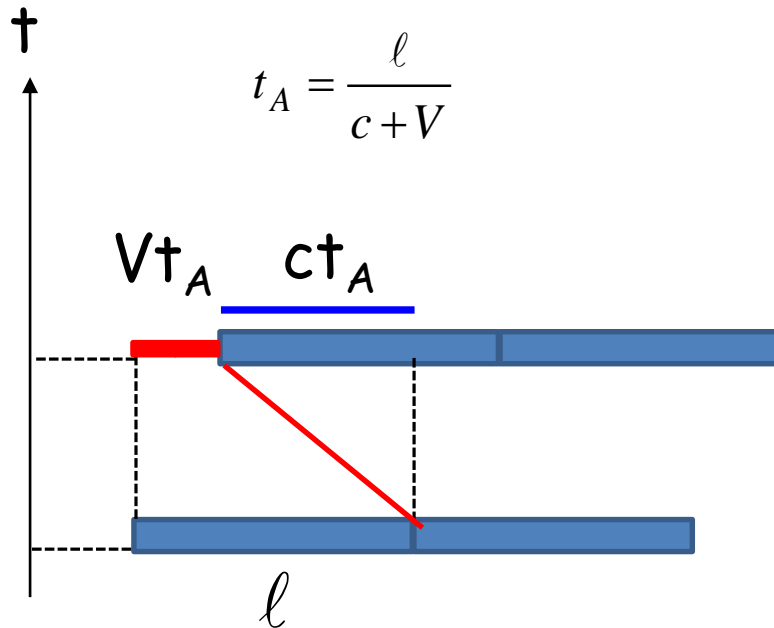
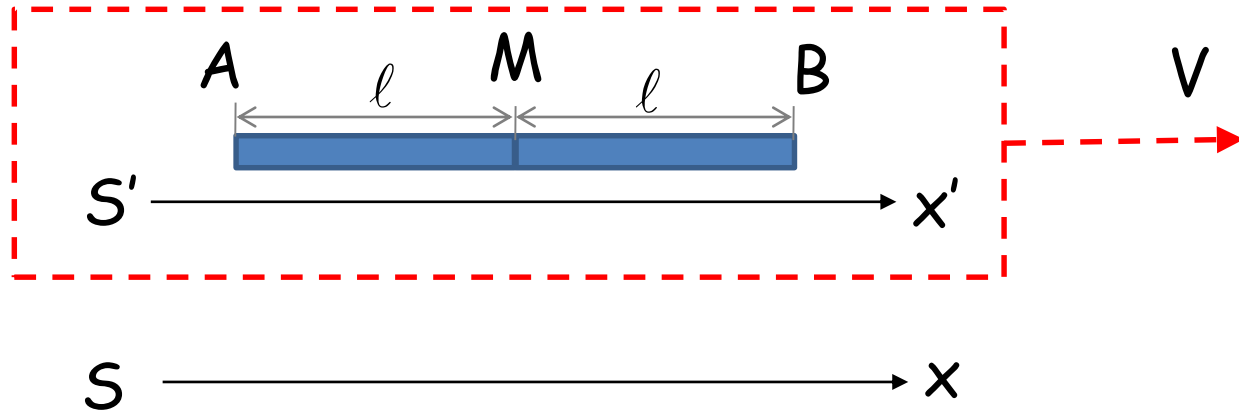
In relativity, $t \neq t'$.

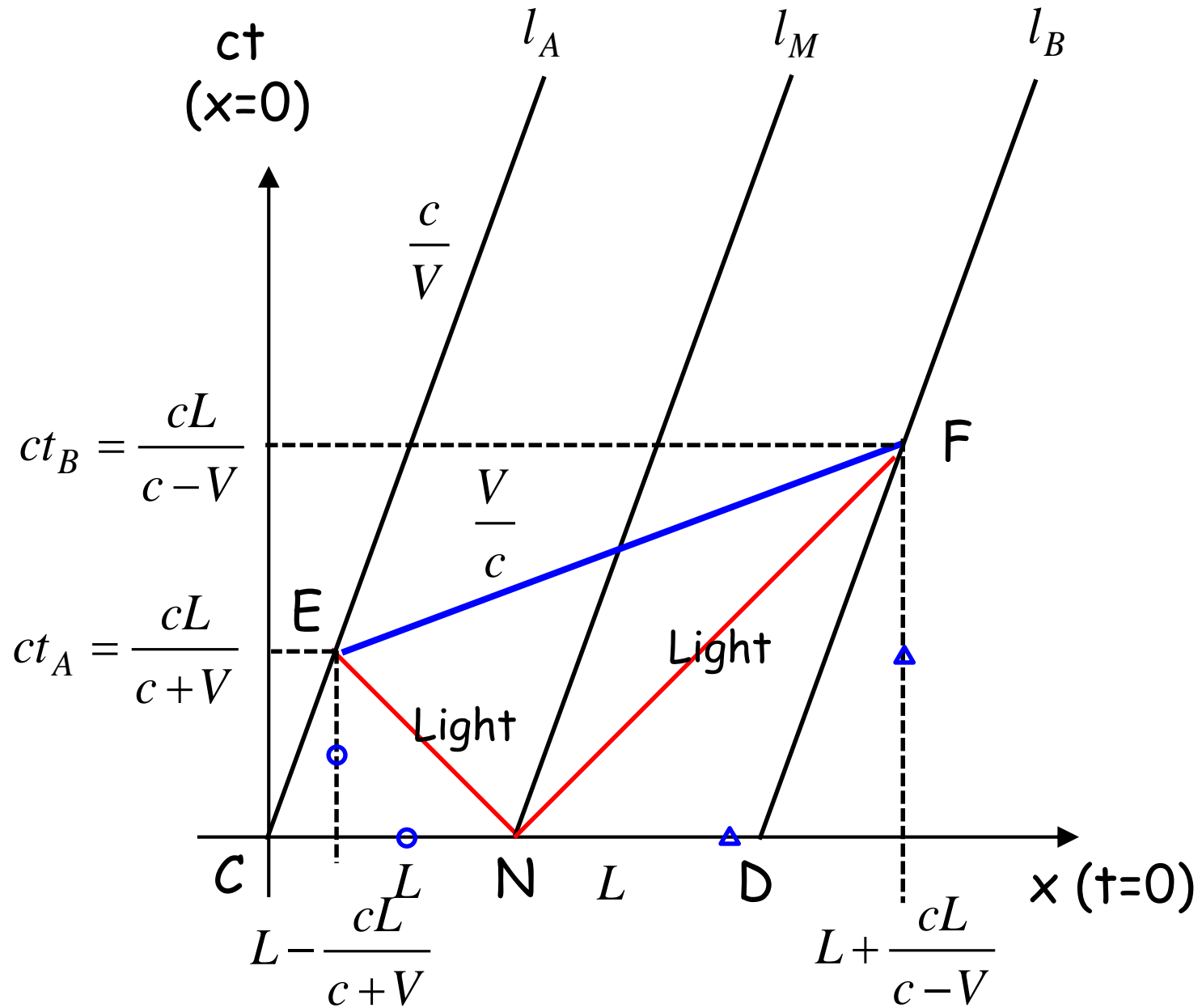
Charts of spacetime

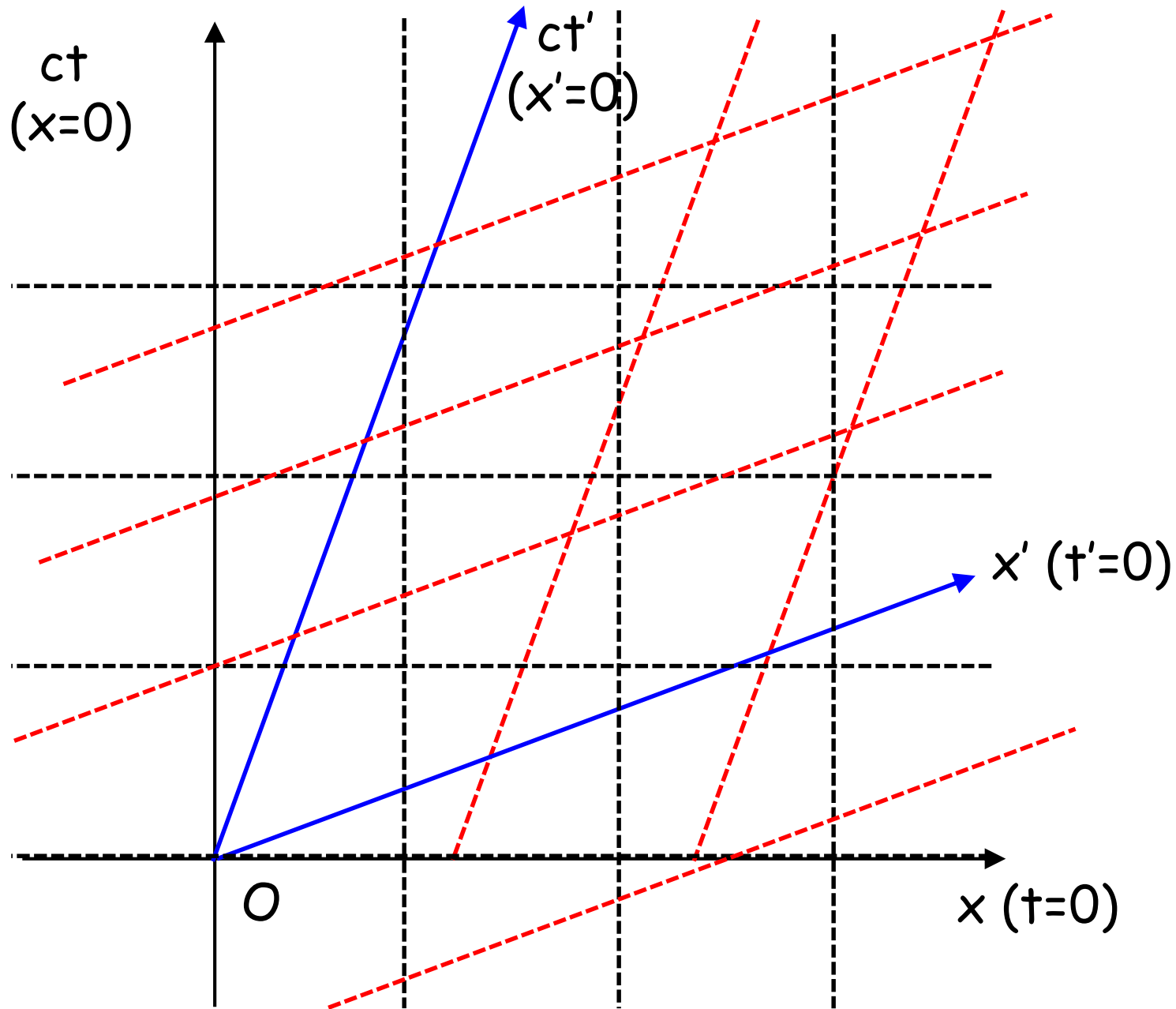




Another derivation of Lorentz transform







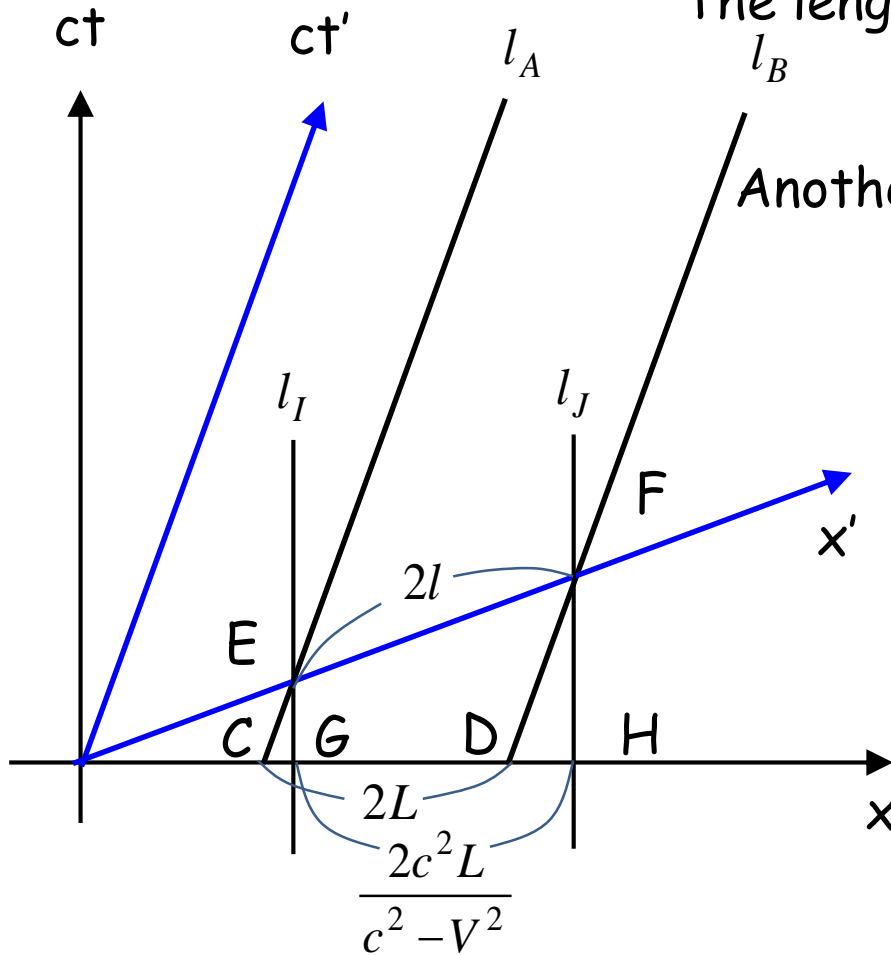
A stick AB traveling V together with the x' axis.

The length of the stick looks shrunk in the tx.

$$L = \alpha(V)\ell$$

Another stick IJ is located in GH in the tx.

$$\ell = \alpha(V)\frac{c^2 L}{c^2 - V^2}$$

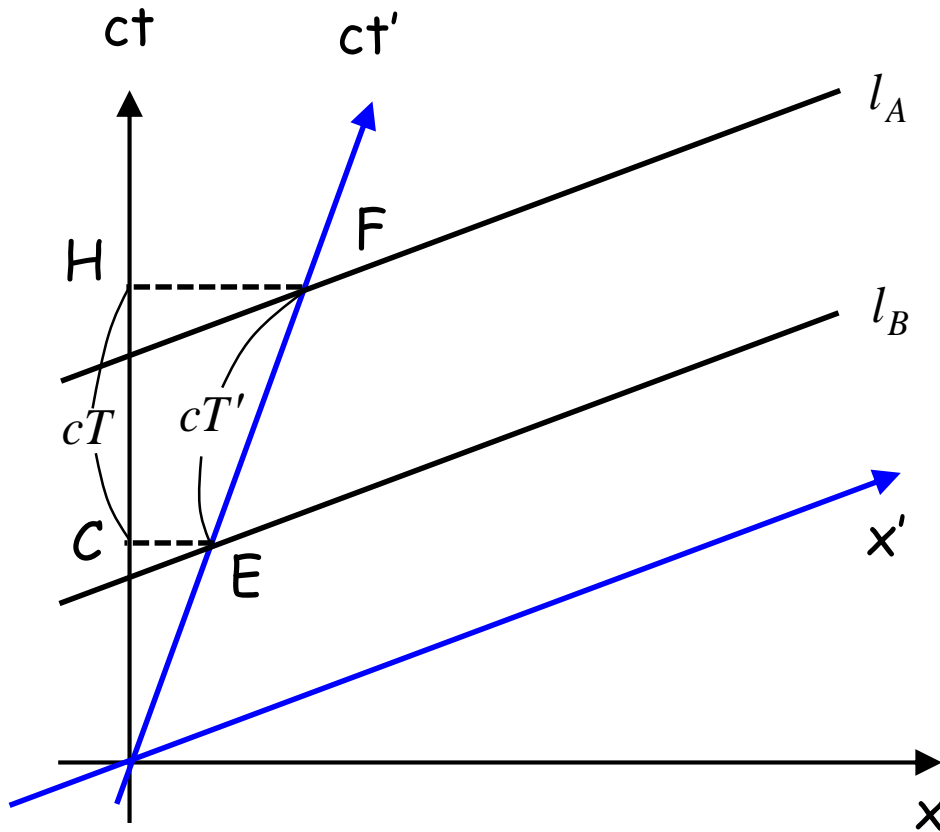


	Stick AB	Stick IJ
--	----------	----------

S coord.	$2L$	$\frac{2c^2 L}{c^2 - V^2}$
----------	------	----------------------------

S' coord.	2ℓ	2ℓ
-----------	---------	---------

$$\alpha(V) = \sqrt{1 - \frac{V^2}{c^2}}$$



$$cT' : cT = 2\ell : \frac{2c^2 L}{c^2 - V^2} = 2\ell : \frac{2c^2 \alpha(V) \ell}{c^2 - V^2}$$

$$= \alpha(V) : \frac{2c^2 \{\alpha(V)\}^2}{c^2 - V^2} = \alpha(V) : 1$$

$$\Rightarrow \alpha(V) T = T'$$

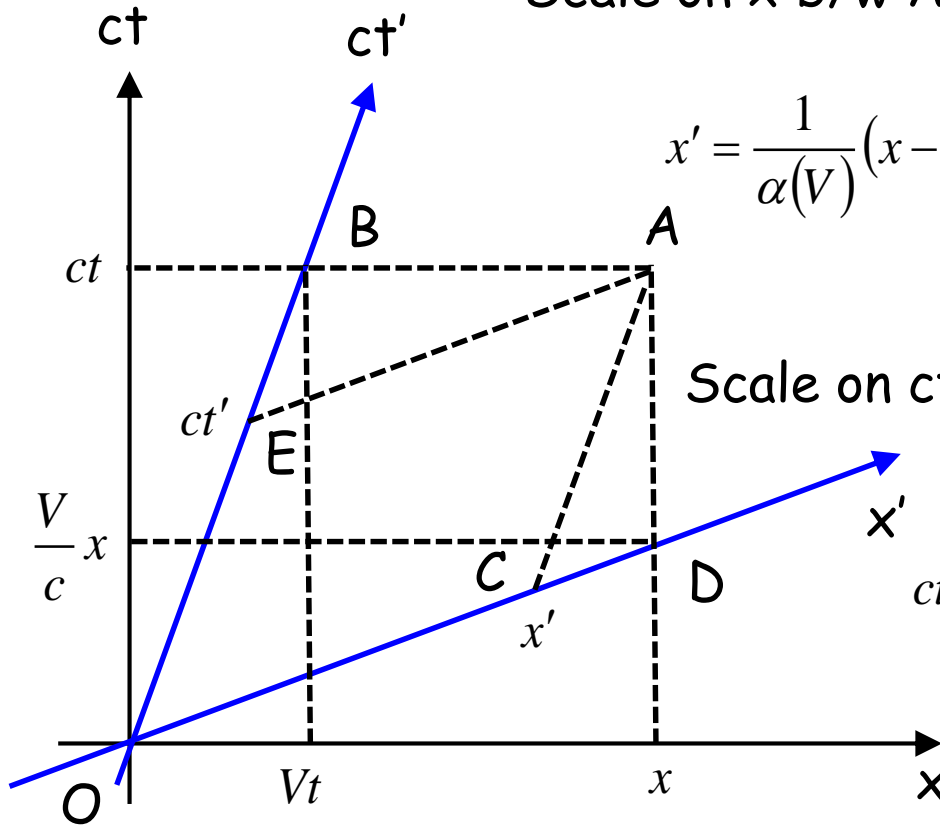
How (x, ct) is expressed by (x', ct') ?

Scale on x b/w AB : Scale on x' b/w OC = $a : 1$

$$x' = \frac{1}{\alpha(V)}(x - Vt) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}(x - Vt) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(x - \frac{V}{c}ct\right)$$

Scale on ct b/w AD : Scale on ct' b/w OE = $a : 1$

$$ct' = \frac{1}{\alpha(V)}\left(ct - \frac{V}{c}x\right) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(ct - \frac{V}{c}x\right)$$



Lorentz transform !!

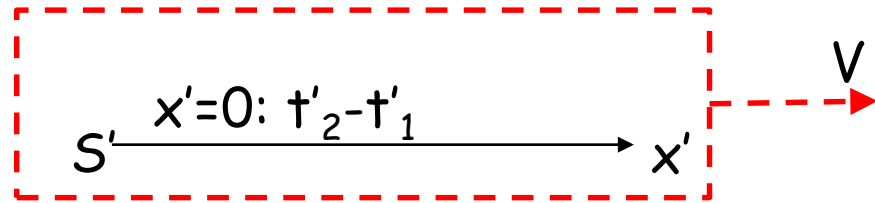
Equality of Lorentz contraction

$$ct' = \gamma(ct - \beta x)$$

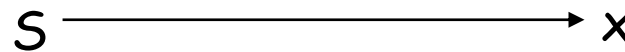
$$x' = \gamma(-\beta(ct) + x)$$

$$y' = y$$

$$z' = z$$



$$\left(\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \beta = \frac{V}{c} \right)$$



$$ct'_1 = \gamma(ct_1 - \beta x_1) \qquad ct'_2 = \gamma(ct_2 - \beta x_2)$$

$$0 = \gamma(-\beta(ct_1) + x_1) \qquad 0 = \gamma(-\beta(ct_2) + x_2)$$

$$ct'_1 = \gamma(ct_1 - \beta(\beta ct_1)) = \gamma(1 - \beta^2)ct_1 = \frac{1}{\gamma}ct_1 \qquad ct'_2 = \frac{1}{\gamma}ct_2$$

$$\Rightarrow \gamma(t'_2 - t'_1) = t_2 - t_1$$



$$ct'_1 = \gamma(ct_1 - \beta 0) = \gamma ct_1 \qquad ct'_2 = \gamma(ct_2 - \beta 0) = \gamma ct_2$$

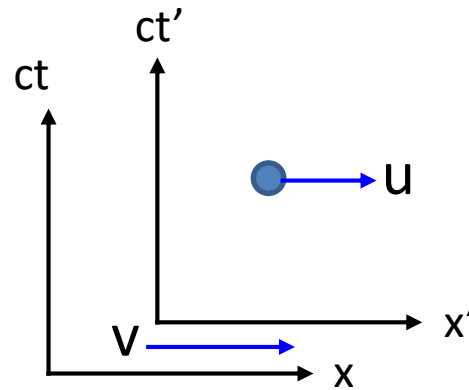
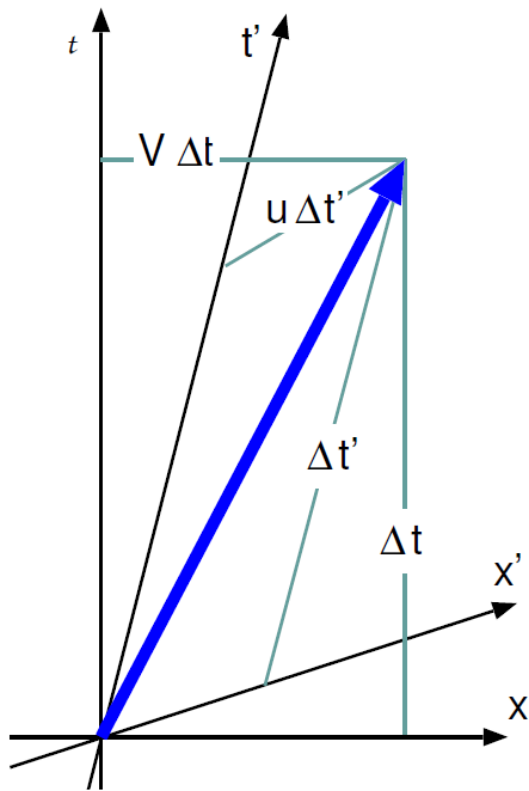
$$x'_1 = \gamma(-\beta(ct_1) + 0) \qquad x'_2 = \gamma(-\beta(ct_2) + 0)$$

$$\Rightarrow t'_2 - t'_1 = \gamma(t_2 - t_1)$$

"The time of the inertial system S' seen from S slowly advances."

"The time of the inertial system S seen from S' slowly advances."

Velocity addition



$$\mathbf{u} = (u_x, u_y, u_z)$$

$$x' = ut'$$

$$\gamma(x - vt) = u\gamma\left(t - \frac{v}{c^2}x\right)$$

$$x - vt = ut - \frac{uv}{c^2}x$$

$$x + \frac{uv}{c^2}x = ut + vt$$

$$x = \frac{u + v}{1 + \frac{uv}{c^2}}t$$

$$y' = u_y t'$$

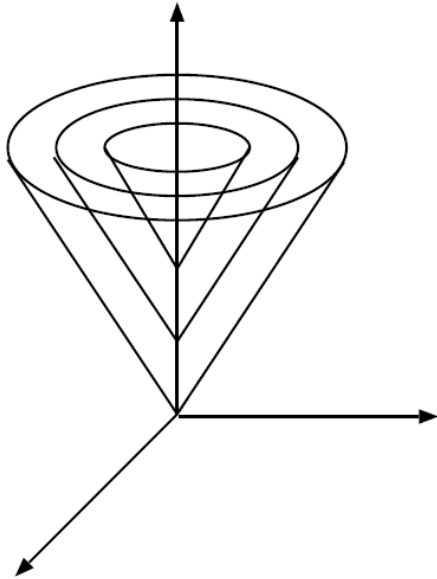
$$y = u_y \gamma\left(t - \frac{v}{c^2}x\right)$$

$$y = u_y \gamma\left(t - \frac{v}{c^2} \frac{u_x + v}{1 + \frac{u_x v}{c^2}} t\right) = u_y \gamma\left(1 - \frac{v}{c^2} \frac{u_x + v}{1 + \frac{u_x v}{c^2}}\right) t$$

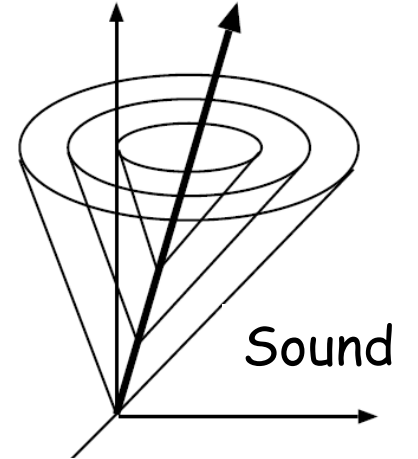
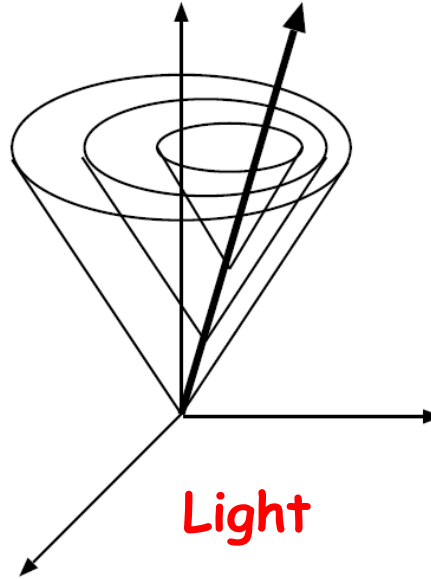
$$y = u_y \gamma\left(\frac{1 + \frac{u_x v}{c^2} - \frac{u_x v}{c^2} - \frac{v^2}{c^2}}{1 + \frac{u_x v}{c^2}}\right) t = u_y \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(\frac{1 - \frac{v^2}{c^2}}{1 + \frac{u_x v}{c^2}}\right) t = u_y \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u_x v}{c^2}} t$$

Doppler effect

Waves emitted from a stationary object

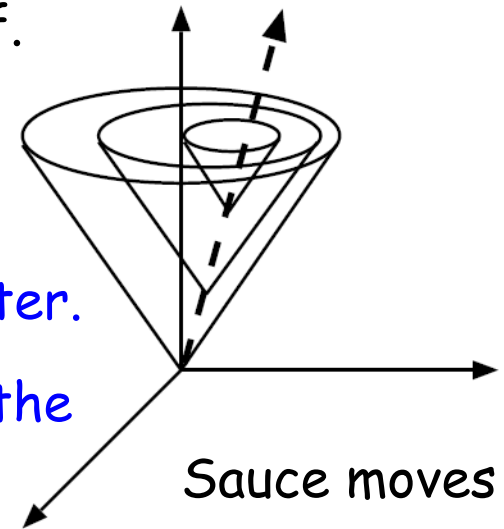


How it looks as you move?



blown by the wind.

cf.



Sound $v = \frac{V}{\lambda} \Rightarrow$ Change when the observers move.
 $\lambda \Rightarrow$ Change when the source moves.

Light $v = \frac{c}{\lambda}$ ✓ The interval between waves is shorter.
✓ Due to the Rip Van Winkle effect, the interval between the light source and the next wave is extended.

Doppler effect of light

Frequency of light: ν_0 Time interval: $\frac{1}{\nu_0}$

$$(ct', x', y', z') = \left(\frac{nc}{\nu_0}, 0, 0, 0 \right) \quad (ct, x, y, z) = \left(\gamma \frac{nc}{\nu_0}, \gamma\beta \frac{nc}{\nu_0}, 0, 0 \right)$$

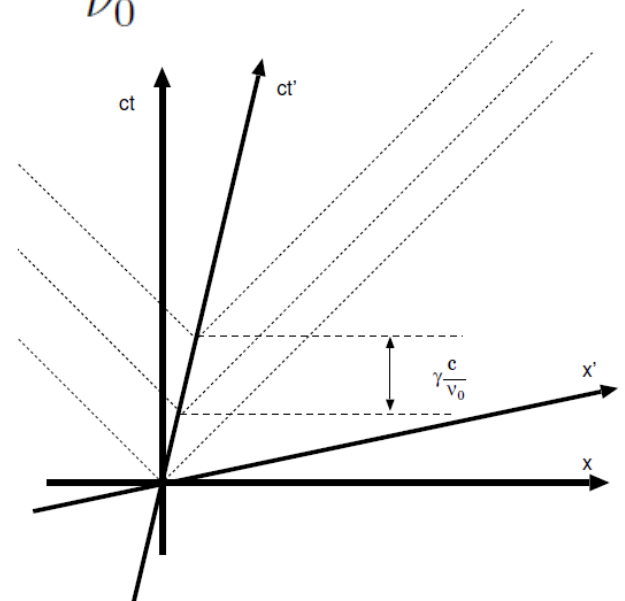
Observer: $(x, y, z) = (L, 0, 0)$

$$\underbrace{\gamma \frac{n}{\nu_0}} + \underbrace{\frac{L - \gamma\beta \frac{nc}{\nu_0}}{c}} = \frac{L}{c} + \gamma(1 - \beta) \frac{n}{\nu_0}$$

Time it takes for light to reach.

Time when the mountain is emitted.

$$\nu = \nu_0 \frac{1}{\gamma(1 - \beta)} = \nu_0 \frac{\sqrt{1 - \beta^2}}{1 - \beta} = \nu_0 \sqrt{\frac{1 + \beta}{1 - \beta}}$$



More generally

Observer: $(L \cos \theta, L \sin \theta, 0)$

$$\begin{aligned} \gamma \frac{n}{\nu_0} + \frac{1}{c} \sqrt{\left(L \cos \theta - \gamma \beta \frac{nc}{\nu_0}\right)^2 + (L \sin \theta)^2} &\simeq \gamma \frac{n}{\nu_0} + \frac{1}{c} \sqrt{L^2 - 2L \cos \theta \gamma \beta \frac{nc}{\nu_0}} \\ &\simeq \gamma \frac{n}{\nu_0} + \frac{1}{c} \left(L - \cos \theta \gamma \beta \frac{nc}{\nu_0}\right) \end{aligned}$$

Change of 1 of n:
$$\frac{\gamma(1 - \beta \cos \theta)}{\nu_0}$$

$$\nu = \nu_0 \frac{\sqrt{1 - \beta^2}}{1 - \beta \cos \theta}$$

$\cos \theta = 0$

→ The light source moves in a transverse direction.

Transverse Doppler effect !!

Relativistic effect !!

Consequences from the principle of light velocity invariance

- ✓ A moving clock ticks more slowly than a clock at rest.
- ✓ The length of what is moving relative to the observer shrinks in the direction of movement.
- ✓ Doppler effect.
- ✓ Velocity addition.
- ... etc.

Minkowski space

In the special theory of relativity the speed of light is an absolute quantity and time and length (scale of space) is not an absolute quantity.

How to reflect the principle of light velocity invariance?

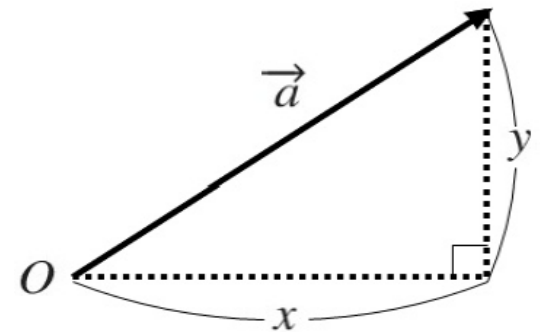
How to define "length"?

For light, the length of anything is 0.

$$t = \frac{x}{c} \quad \|\vec{a}\| = \sqrt{|x^2 - (ct)^2|}$$

$$\|\vec{a}\| = \sqrt{|x^2 + y^2 + z^2 - (ct)^2|}$$

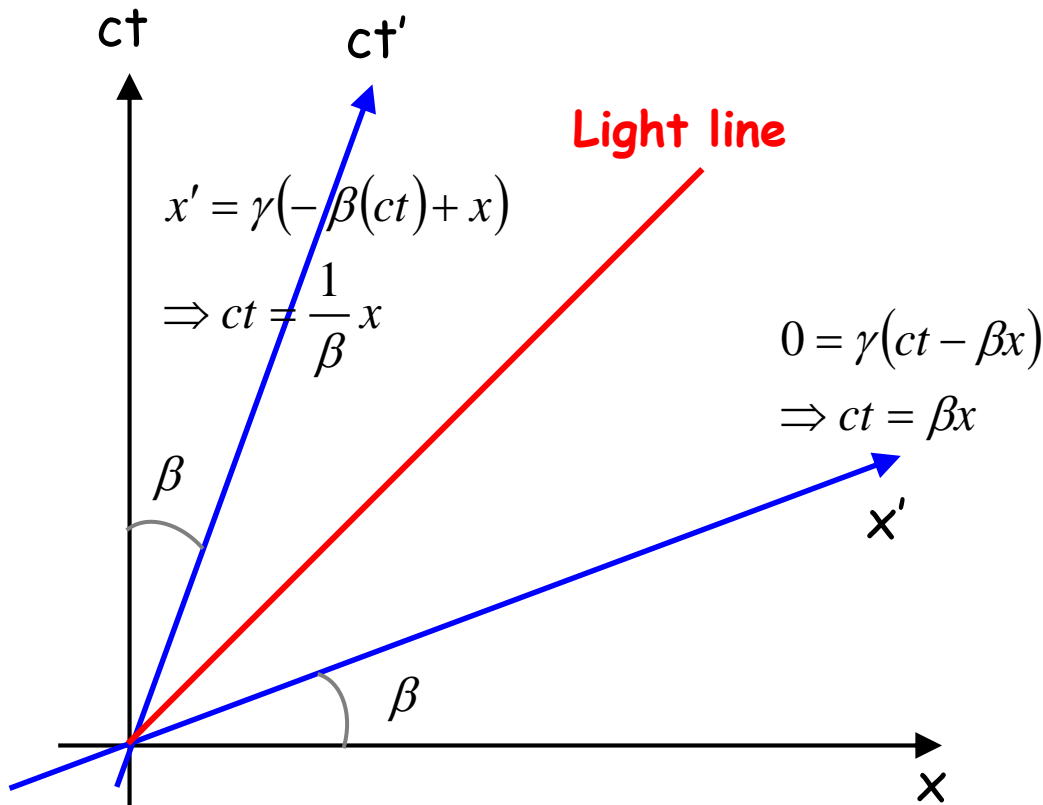
$$|\vec{a}| = \sqrt{x^2 + y^2}$$



Euclidean space

Lorentz transform

Coordinate transform in Minkowski space



$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(-\beta(ct) + x)$$

$$y' = y$$

$$z' = z$$

$$\left(\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \beta = \frac{V}{c} \right)$$

Minkowski's length is invariant by Lorentz transformation

$$x'^2 - (ct')^2 = x^2 - (ct)^2$$

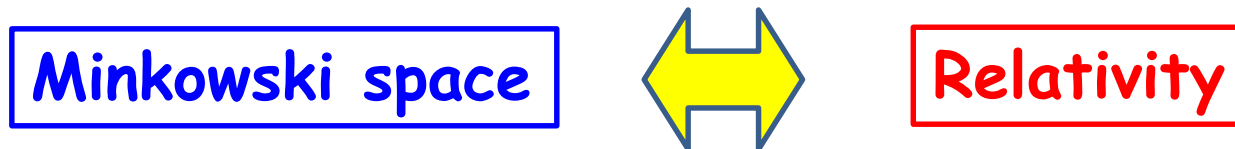
$$\begin{aligned}x'^2 - (ct')^2 &= \gamma^2 \left\{ (x - vt)^2 - c^2 \left(-\frac{v}{c^2}x + t \right)^2 \right\} \\&= \gamma^2 \left\{ x^2 - 2xvt + v^2t^2 - c^2 \left(\frac{v^2}{c^4}x^2 - 2\frac{v}{c^2}xt + t^2 \right) \right\} \\&= \gamma^2 \left\{ x^2 - 2vxt + v^2t^2 - \frac{v^2}{c^2}x^2 + 2vxt - c^2t^2 \right\} \\&= \frac{1}{1 - \frac{v^2}{c^2}} \left\{ \left(1 - \frac{v^2}{c^2} \right) x^2 + (v^2 - c^2)t^2 \right\} \\&= x^2 - \frac{c^2}{c^2 - v^2} (c^2 - v^2)t^2 \\&= x^2 - (ct)^2\end{aligned}$$

Relativity & geometry

Euclidean space: Space where the length and angle that we normally use are fixed.

Minkowski's space: length is invariant by Lorentz transform.

Relativity theory is a physical theory that unifies time, space and matter.



Special relativity theory is a physical interpretation of the Minkowski space theory as geometry.