Data Structures

CH6 Graphs

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NTHU EE

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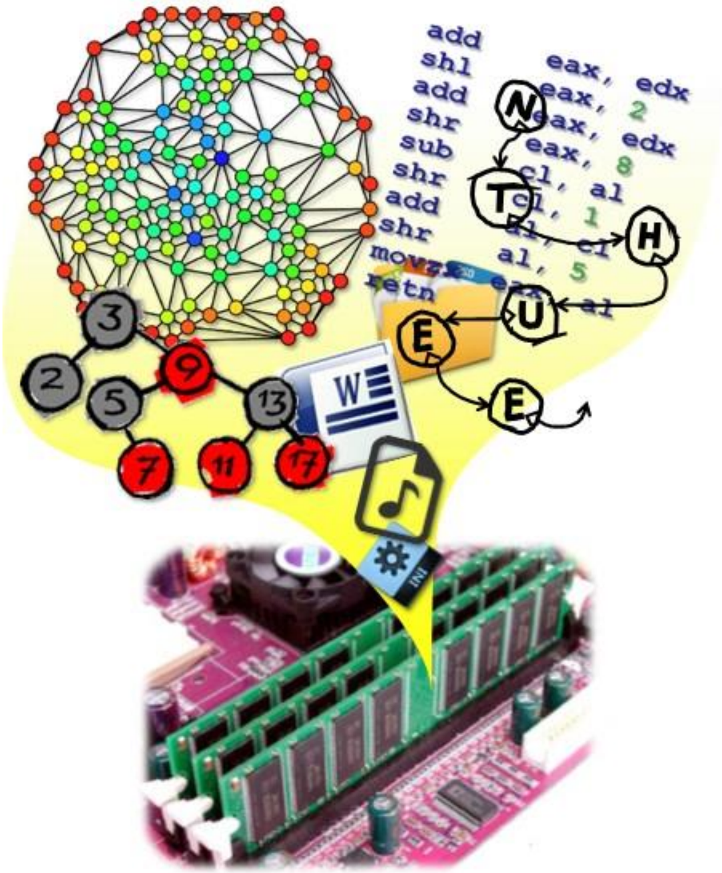
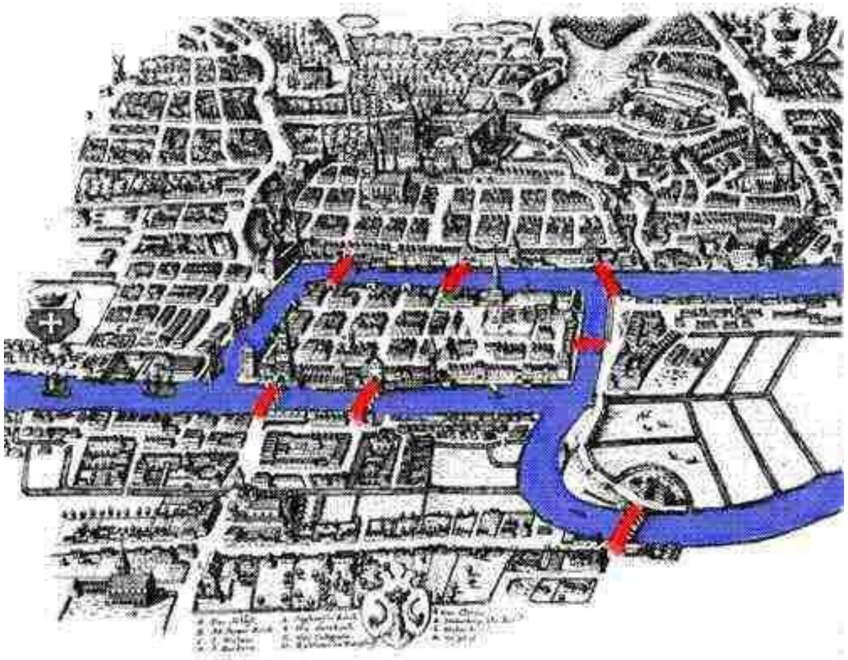


Outline

- **6.1 Introduction and the graph abstract data type**
- 6.2 Elementary graph operations
- 6.3 Minimum-cost spanning trees
- 6.4 Shortest paths (and transitive closure)
- (6.5 Activity networks)

Konigsberg Bridge Problem

- Also known as "一筆畫問題" or "七橋問題"
  - Four land areas are interconnected by seven bridges
  - Is it possible to walk across seven bridges exactly once in returning to the starting place?



Konigsberg Bridge Problem

- Also known as "一筆畫問題" or "七橋問題"
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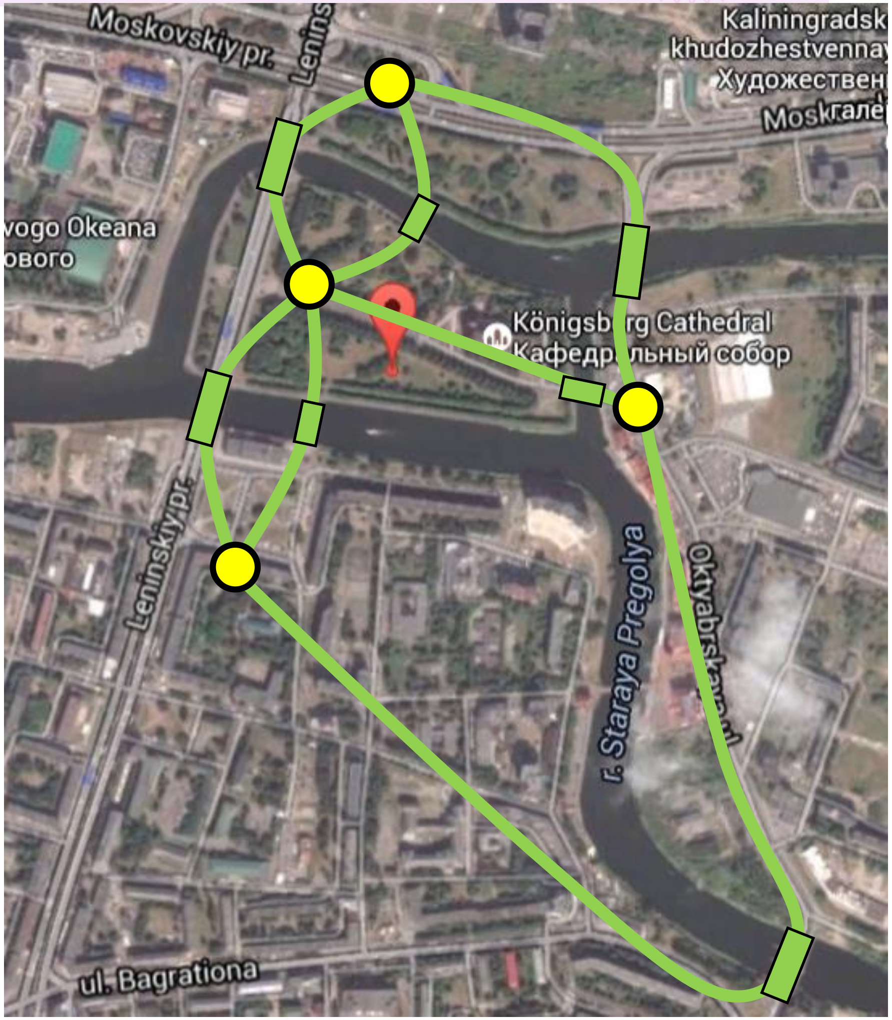
Google Map 54.706 N, 20.510 E

Konigsberg Bridge Problem

- Euler solved the problem by representing the land areas as vertices and the bridges as edges (1736)
  - First recorded evidence of the use of graphs

- Since then, graphs have been used in a wide variety of applications
  - Analysis of circuits, genetics, social networks…

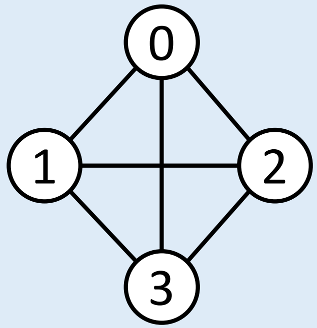


Google Map 54.706 N, 20.510 E

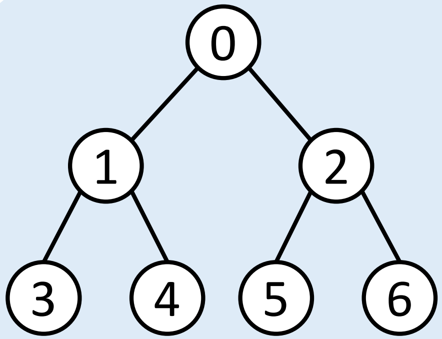
Graphs

- Definition : A graph, G, consists of two sets, V and E
  - G = (V, E)
- V is a finite, nonempty set of vertices
- E is a set of pairs of vertices, called edges
  - Undirected graphs (無向圖)
    - Pair of vertices representing any edge is unordered
    - (u, v) and (v, u) represent the same edge
  - Directed graphs (digraphs) (有向圖)
    - Each edge is represented by a directed pair <u, v>
    - u is the tail and v the head of the edge

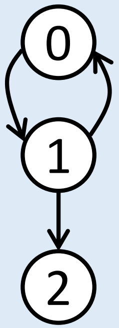
Graphs



G_1



G_2



G_3

$V(G_1) = \{0, 1, 2, 3\}$

$E(G_1) = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$

$V(G_2) = \{0, 1, 2, 3, 4, 5, 6\}$

$E(G_2) = \{(0, 1), (0, 2), (1, 3), (1, 4), (2, 5), (2, 6)\}$

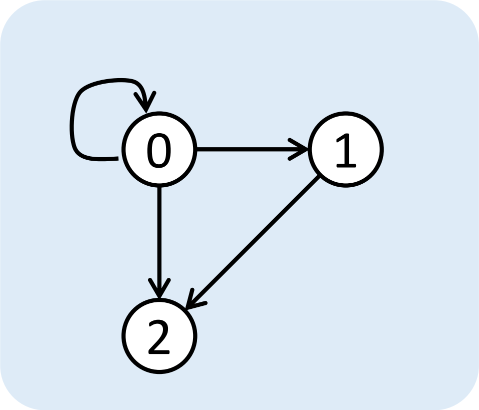
$V(G_3) = \{0, 1, 2\}$

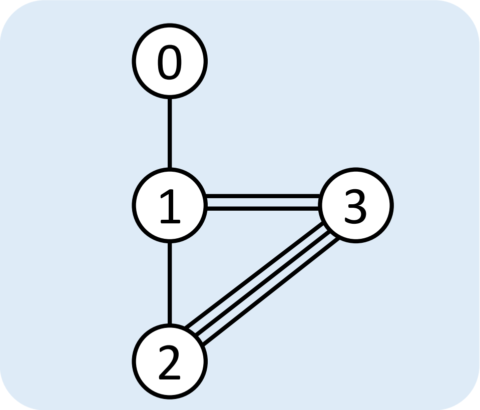
$E(G_3) = \{<0, 1>, <1, 0>, <1, 2>\}$

Simple Graphs (Strict Graphs)

- This book only considers simple graphs (strict graphs)

- The followings are not allowed in simple graphs
  - Self edges / self loops
    - (v, v)
    - <v, v>
  - Multiple occurrences of the same edge





- Therefore, the max number of edges of an n-vertex simple graph
  - n(n-1)/2 for an undirected graph
  - n(n-1) for an directed graph

Terminologies

- Complete graphs (also called as cliques (團))
  - A graph having the max possible number of edges
    - n(n-1)/2 for an undirected graph
    - n(n-1) for an directed graph

- Adjacency and incidence
  - u and v are adjacent if (u, v) ∈ G
  - (u, v) is incident on (關聯) u and also v

- A subgraph of G is a graph G' such that
    - V(G') ⊆ V(G)
    - E(G') ⊆ E(G)

Terminologies

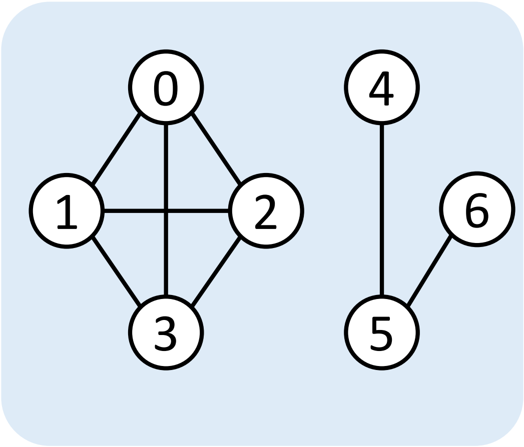
- A path from u to v in a graph G is
  - a sequence of vertices: u, i_1, i_2, …, i_k, v
    - (u, i_1), (i_1, i_2), …, (i_k, v) ∈ E(G), G is undirected
    - <u, i_1>, <i_1, i_2>, …, <i_k, v> ∈ E(G), G is directed
- A simple path is
  - a path in which all vertices except possibly the first and last are distinct
- A cycle is
  - a simple path in which the first and last are the same

Connectivity

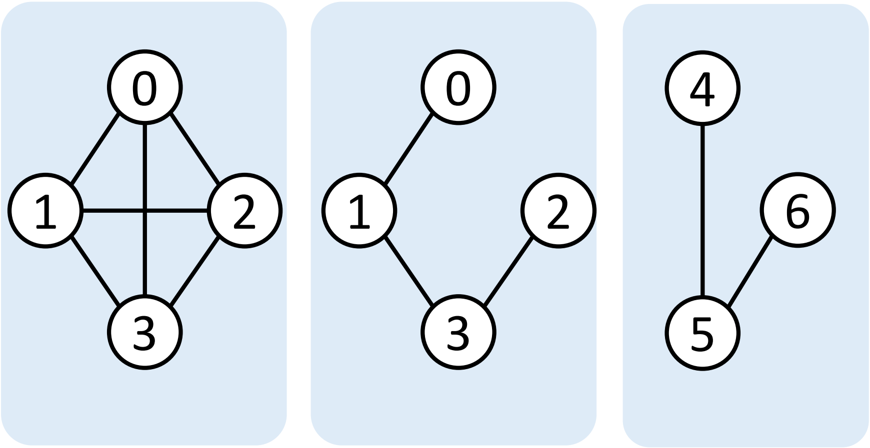
- In an undirected graph, vertices u and v are connected iff there is a path from u to v
- An undirected graph is connected iff every pair of distinct vertices u and v in V(G) is connected
  - A tree is a connected acyclic graph

Connected Components

- A connected component (or component for short), H, of an undirected graph is
  - the maximal connected subgraph
  - By maximal, we mean that G contains no other subgraph that is both connected and properly contains H



G_1

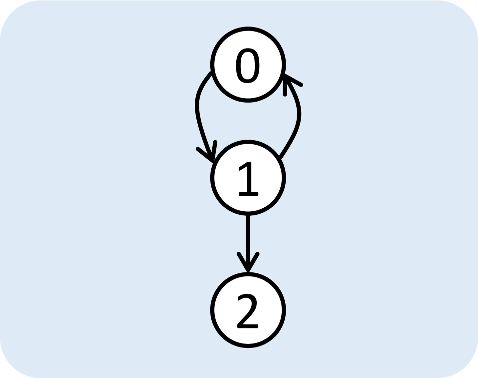


Some connected components of G_1

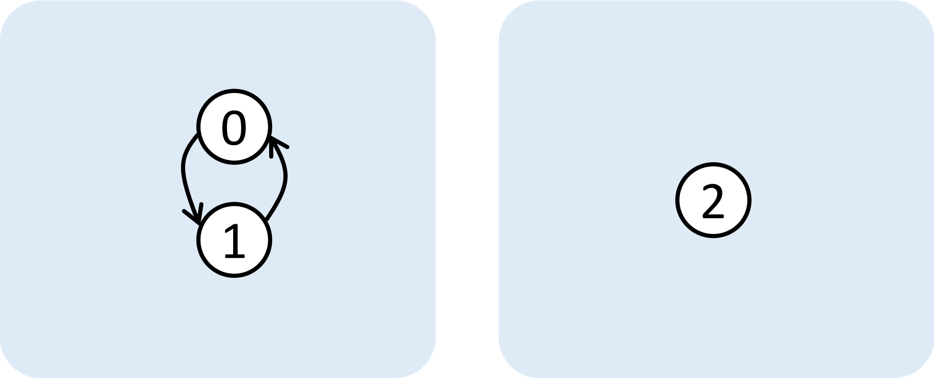
Strongly Connected Graphs

- A digraph G is said to be strongly connected iff for every pair of distinct vertices u and v in V(G) there is a directed path from u to v and also from v to u

- A strongly connected component is a maximal subgraph that is strongly connected



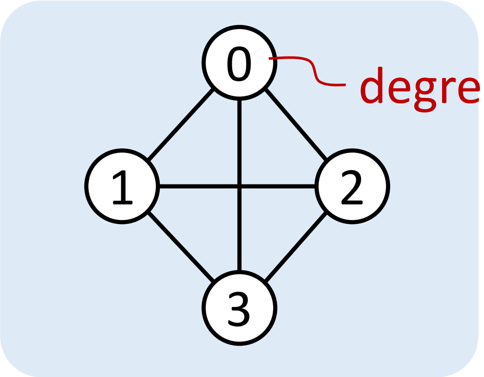
G_3



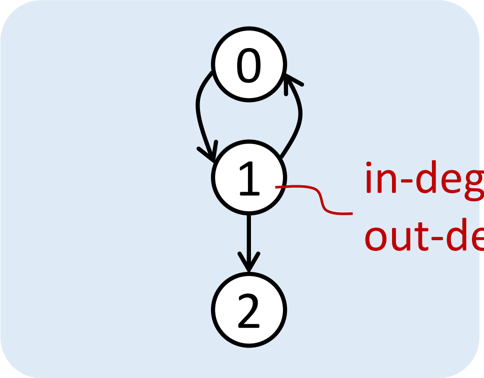
Strongly connected components of G_3

Degree

- The degree of a vertex is the number of edges incident to that vertex
- For digraph
  - The out-degree of a vertex v is the number of edges for which v is the tail
  - The in-degree of a vertex v is the number of edges for which v is the head



degree=3



in-degree=1
out-degree=2

Graph ADT

```
class Graph
{
public:
    virtual ~Graph() {} // virtual destructor
    bool IsEmpty() const           {return n == 0};
    int NumberOfVertices() const   {return n};
    int NumberOfEdges() const      {return e};

    virtual int  Degree(int u) const            = 0;
    virtual bool ExistsEdge(int u, int v) const = 0;
    virtual void InsertVertex(int v)            = 0;
    virtual void InsertEdge(int u, int v)       = 0;
    virtual void DeleteVertex(int v)            = 0;
    virtual void DeleteEdge(int u, int v)       = 0;
private:
    int n;     // number of vertices
    int e;     // number of edges
};
```

Inheritance vs. Template

- Key question: do types affect the behaviors of a class according to your expectation?

- Inheritance: types may affect behaviors
  - Rectangle and Circle can calculate their areas but have different calculating mechanisms
  - According to this expectation, we design a base class, Shape, with a virtual GetArea() method and let specific shape classes to inherit

- Template: types do not affect behaviors
  - Stack exhibits a last-in-first-out behavior
    - Both Stack of Rectangle and Stack of Circle do so
  - According to this expectation, we design a template stack instead of a base stack and different inherited classes

Non-Virtual vs Virtual Functions

- Non-virtual
  - Static-binding (at compile time) according to the type of a object pointer or reference
- Virtual
  - Dynamic-binding (resolved at run time) according to hidden information in each object
  - Polymorphism: derived classes exhibit their specific behavior even if they are referred to using the base class pointer/reference

```
int main()
{
    Rectangle r;
    Circle c;
    cout << AreaRatio(r, c);
    cout << AreaRatio(c, r);
}
```

```
float AreaRatio(Shape& s1, Shape& s2)
{
    return s1.GetArea() / s2.GetArea();
}
```

Pure Virtual Functions

- Sometimes we want derived classes NOT to inherit the implementation of a virtual function by default
  - Implementation of GetArea()
    - Maybe impractical to have one method to calculate the area of both Rectangle and Circle
    - Maybe error-prone if someone inherits from Shape another specific shape, say Star, without redefining Star's GetArea

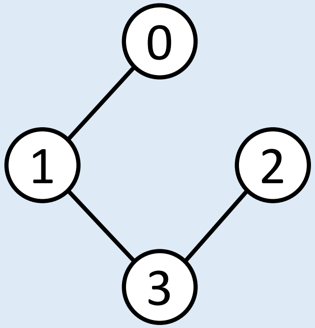
Graph Representations

- Three categories of most commonly used representations
  - Adjacency matrices
  - Adjacency lists
  - Adjacency multi-lists

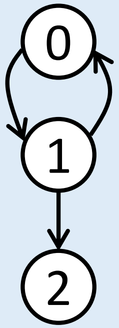
- The choice of a particular representation depends upon the application one has in mind and the functions one expects to perform on the graph

Adjacency Matrices

- The adjacency matrix of an n-vertex graph, G, is a 2D n×n array, say array A
    - A[u][v] = 1 iff (u, v) (or <u, v>) is in E(G)
    - A[u][v] = 0 otherwise
- Adjacency matrices of undirected graphs are always symmetric
    - This allows optimization that halves the space requirement
- Adjacency matrices are wasteful of space for sparse graphs (i.e., graphs with only few edges)



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \end{array}$$



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{array}$$

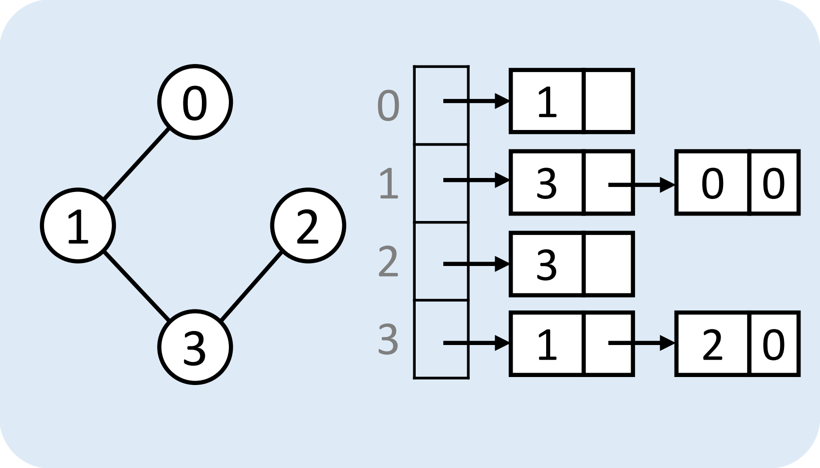
Adjacency Matrix Operations

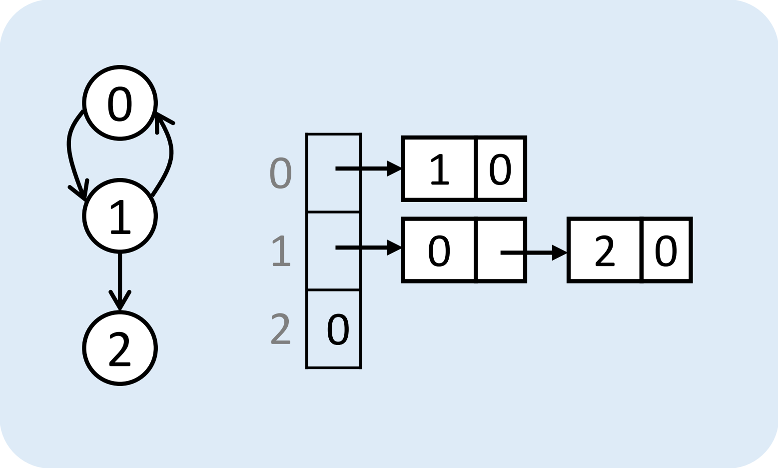
Degree (int u)	Return Σ a[u][i];	O(n)
Out-degree(int u)	Return Σ a[u][i];	O(n)
In-degree(int u)	Return Σ a[i][u];	O(n)
ExistsEdge(int u, int v)	Return a[u][v];	O(1)
InsertEdge(int u, int v)	Set a[u][v] = 1;	O(1)
DeleteEdge(int u, int v)	Set a[u][v] = 0;	O(1)
IsConnectedGraph()		$O(n^2)$

(Linked) Adjacency Lists

- The adjacency list of an n-vertex, e-edge graph, G
    - Contains an n-element array, n chains, and 2e chain nodes
        - Nodes in chain i represent the vertices adjacent from vertex i
        - Nodes in each chain are not required to be ordered

- (Recall the equivalence class problem)





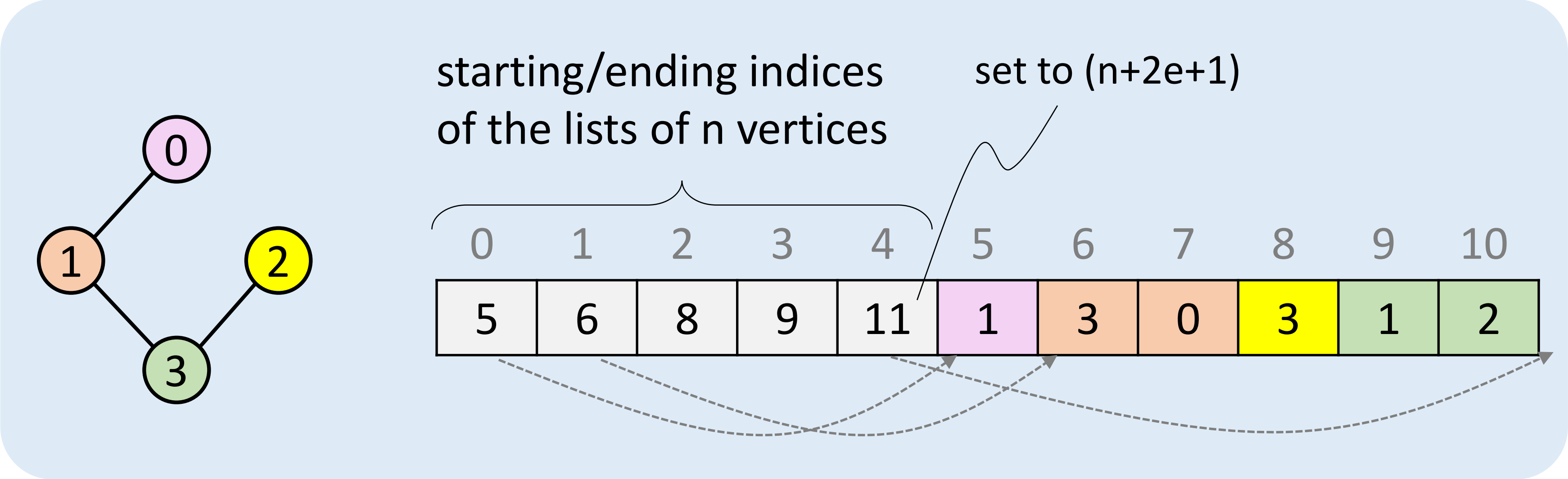
(Linked) Adjacency List Operations

Degree (int u)	Count the # of nodes in chain u;	O(e)
Out-degree(int u)	Count the # of nodes in chain u;	O(e)
In-degree(int u)	Count the # of u's in all the chains;	O(n+e)
ExistsEdge(int u, int v)	Look for v in chain u;	O(e)
InsertEdge(int u, int v)	• Check the existence of v in chain u (and u in v); • Push v onto chain u (and u onto v);	O(e) O(1)
DeleteEdge(int u, int v)	• Find v in chain u (and u in v); • Remove v from chain u (and u from v);	O(e) O(1)
IsConnectedGraph()		O(n+e)

Facebook would need to scan its billions of users to calculate how many people follows your page if Facebook uses the simple Adjacency List representation

Sequential Adjacency Lists

- Sequential adjacency list of an n-vertex, e-edge graph, G
  - Contains an (n+2e+1)-element array
  - n+1 for indexing the list of each vertex
  - 2e for adjacency information



starting/ending indices of the lists of n vertices

set to (n+2e+1)

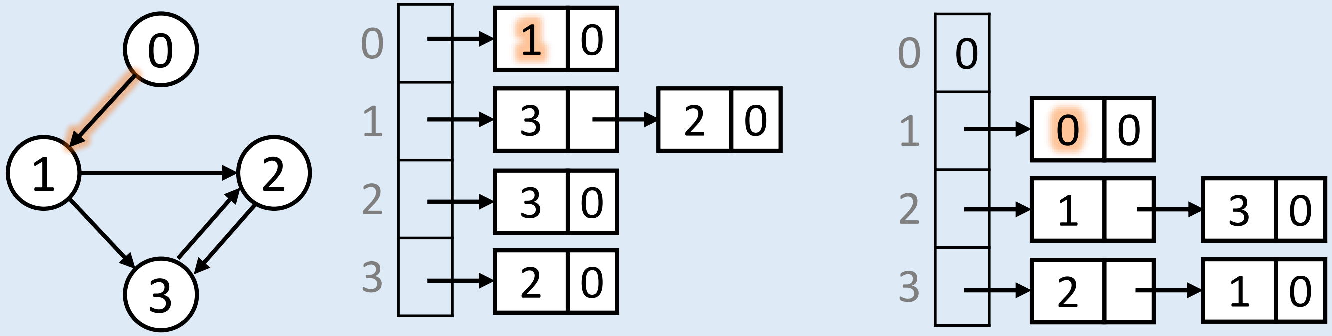
0	1	2	3	4	5	6	7	8	9	10
5	6	8	9	11	1	3	0	3	1	2

Sequential Adjacency List Operations

Degree (int u)	Return the index difference between u and u+1	O(1)
Out-degree(int u)	Return the index difference between u and u+1	O(1)
In-degree(int u)	Count the # of u's in the entire graph;	O(n+e)
ExistsEdge(int u, int v)	Look for v in list u;	O(e)
InsertEdge(int u, int v)	Make space and insert the edge	O(n+e)
DeleteEdge(int u, int v)	Delete the edge and compact the array	O(n+e)
IsConnectedGraph()		O(n+e)

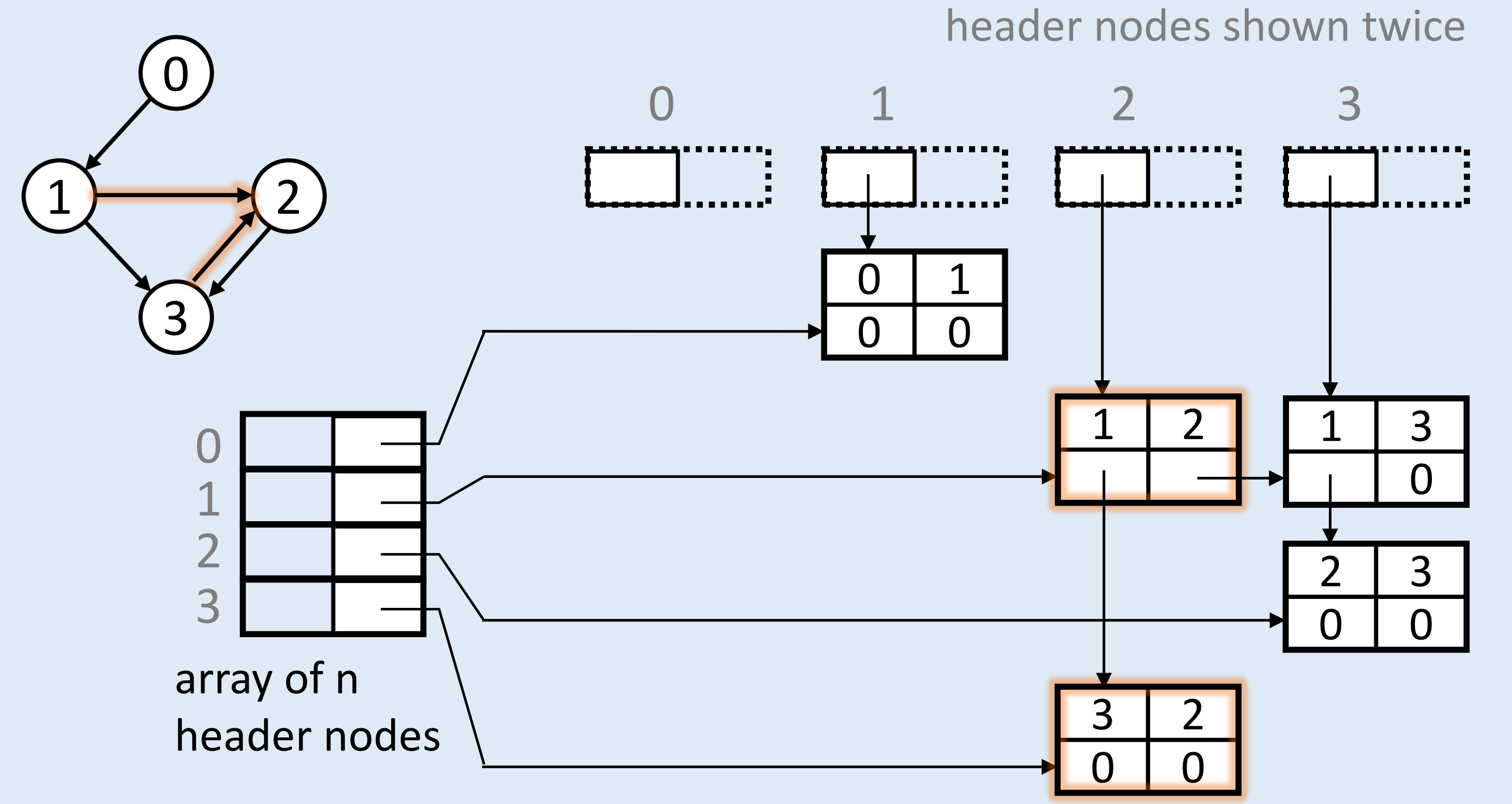
Inverse Adjacency Lists

- Ease repeatedly accessing all vertices adjacent to and from another vertex in a digraph
  - E.g., in-degree and out-degree
  - Keep an additional inverse adjacency list
    - List i stores edges of the form <x, i>
- An alternative is to use orthogonal adjacency lists



Orthogonal Adjacency List

- Use an n×n orthogonal (正交) list to store the adjacency information
  - Terms correspond to edges, (u, v) or <u, v>
- (Recall p.218 sparse matrices)

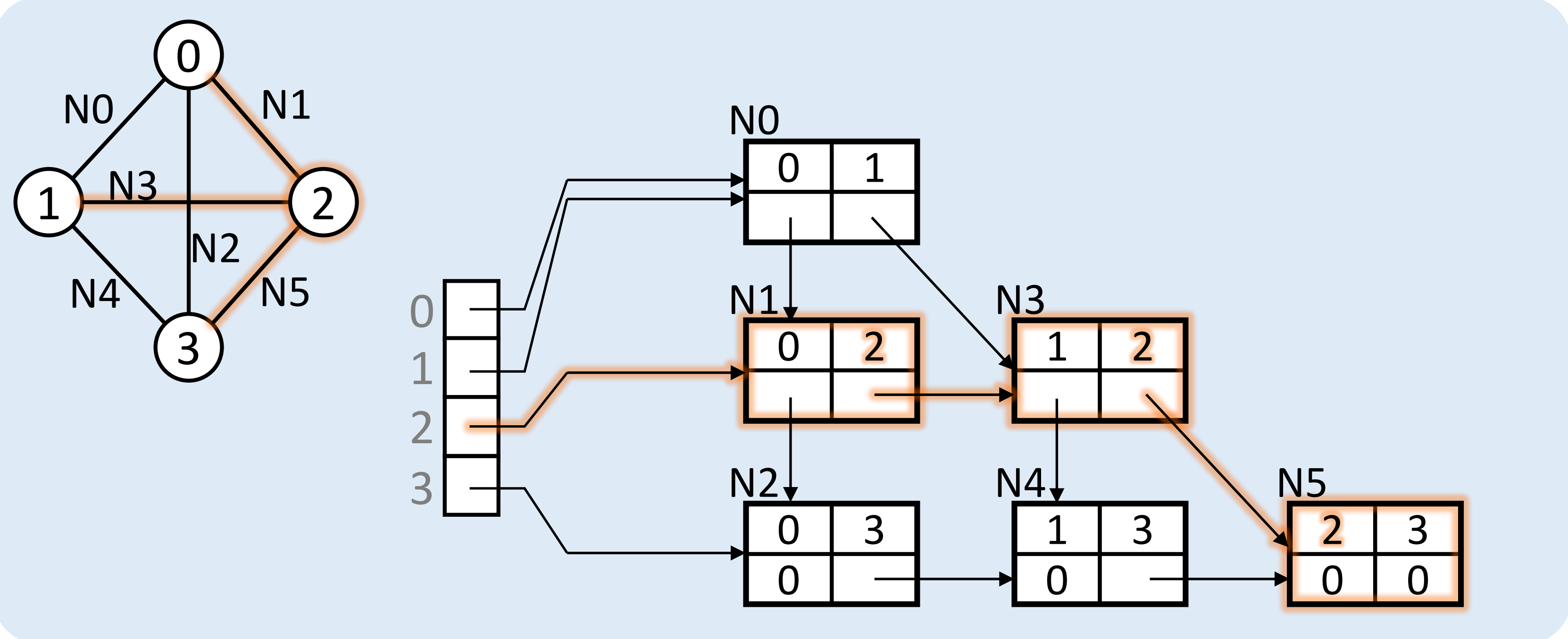


Orthogonal Adjacency List Operations

Out-degree(int u)	Count the # of nodes in chain u	O(e)
In-degree(int u)	Count the # of nodes in chain u	O(e)
ExistsEdge(int u, int v)	Look for v in chain u ;	O(e)
InsertEdge(int u, int v)	• Check the existence of v in chain u (and u in v); • Insert (instead of push or append) v into chain u (and u into v);	O(e) O(e)
DeleteEdge(int u, int v)	• Locate v in chain u (and u in v); • Remove v from chain u (and u from v);	O(e) O(1)
IsConnectedGraph()		O(n+e)

Adjacency Multi-Lists

- Multi-lists
  - One node can be shared among multiple lists
- Adjacency multi-lists
  - An edge is represented by a node
  - Support accessing all edges incident on a vertex in undirected graphs



Discussion:
Adjacency Matrices vs Adjacency Lists

Operation	**Which performs better**
Determining if (u, v) is an edge in G	
Degree of vertex u	
Determining if there is a path from u to v	
Adding an edge to G	
Space to store a dense graph	
Space to store a sparse graph	
Konigsberg Bridge Problem	

Outline

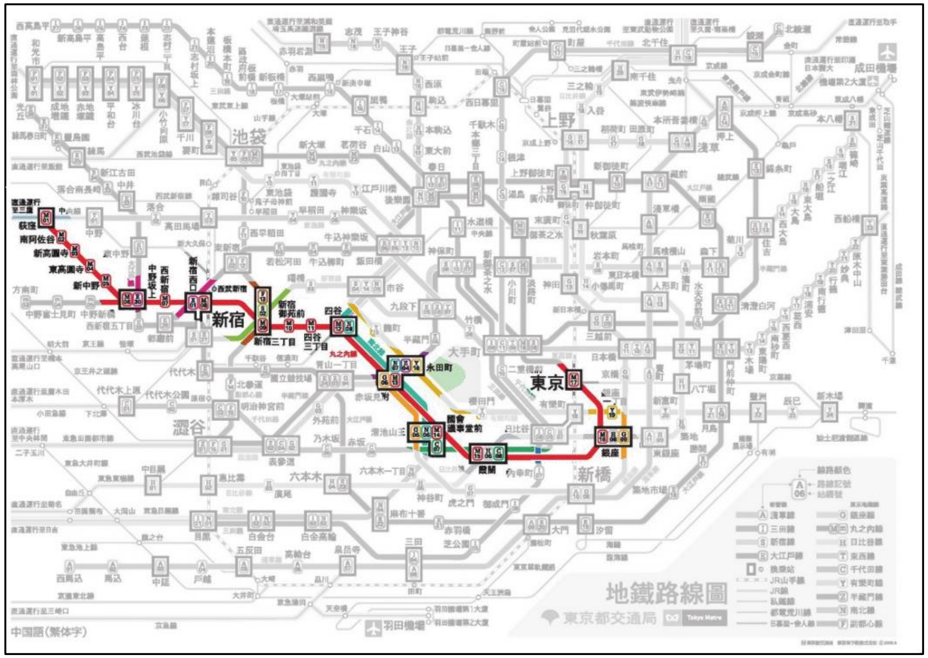
- 6.1 The graph abstract data type
- **6.2 Elementary graph operations**
- 6.3 Minimum-cost spanning trees
- 6.4 Shortest paths and transitive closure
- 6.5 Activity networks

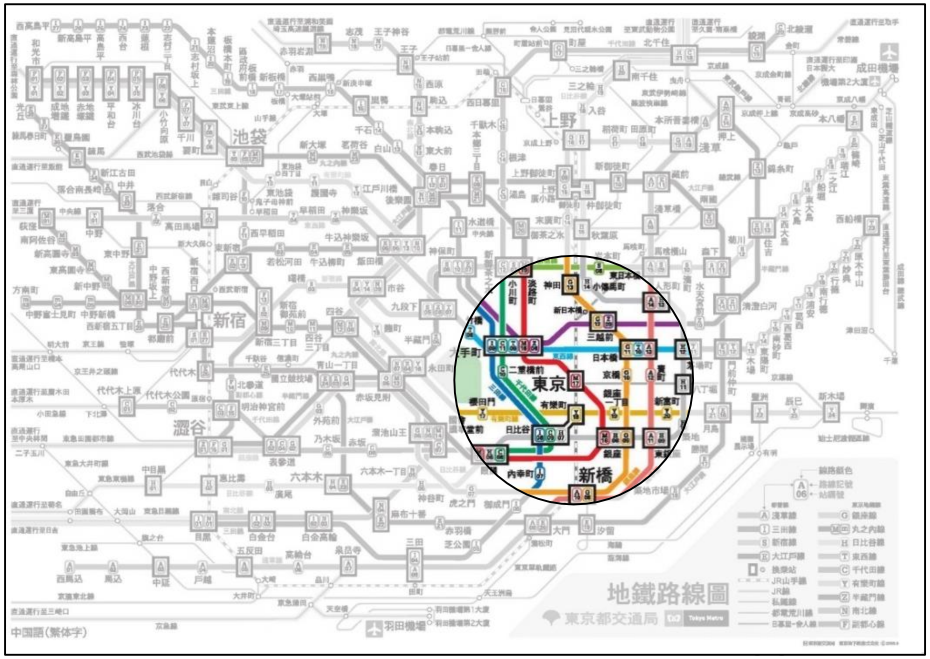
Elementary Graph Operations

- Depth-first search (DFS)
- Breadth-first search (BFS)
- Connected components
- Spanning trees
- Biconnected components

Concept of Search

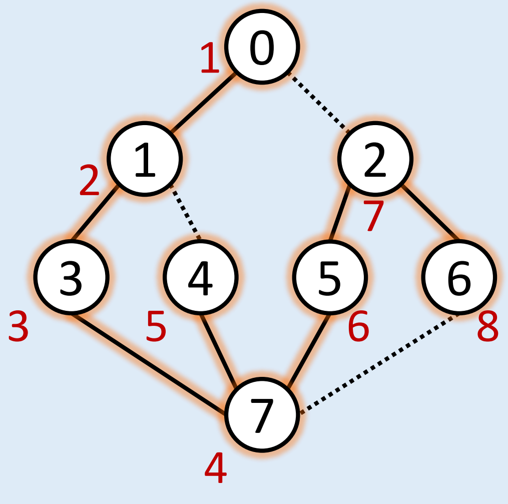
- Suppose we want to systematically traverse a city with a subway map in hand
  - Depth-first style
    - Following a subway path and visiting the places one after one
  - Breadth-first style
    - Visiting all places within a certain traveling distance





Depth-First Search (DFS)

- Begin by visiting the start vertex v
- Next an unvisited vertex w adjacent to v is selected
- A depth-first search from w is initiated
  - Recursion
- Backtrack if no unvisited vertices are reachable

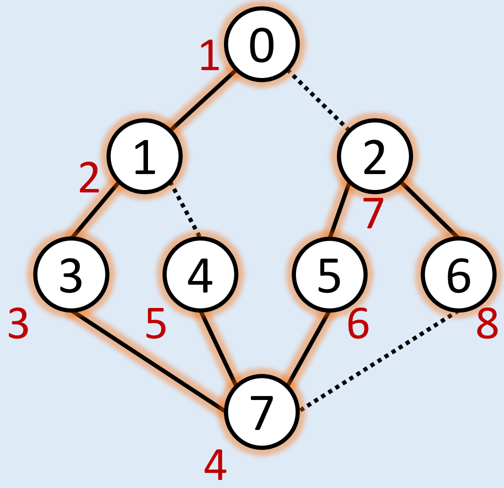


DFS

Depth-First Search (DFS)

```
virtual void Graph::DFS() // Driver
{
  visited = new bool[n];
  fill (visited, visited + n, false);
  DFS(0); // start search at vertex 0
  delete [] visited;
}
```

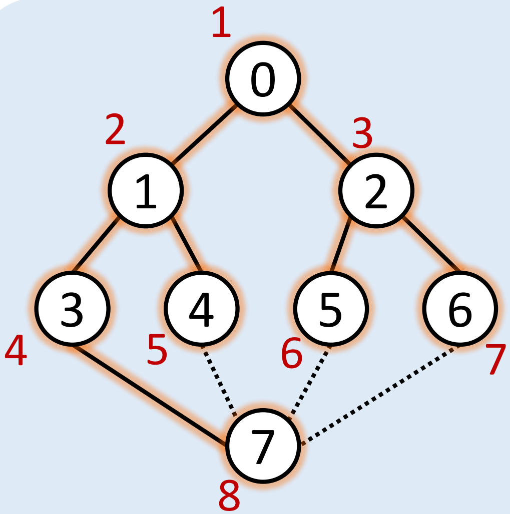
```
virtual void Graph::DFS(const int v)
{
  visited[v] = true;
  for (each vertex w adjacent to v)
    if (!visited[w])
      DFS(w);
}
```



DFS

Breadth-First Search

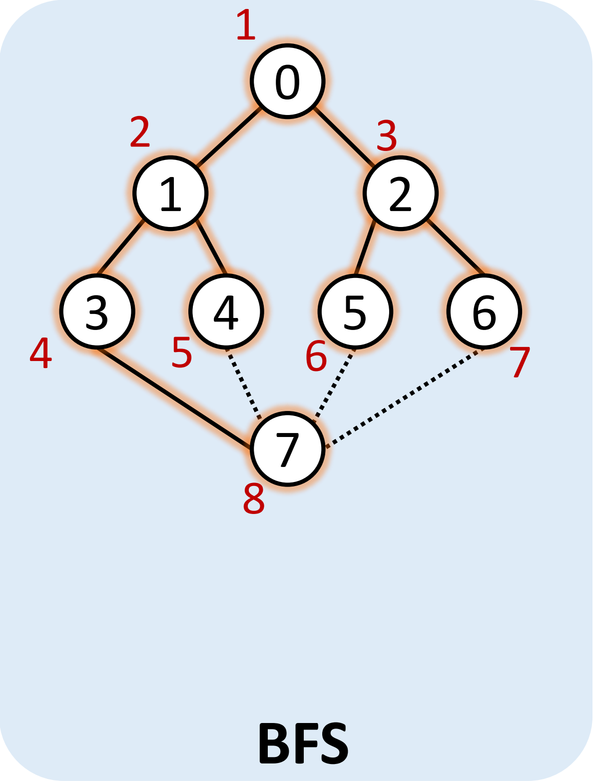
- Begin by visiting the start vertex v
- All unvisited vertices adjacent to v are visited
- Unvisited vertices adjacent to these newly visited vertices are then visited, and so on



BFS

Breadth-First Search

```
virtual void Graph::BFS(int v)
{
  visited = new bool [n];
  fill (visited, visited + n, false);
  visited[v] = true;
  Queue<int> q;
  q.Push (v);
  while (!q.IsEmpty ()) {
    v = q.Front ();
    q.Pop ();
    for (all vertices w adjacent to v)
      if (!visited [w]) {
        q.Push (w);
        visited[w] = true;
      }
    }
    delete [] visited;
}
```



Concept of Connect Component

- Determine whether a graph is connected
  - Call DFS of BFS and then determine if there is any unvisited vertex

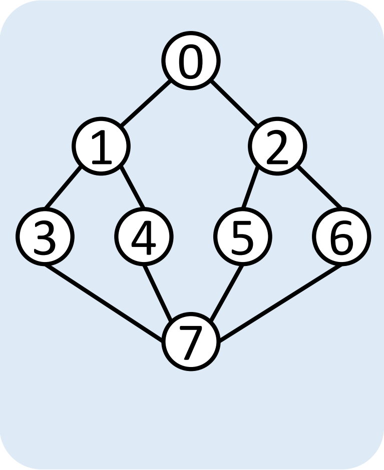
- Find connected components in a graph
  - Make repeated calls to either DFS(v) or BFS(v)
    - where v is a vertex that has not yet been visited

Connect Components

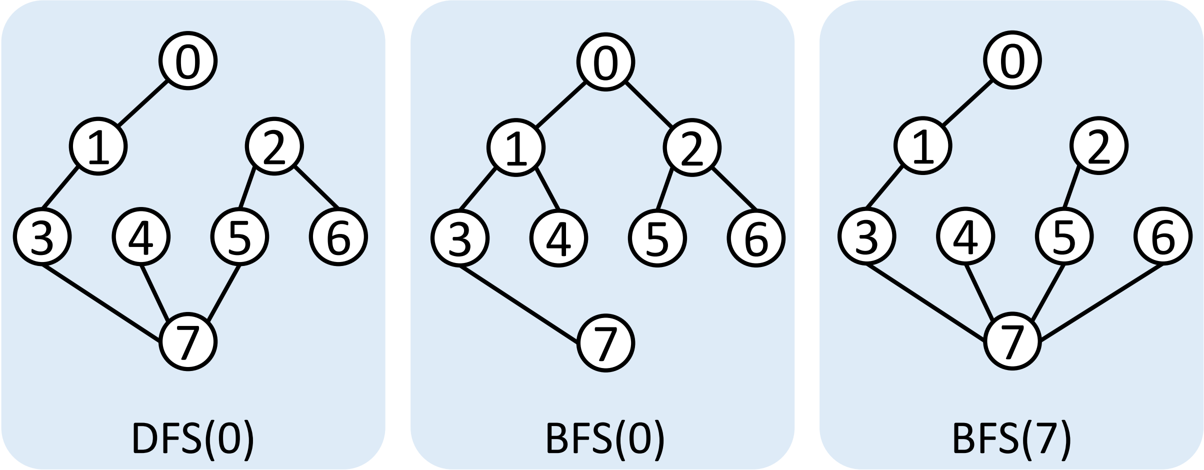
```
virtual void Graph::Components()
{
    visited = new bool [n];
    fill (visited, visited + n, false);
    for (i = 0 ; i < n ; i++){
        if (!visited[i]) {
            DFS(i); // find the component containing i
            OutputNewComponent ();
        }
    }
    delete [] visited;
}
```

Concept of Spanning Tree

- Any tree consisting solely of edges in G and including all vertices in G is called a spanning tree
  - Tree is a connected graph without loops
  - Graph has multiple spanning trees
  - Traversing a graph can produce a spanning tree
    - Depth-first spanning trees or breadth-first spanning trees



G



DFS(0)

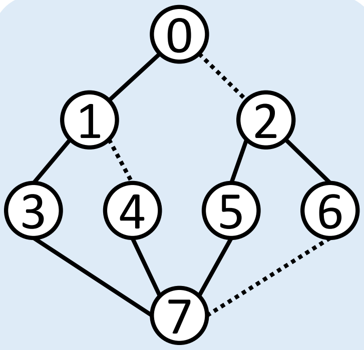
BFS(0)

BFS(7)

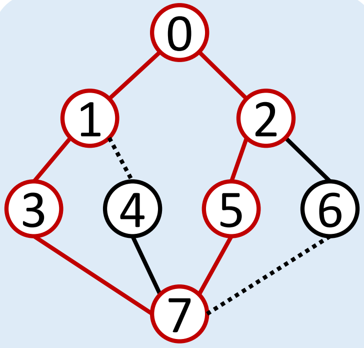
Some spanning trees of G

Spanning Tree → Independent Cycles

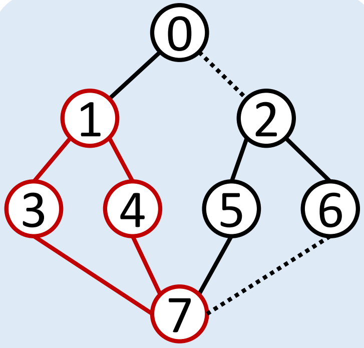
- Introducing a nontree edge (v, w) into a spanning tree produces a cycle
- These cycles are independent
  - Each introduced nontree edge is not contained in any other cycle
  - We cannot obtain any of these cycles by taking a linear combination of the remaining cycles
  - (# of independent cycles) = (# edges) - (# vertices - 1)



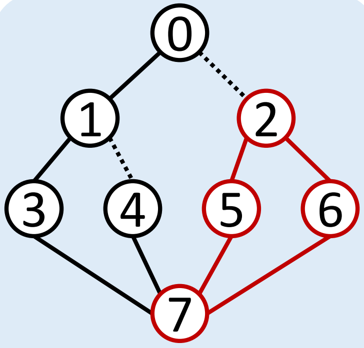
DFS(0)



DFS(0) + (0, 2)



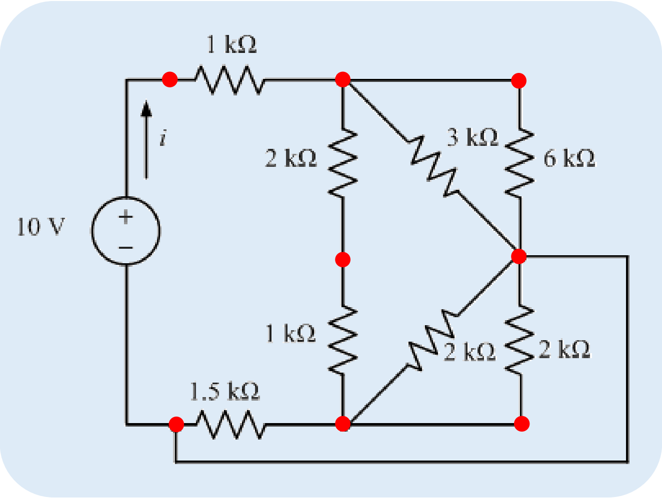
DFS(0) + (1, 4)



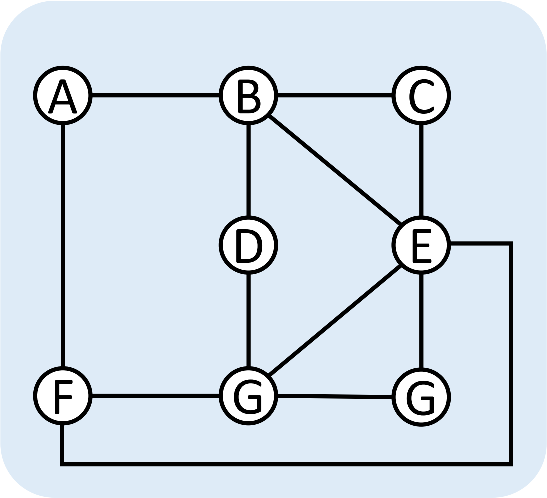
DFS(0) + (6, 7)

Spanning Tree → Independent Cycles

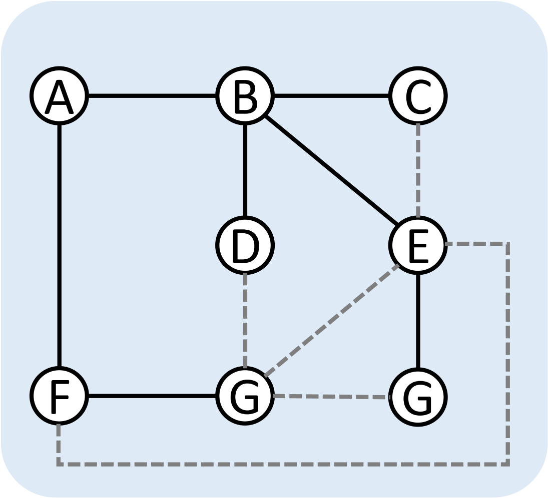
- Producing independent Kirchhoff's voltage equations of an electrical circuit network



Circuit



Graph



BFS(A)

Equations:

- $V_{BC}+V_{CE}+V_{EB} = 0$
- $V_{AB}+V_{BD}+V_{DG}+V_{GF}+V_{FA}= 0$
- …

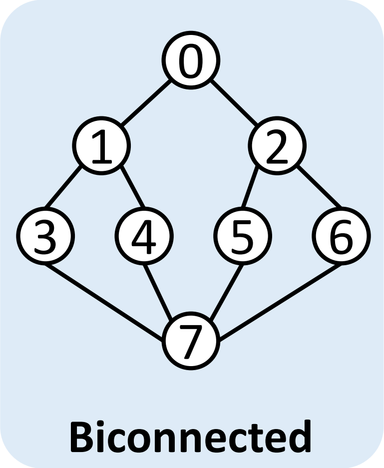
Biconnected Components

- A vertex v of undirected, connected graph G is an articulation (關節) point iff
  - Deleting v and all edges incident to v makes G disconnected

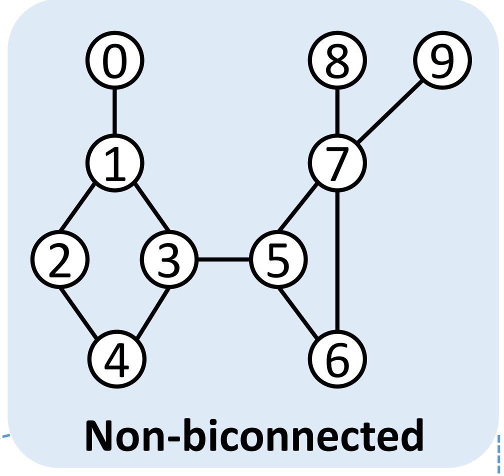
- A biconnected graph is a connected graph with no articulation points
  - No single point of failure
  - A desired property for, say, a communication network

- A biconnected component is a maximal binconected subgraph

Biconnected Components

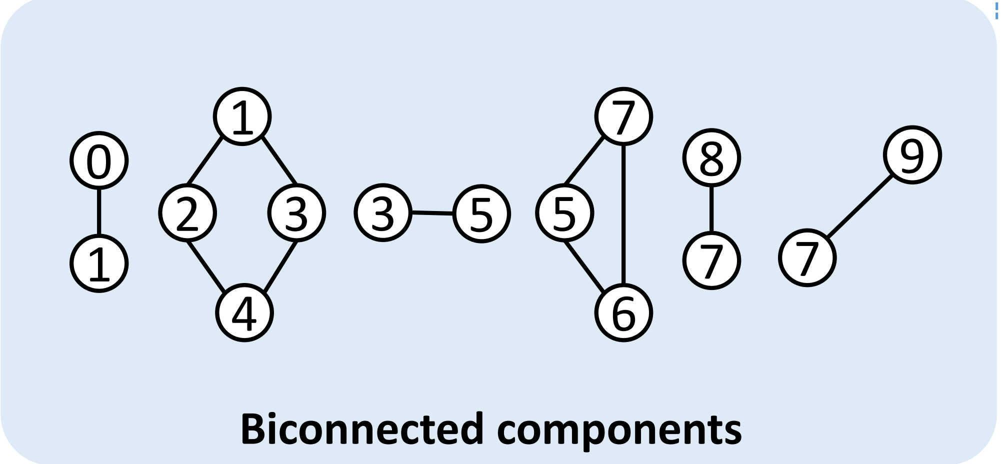


Biconnected

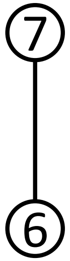


Non-biconnected

Vertices 1, 3, 5, 7 are articulation points



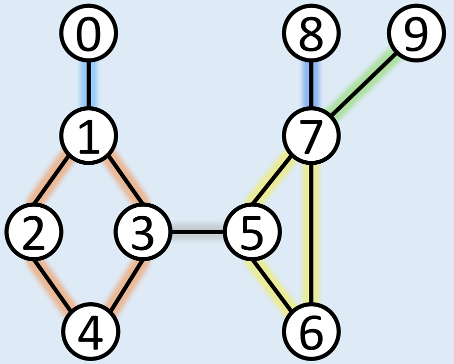
Biconnected components

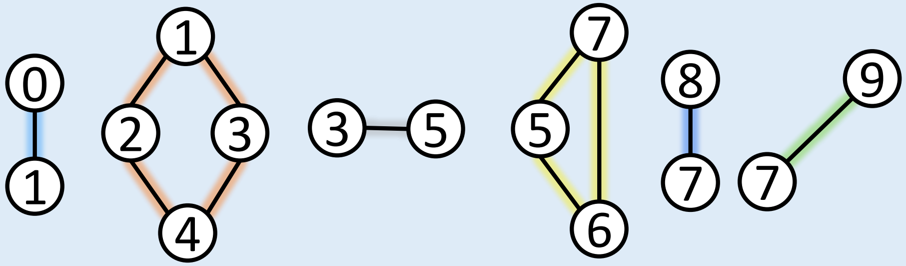


This is not a biconnected component because it is not maximal

Biconnected Components

- A binnected graph has just one biconnected component: the whole graph
- Two biconnected components of the same graph can have at most one common vertex
  - Therefore, no edges can be in two or more biconnected components
- Biconnected components of G partition the edges of G

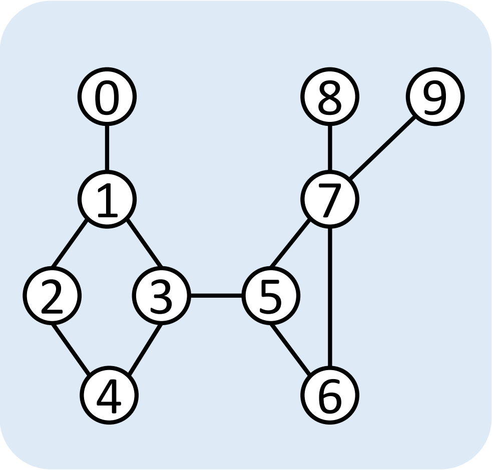




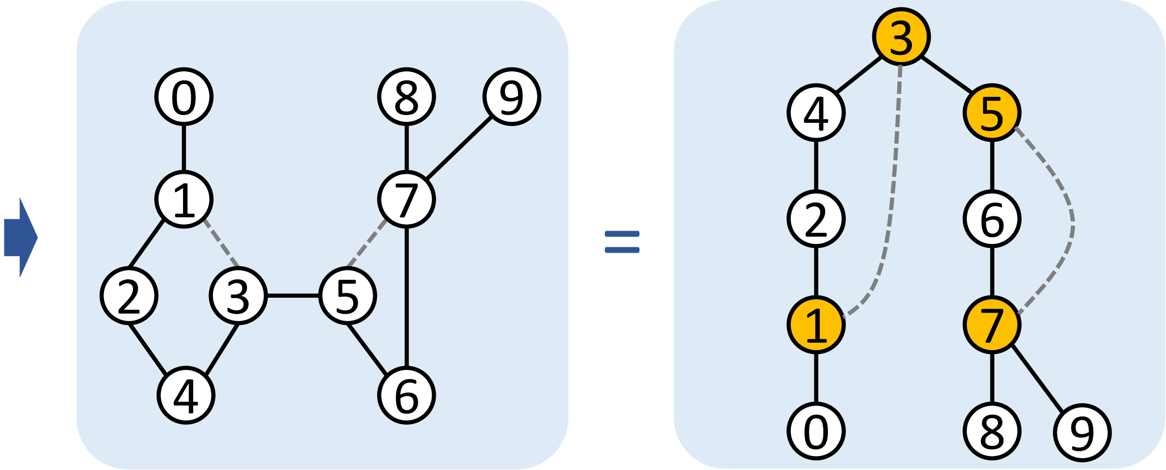
Biconnected components

Find Biconnected Components

- Depth-first spanning tree can be used to find articulation points, which indicate biconnected components



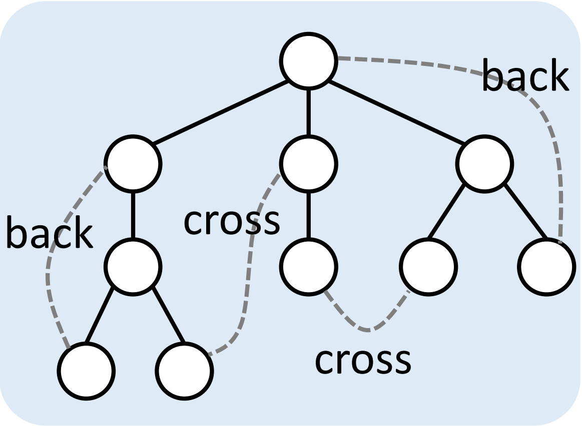
Graph



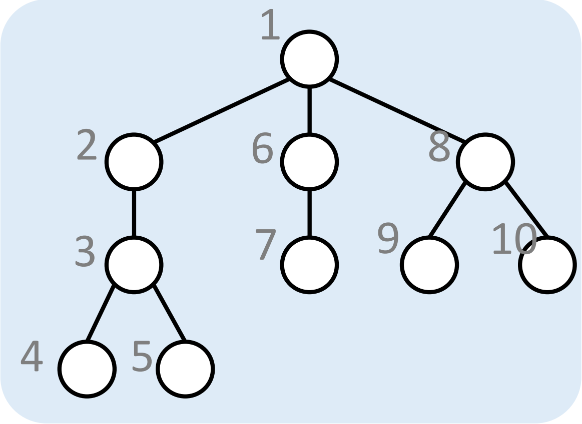
DFS(3) spanning tree
(shaded vertices are articulation points)
(We can pick any vertex as the root and find the same articulation points)

Depth-First Spanning Trees of Graphs

- Nontree edges
  - Back edge
    - A nontree edge (u, v) in which either u is an ancestor of v or v an ancestor of u
  - Cross edge
    - A nontree edge that is not a back edge
  - From the definition of DFS, a graph has no cross edges with respect to its depth-first spanning trees

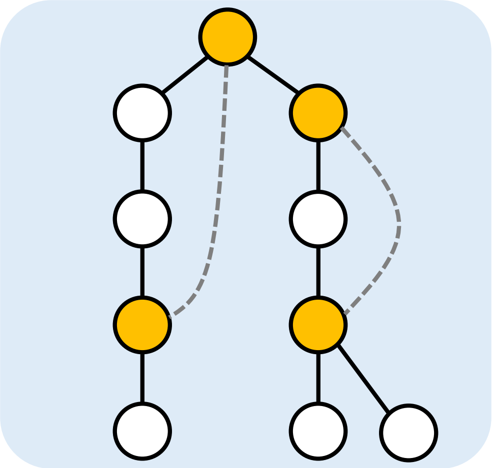


- Depth-first number, dfn,
  - The sequence in which the vertices are visited during the DFS



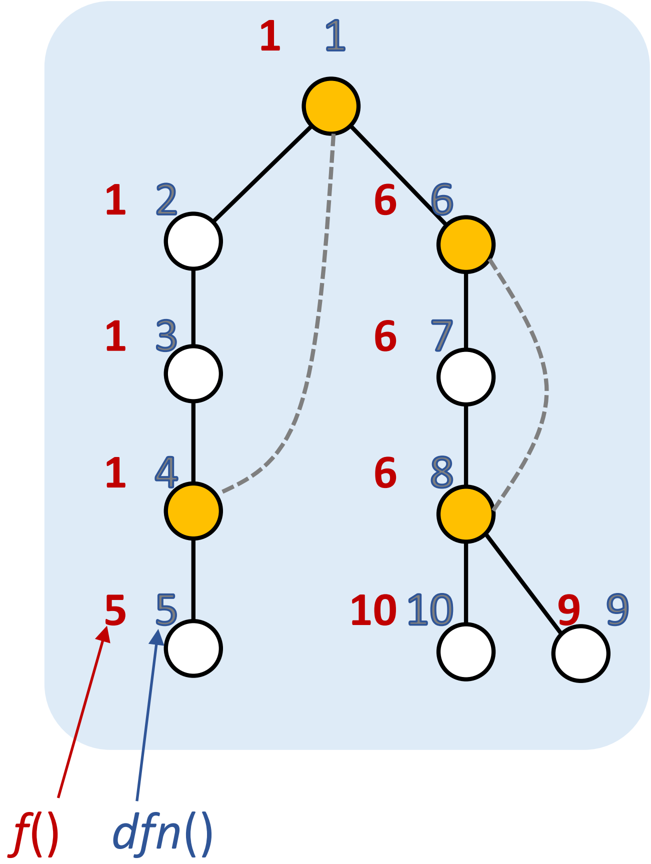
Identifying Articulation Points

- Analyzing any depth-first spanning tree of the graph
  - Leaf
    - Cannot be an articulation point
  - Root
    - Is an articulation point iff it has ≥ 2 children
    - since there are no cross edges among the root's subtrees
  - Other (non-root ,non-leaf) vertex, *u*
    - Is an articulation point iff u's ancestors lacks a non-tree edge to any of u's subtrees
    - Without the non-tree edge, *u* separates the ancestors from the subtrees



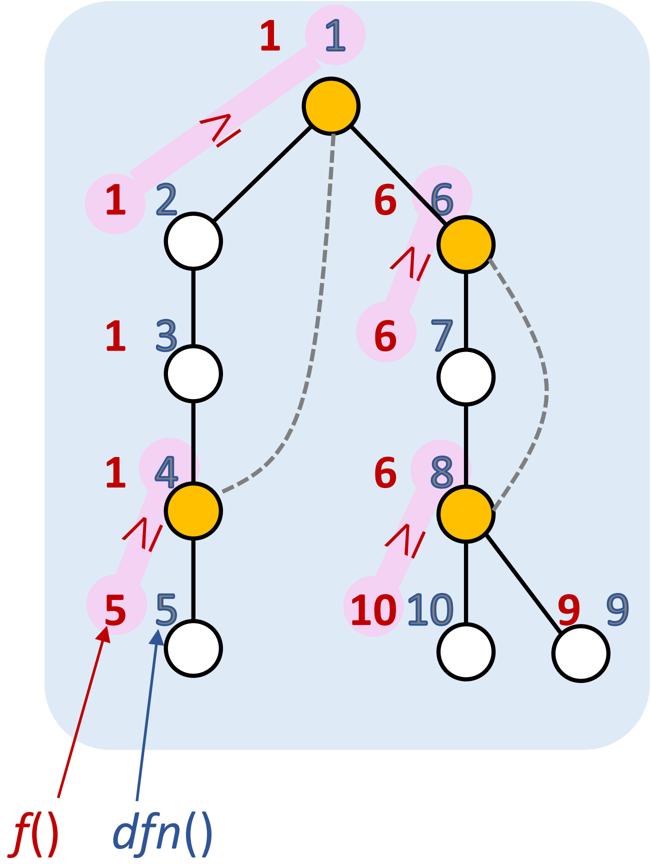
Algorithm

- Technique
  - Find the lowest reachable ancestor through descendants and one back edge
  - Define *f*(w) for a vertex w as the minimum of the following values
    - *dfn*(w)
    - *dfn*(x | (w,x) is a nontree edge)
    - *f*(w's children)
  - *f*(w)
    - 由 root 出發到 w，替代路徑出發點的深度
    - 若 *f*(w)=*dfn*(w) 代表無替代路徑



Algorithm

- u has any child w such that *f*(w) ≥ *dfn*(u)
  - →由 root 出發到達 w 必經過 u，沒有替代的 non-tree edge
  - → u is an articulation point
- In the textbook, *f*() is called as *low*()

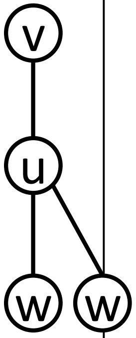


Computing dfn, low, and Outputting Biconnected Components

```
virtual void Graph::Biconnected()
{
    num = 1;          // num is an int data member of Graph
    dfn = new int[n]; // dfn is declared as int* in Graph
    low = new int[n]; // low is declared as int* in Graph
    fill(dfn, dfn + n, 0);
    fill(low, low + n, 0);
    rBiconnected(0, -1);
    delete [] dfn;
    delete [] low;
}
```

Computing dfn, low, and Outputing Biconnected Components

```
void Graph::rBiconnected (const int u, const int v)
{
    dfn[u] = low[u] = num++;
    for (each vertex w adjacent from u){  // (u, w)
        if ((v != w) && (dfn[w] < dfn[u]))
            add (u, w) into stack s;
        if (dfn[w] == 0) { // w is an unvisited vertex, a child
            rBiconnected(w, u);
            low[u] = min(low[u], low[w]);
            if (low[w] >= dfn[u]) { // u is an articulation point
                cout << “New Biconnected Component:” << endl;
                do {
                    delete an edge from the stack s;
                    let this edge be (x, y);
                    cout << x << “,” << y << endl;
                } while ( (x, y) and (u, w) are not the same edge)
            }
        }
        else if (w != v)
            low[u] = min(low[u], dfn[w]); // back edge
    }
}
```

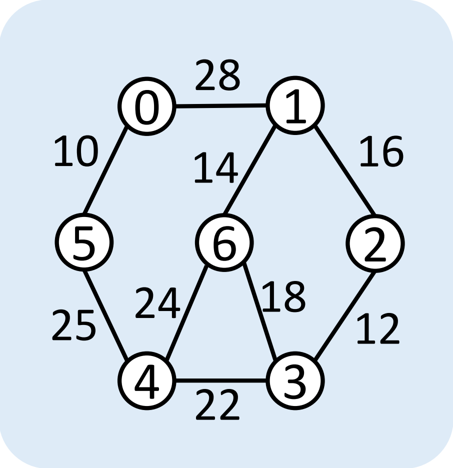


Outline

- 6.1 The graph abstract data type
- 6.2 Elementary graph operations
- **6.3 Minimum-cost spanning trees (MSTs)**
- 6.4 Shortest paths and transitive closure
- 6.5 Activity networks

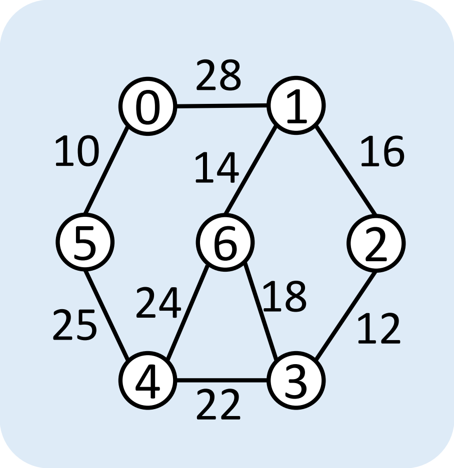
Minimum-Cost Spanning Trees (MSTs)

- An graph can have many spanning trees
- For a weighted, connected, and undirected graph
  - We define the cost of a spanning tree is the sum of the weights of the edges in the spanning tree
- We may want to minimize the cost
  - Possible applications: road construction, circuit layout, internet routing



Minimum-Cost Spanning Trees

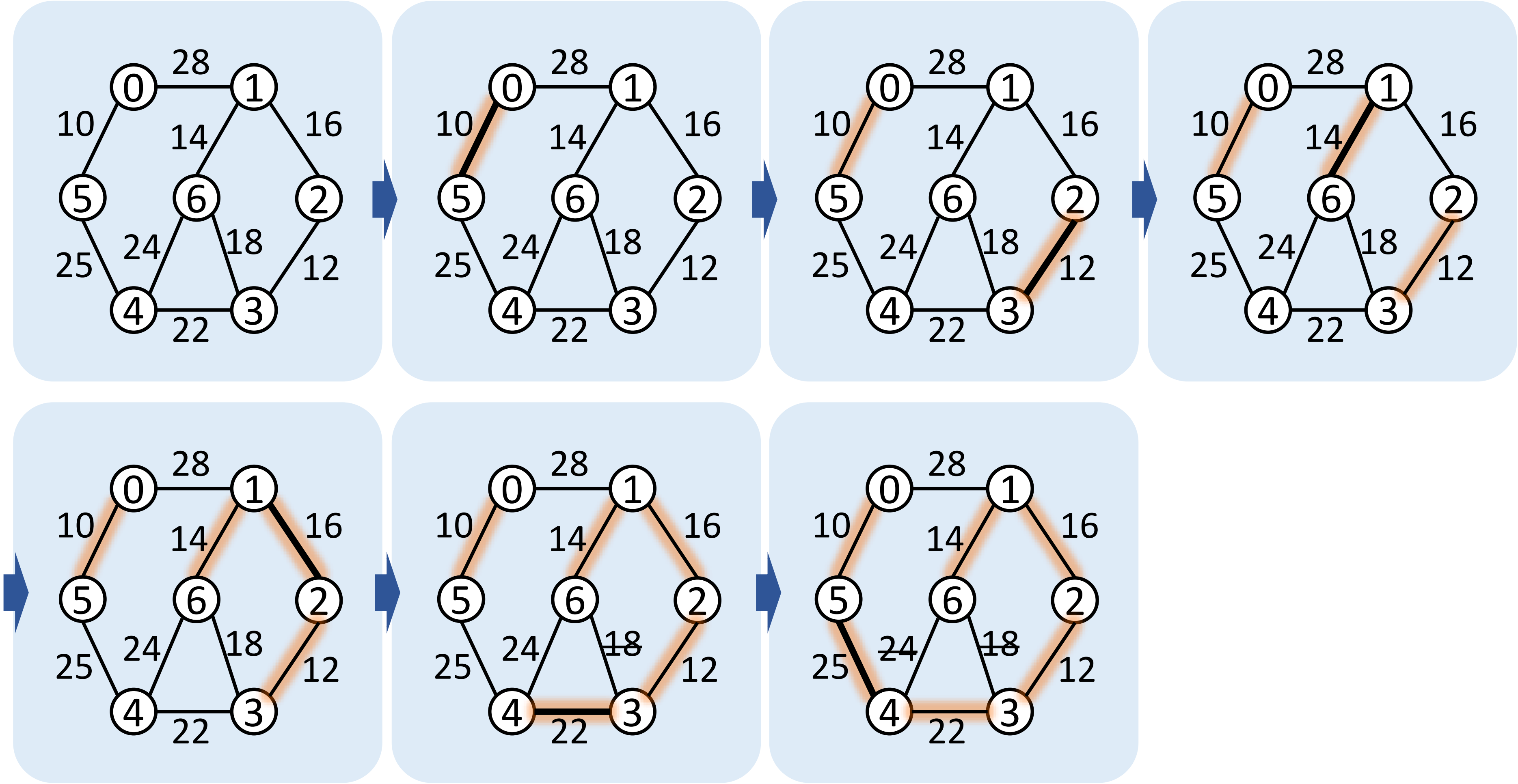
- Three greedy methods
  - Kruskal's
  - Prim's
  - Sollin's
- In a greedy method, we construct an optimal solution in stages
  - At each stage, we make the best decision possible at the time
    - (e.g., the least-cost edge is chosen for building a minimum-cost spanning tree
  - We do not change this decision later
- Greedy strategy can lead to MST construction



Kruskal's Algorithm

- Create an empty graph, T
- Sort edges according to weights
- Add edges to T one at a time
    - The least-cost edge that does not form a cycle with T's edges
- Exactly n-1 edges are added, where n is the number of vertices

Kruskal's Example



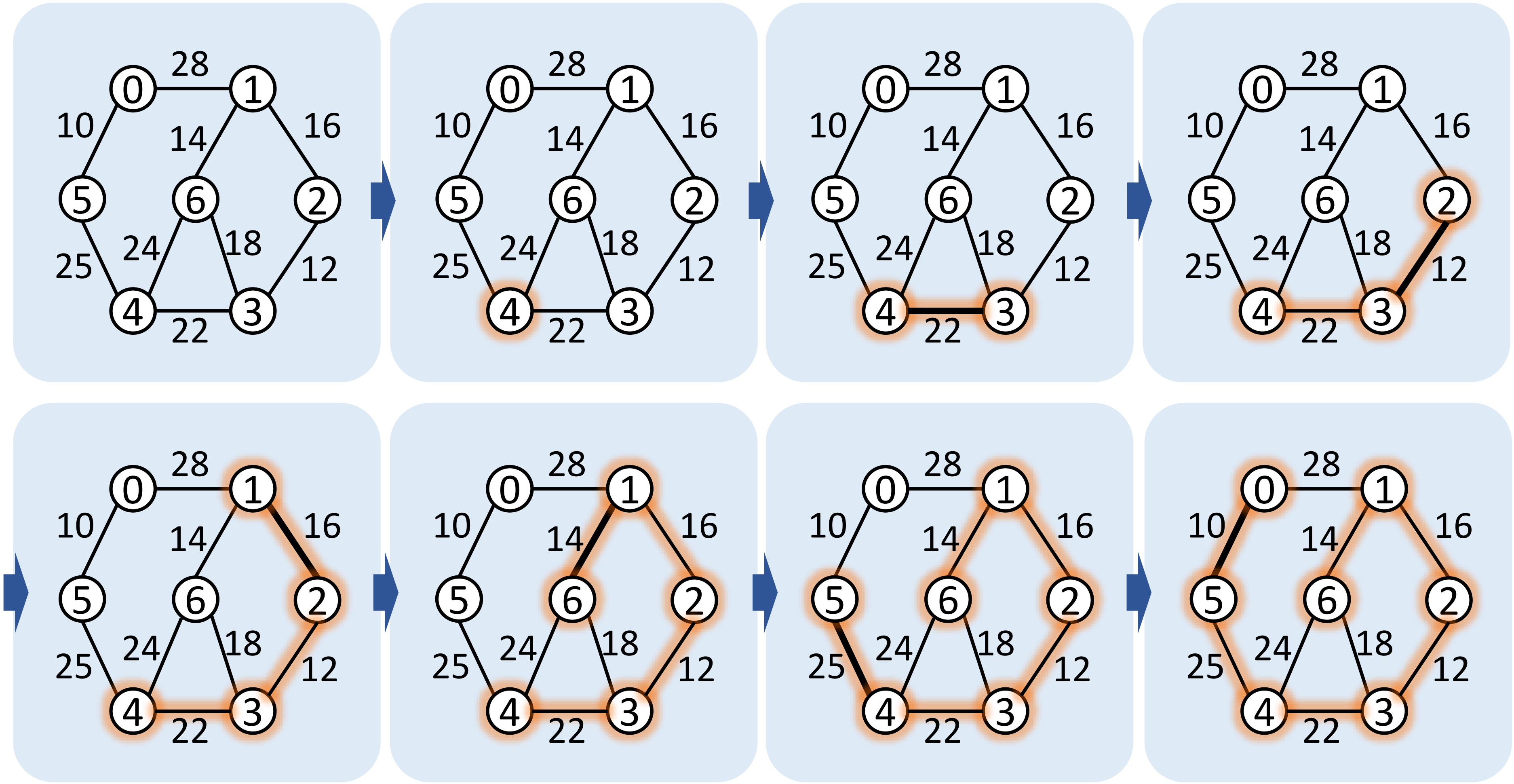
Kruskal's Algorithm

```
T = Φ;
while ( ( T contains less than n-1 edges) &&
        (E is not empty) ) {
  choose an edge (v, w) from E of lowest cost;
  delete (v, w) from E;
  if ( v and w belong to diff. sets){ // no loop
    add (v, w) to T;
    merge v's and w's sets;
  }else{
    discard (v, w);
  }
}
if ( T contains fewer than n-1 edges)
    cout << "no spanning tree" << endl;
```

Prim's Algorithm

- Begin with an empty tree, T
- Sort edges according to weights
- Add to T any vertex of the graph
- Add vertices to T one at a time
  - The vertex is adjacent to a vertex in T
  - The vertex corresponds to the least-cost edge

Prim's Example



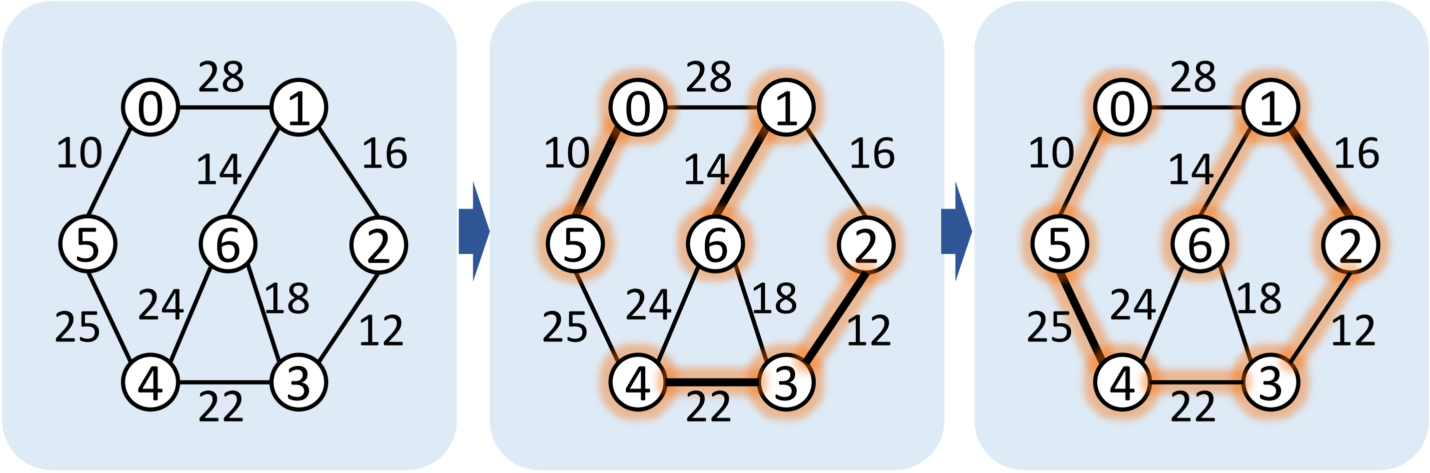
Prim's Algorithm

```
if (G has at least one vertex)
    cout << "no spanning tree" << endl;

TV = {0}; // start with vertex 0 and no edges
for (T = Φ；T contains less than n–1 edges；add (u, v) to T)
{
    Let (u, v) be a least-cost edge with u in TV && v not in TV;
    if (there is no such edge)
        break;
    add v to TV;
}

if ( T contains fewer than n-1 edges)
    cout << "no spanning tree" << endl;
```

Sollin's Algorithm

- Create n subgraphs, each subgraph having a single vertex
- Sort edges according to weights
- Add edges to each subgraph one at a time
  - The least-cost edge that does not form a cycle with each subgraph 's edges
  - Duplicate edges are discarded
- A total of exactly n-1 edges are added, where n is the number of vertices

Sollin's Example



Outline

- 6.1 The graph abstract data type
- 6.2 Elementary graph operations
- 6.3 Minimum-cost spanning trees
- **6.4 Shortest paths and transitive closure**
- 6.5 Activity networks

Shortest Path Problem

- Let's consider a GPS device using graph data structures to represent the highway structures of a state
  - Vertices representing cities
  - Edges representing sections of highway
  - Edge weights representing lengths of the highway sections

- Important questions
  - Is there a path from A to B
  - What is the shortest path from A to B

Various Flavors of Path Problems

- Edge costs
  - Non-negative costs (e.g., traveling distance, spent time)
  - General costs (e.g., spent/obtained fuels)
- Number of sources and destinations
  - Single source single destination
  - Single source all destinations
  - All sources single destination
  - All pairs
- Textbook covers

$$\left.\begin{array}{l}\bullet\ \text{Single source all destinations}\\ \bullet\ \text{All pairs}\end{array}\right\}\times\left\{\begin{array}{l}\text{Non-negative costs}\\ \text{General costs}\end{array}\right.$$

Comparisons

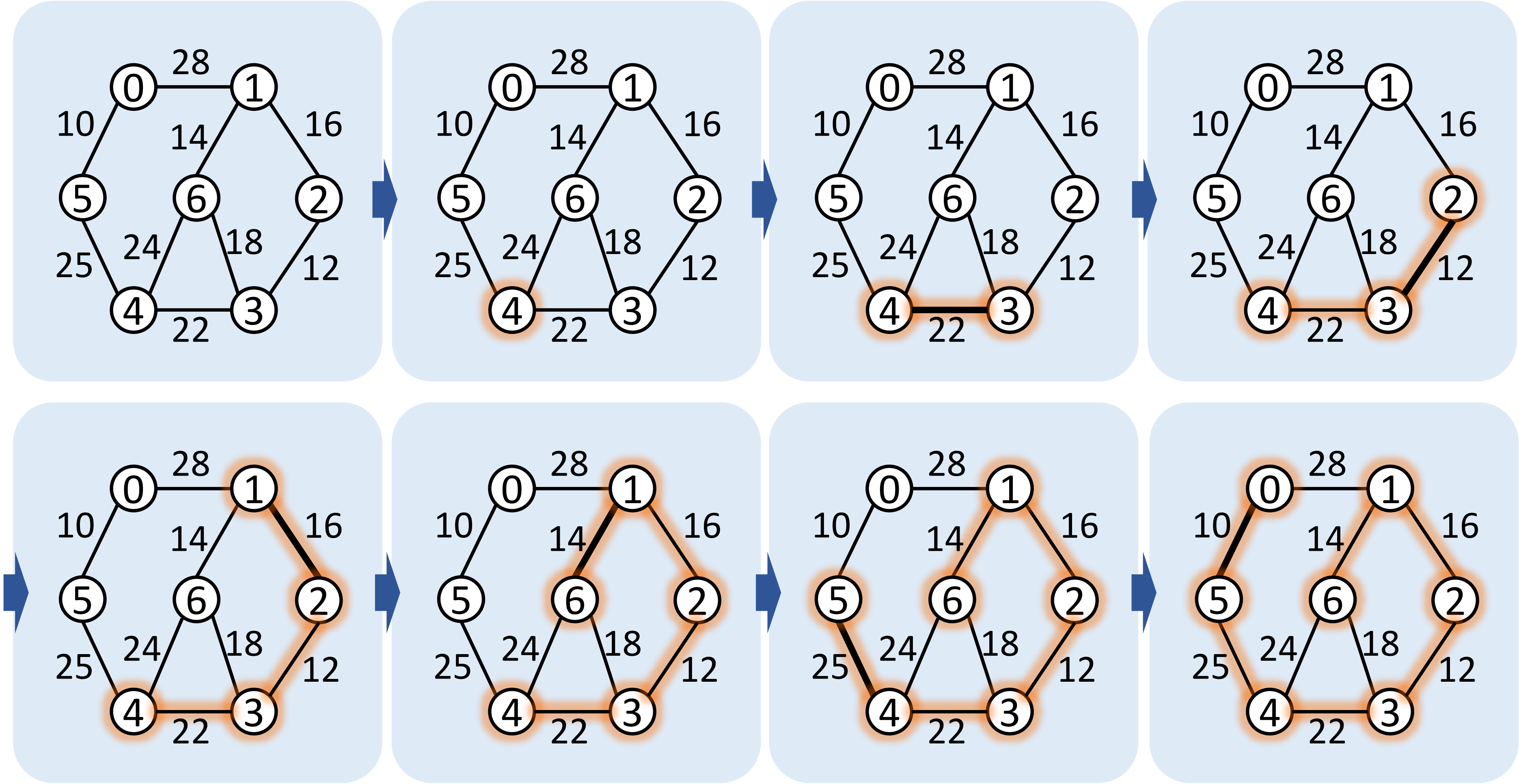
- Prim's (for MST)
- Dijkstra's (for shortest paths)

Strategy: greedy

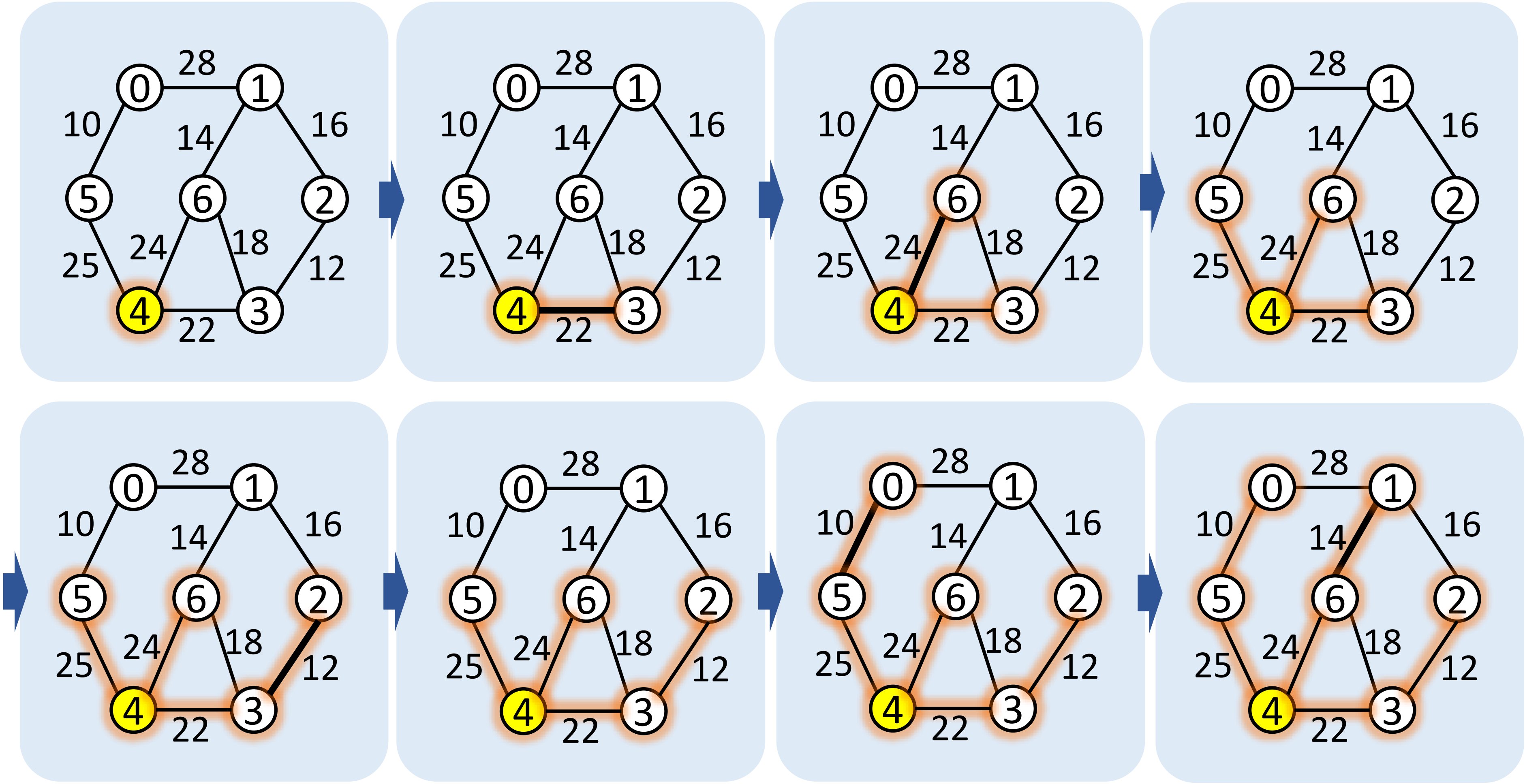
- Bellman-Ford (for shortest paths)
- All pairs (for shortest paths)

Strategy: table

Prim's Example

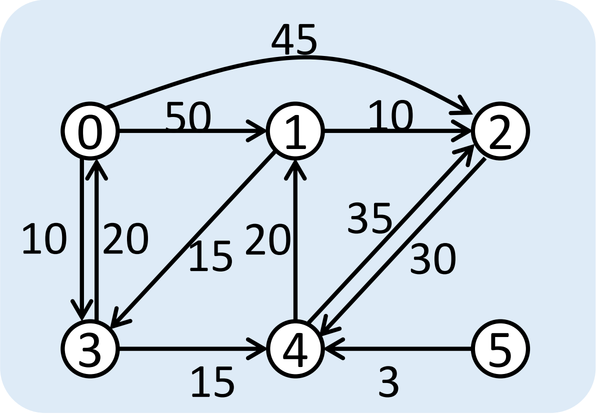


Dijastra's Example



Single Source, All Destinations, and Nonnegative Costs

- Input
  - A directed graph G = (V, E)
  - Length(i, j) for the edges of G
  - A source vertex v
- Output
  - Determine a shortest path from v to each of the remaining vertices of G in non-decreasing length order



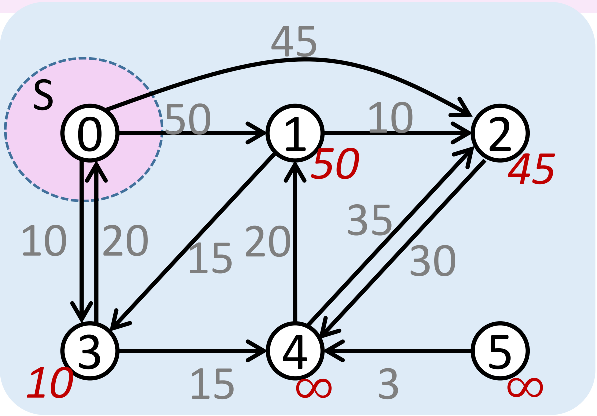
Graph

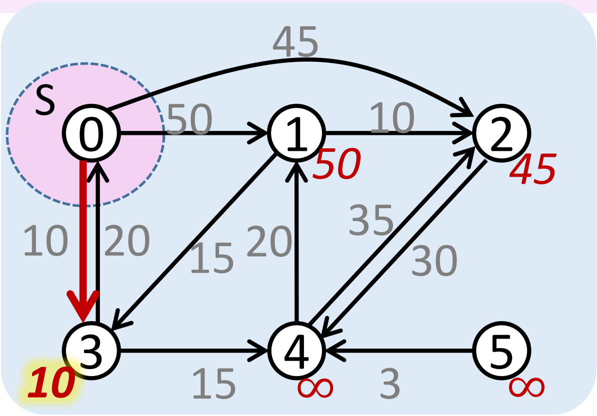
Path	**Length**
0, 3	10
0, 3, 4	25
0, 3, 4, 1	45
0, 2	45

Non-decreasing

All-destinations shortest path from 0

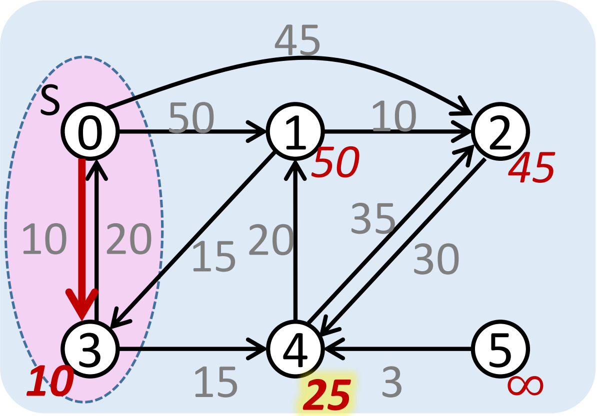
Dijkstra's Algorithm Example



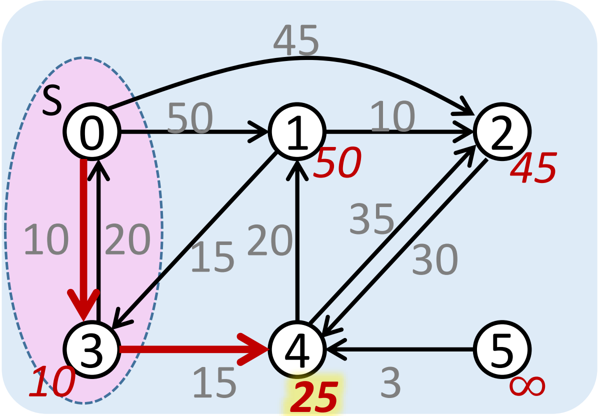


Path	Length
0, 3	10

Vertex 3 has the least-cost dist , i.e., 10.
So, output the 0-3 path.



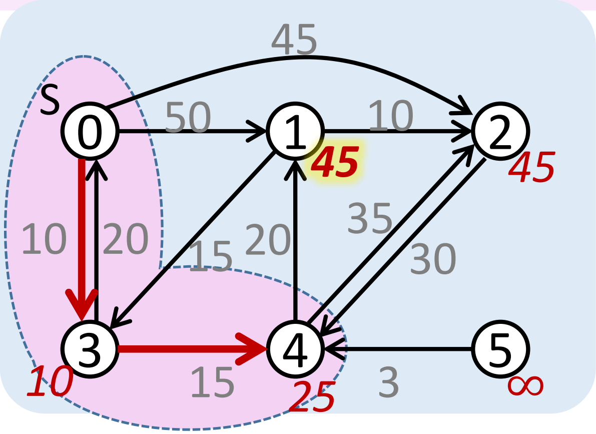
Include 3 in to S and update vertices adjacent from 3.



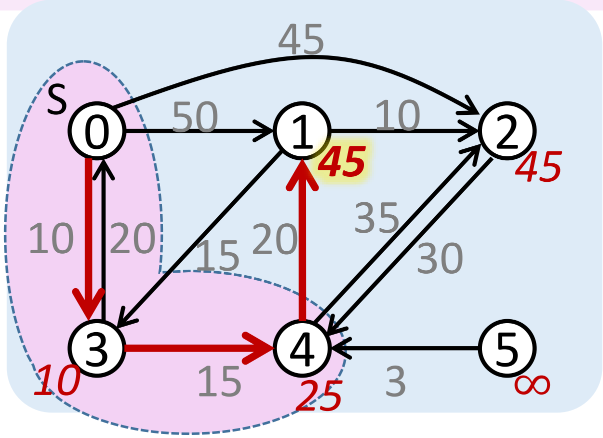
Path	Length
0, 3	10
0, 3, 4	25

Vertex 4 has least-cost dist , i.e., 25.
Output the 0-4 path.

Dijkstra's Algorithm Example

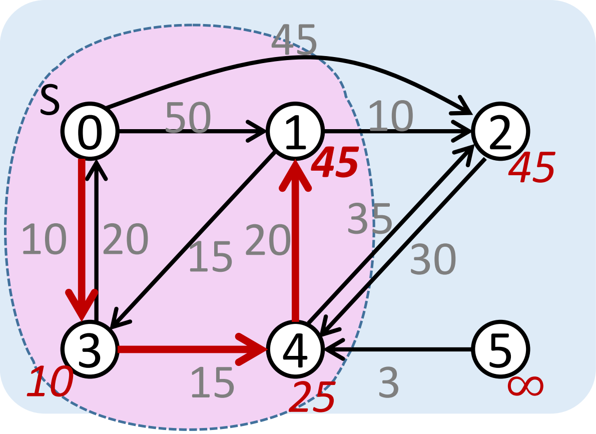


Include 4 in to S and update vertices adjacent from 4.

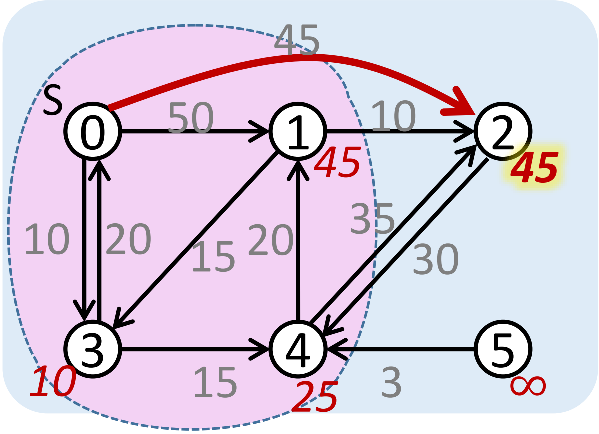


Both vertices 1 and 2 have the least-cost dist. Output one of them, e.g., 1.

Path	Length
0, 3	10
0, 3, 4	25
0, 3, 4, 1	45



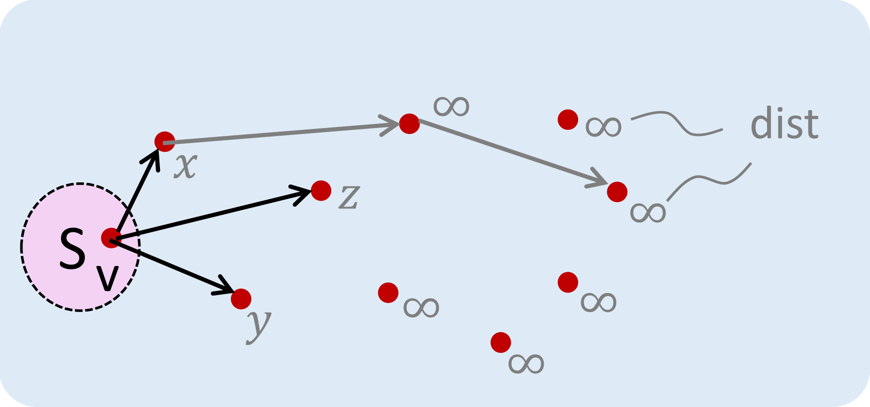
Include 1 in to S and update vertices adjacent from 1.



Include 1 in to S and update vertices adjacent from 1.

Path	Length
0, 3	10
0, 3, 4	25
0, 3, 4, 1	45
0, 2	45

Dijkstra’s Algorithm



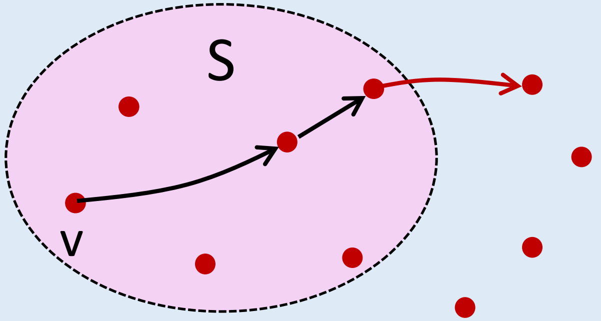
Edsger W. Dijkstra

- Dutch computer scientist
- Turing award recipient

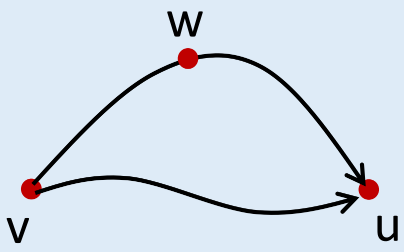
"Dijkstra" pronounces similar to /dye-k-stla/

- **S**: A set of vertices to which the shortest paths have already been found
  - S = {v} in the beginning
- **dist[u]**: Shortest distance from v, through vertices in S, to a vertex u not in **S**
  - <v, u> exists → dist[u] = edge weight
  - <v, u> doesn't exist → dist[u] = ∞
  - dist[v] is considered as 0

Dijkstra’s Algorithm

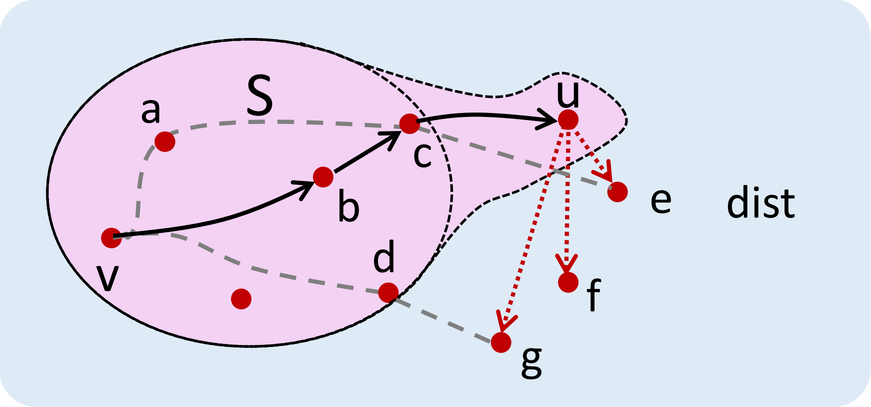


- When **S** contains n≥1 vertices
- A next shortest path must contain only vertices in **S** plus the destination
  - There may be multiple equal-length shortest path. At least one of them is so



- Let u be the destination of a next shortest path
- Assume said path contains an intermediate vertex w not in **S**
  - The length of the v-w path is no greater than the v-u path
  - u should not be the destination of a next shortest path (→←)

Dijkstra's Algorithm

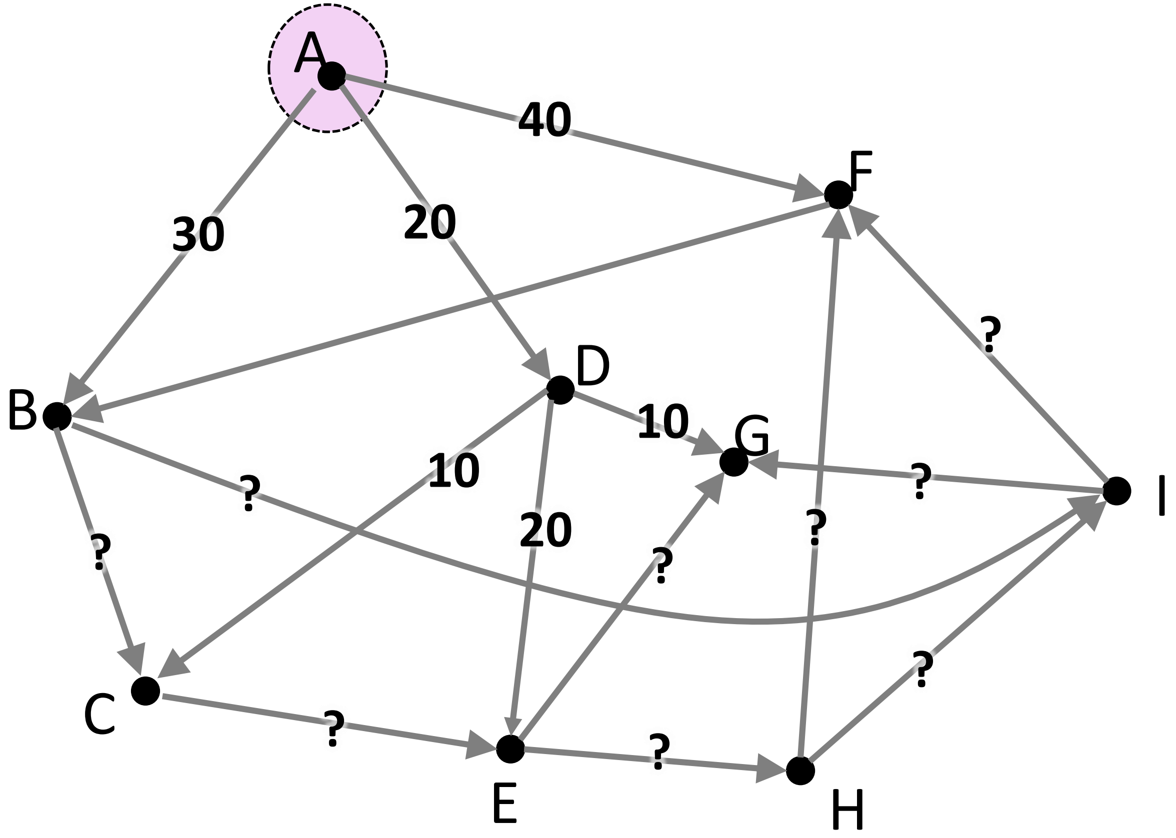


dist[u]: Shortest distance from v, through vertices in S, to a vertex u not in **S**

- Greedy: among vertices not in S, find a vertex u with the lowest dist[]
  - u becomes a new member of **S**
- Keep dist[] updated
  - u may lower **dist[]** of vertices that are not in **S** and adjacent from u
- The algorithm stops when **S** contains all n vertices

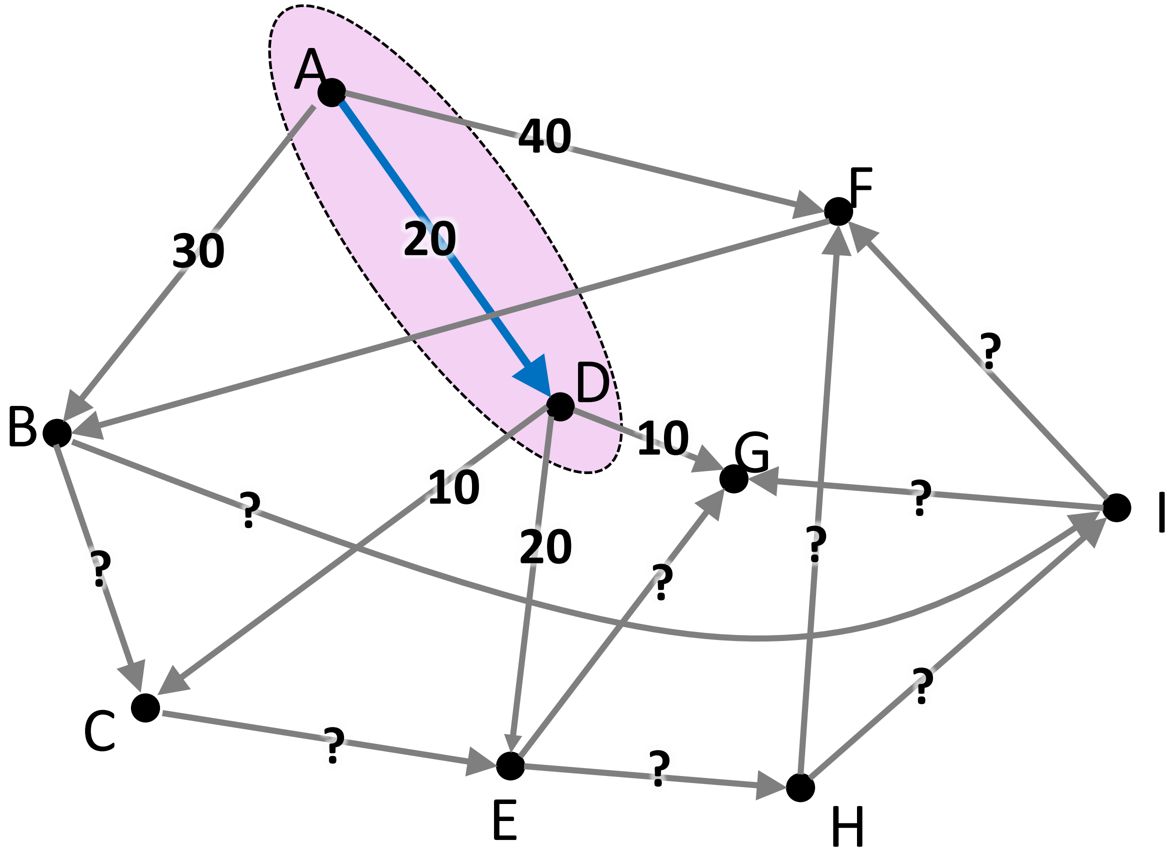
Quick Questions

- Given a graph with non-negative weights, we want to find shortest paths starting from A.



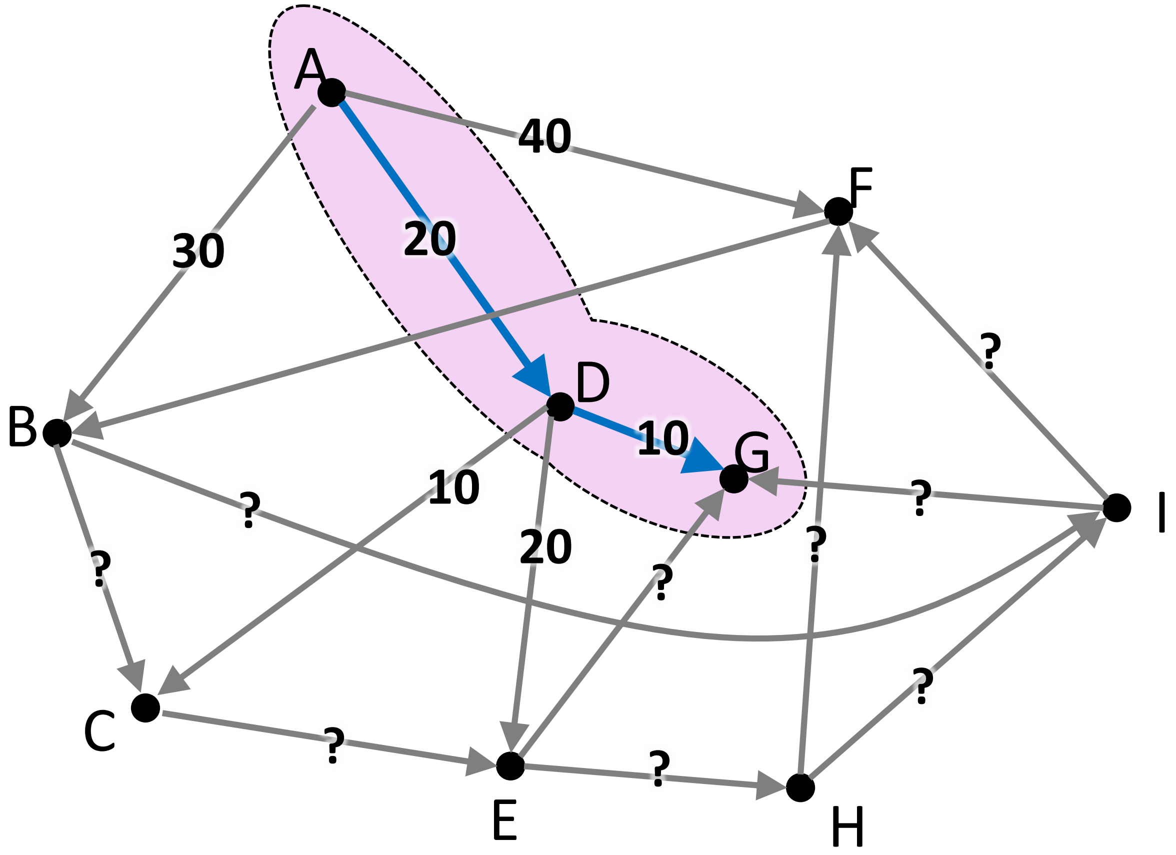
Quick Questions

- A-D must be a shortest path. Why can we be so sure?



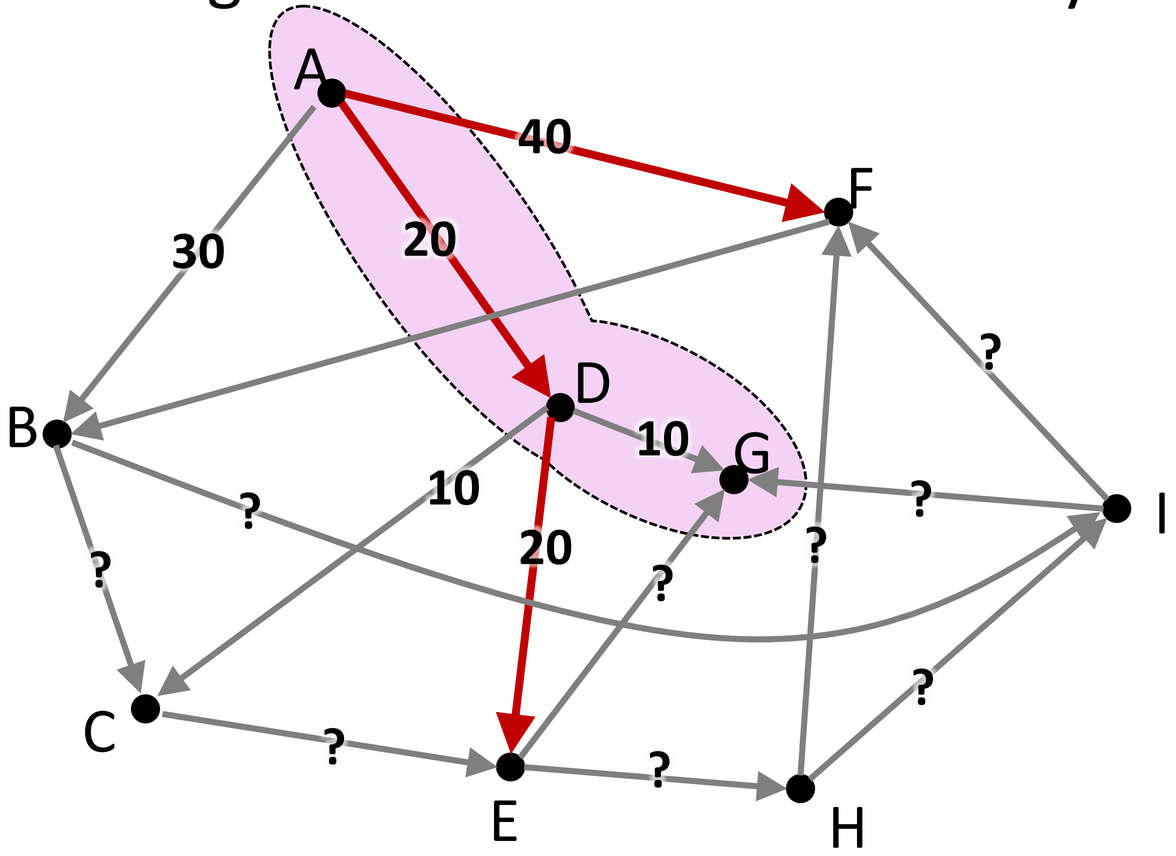
Quick Questions

- A-D-G must be a shortest path. Why can we be so sure?



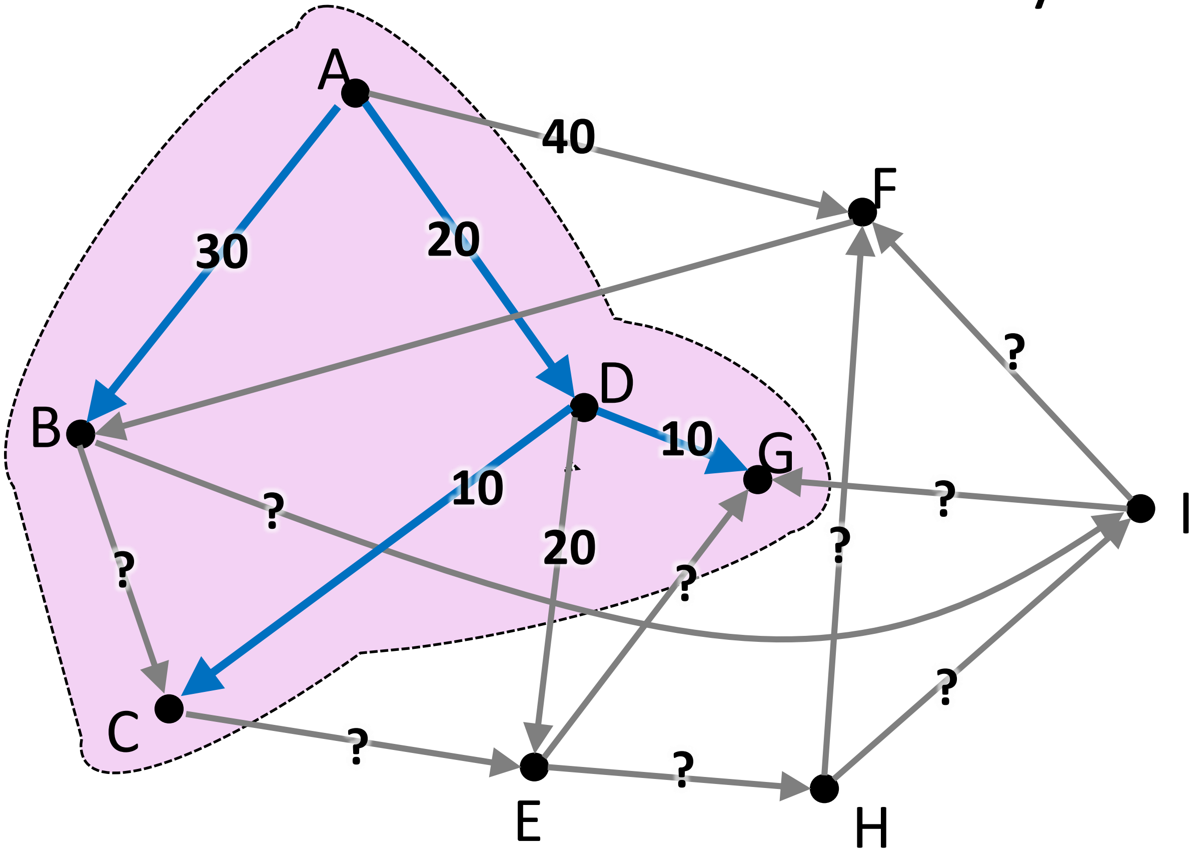
Quick Questions

- Whether A-F, A-D-E are shortest paths depends on the edges with unknown cost. Why?



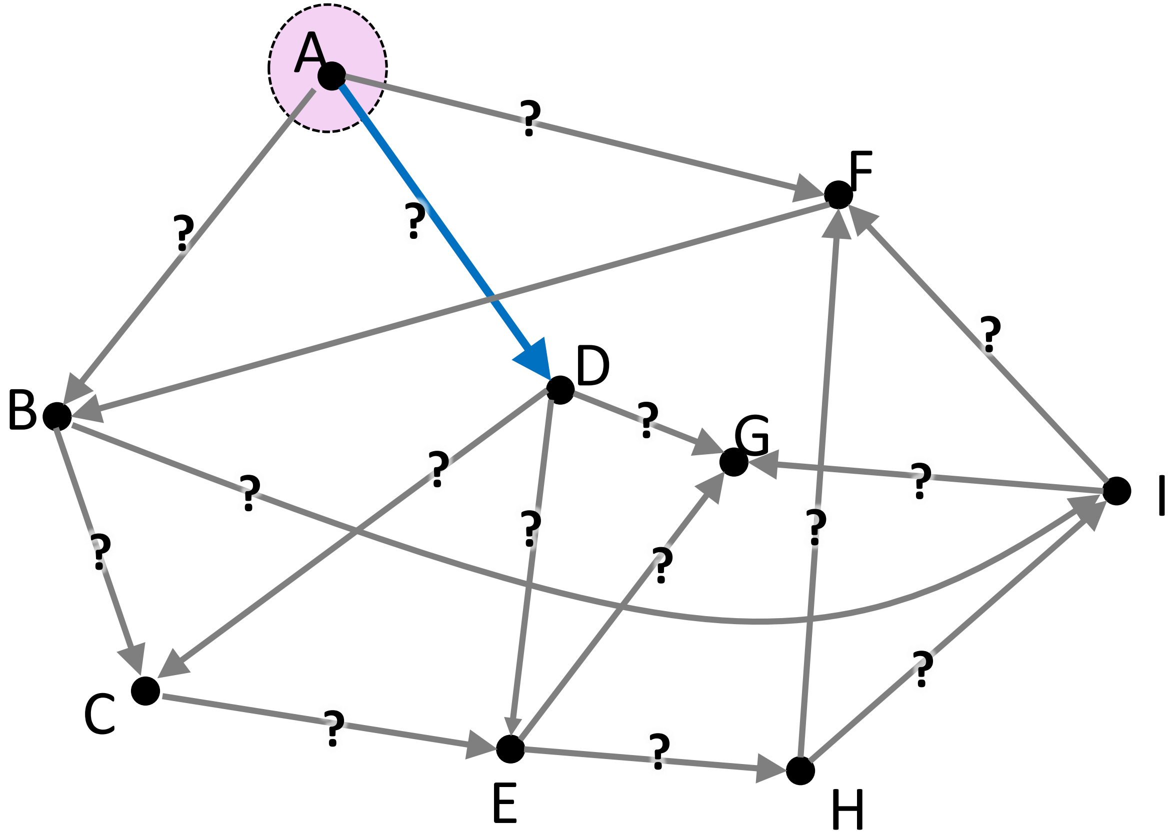
Quick Questions

- Shortest paths from A to B, C, D, G are known, but that to others are unknown. Why?



Quick Questions

- If the following table lists all the shortest paths from. A-D edge cost must be 20. Why?



Destination	Cost
B	30
C	30
D	20
E	35
F	40
G	30
H	50
I	50

Dijkstra's Algorithm

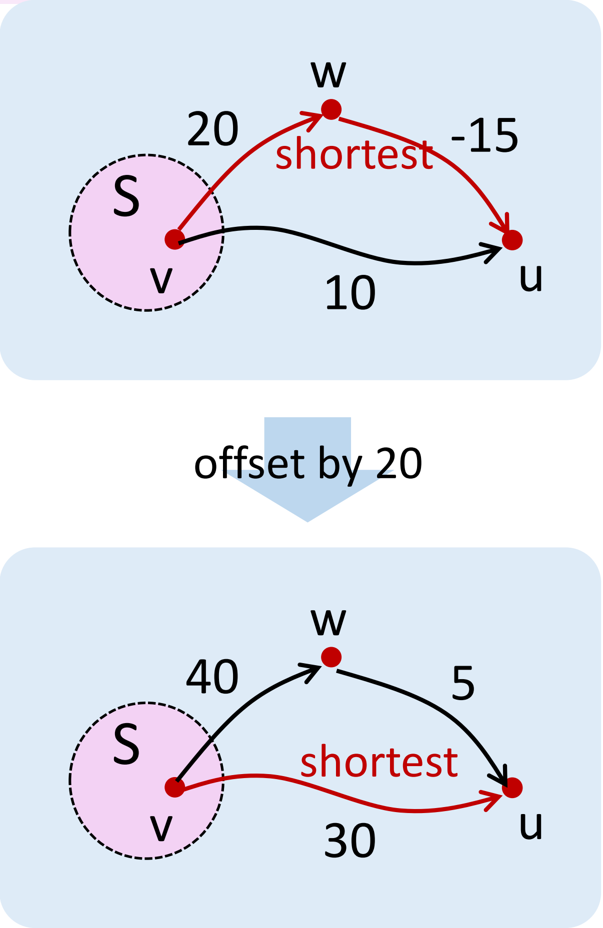
```
void MatrixWDigraph::ShortestPath(const int n, const int v)
{
    for (int i = 0; i < n ; i++) { // initialization
        s[i] = false;              // the set, S
        dist[i] = length[v][i];    // dist[]
    }
    s[v] = true;
    dist[v] = 0;
    for (i = 0; i < n-1 ; i++) { // n-1 shortest paths from v
        choose u that is not in S and has smallest dist[u];
        s[u] = true;  // u becomes a member of S
        for (each <u, w> in the graph) // update dist[w]
             if (!s[w] && (dist[u] + length[u][w]) < dist[w])
                 dist[w] = dist[u] + length[u][w];
    }
 }
```

"choosing the smallest dist[u]" is typically in O(n).
So, the overall complexity is $O(n^2)$

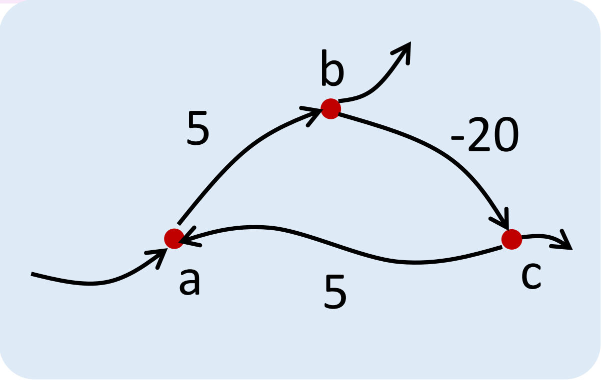
Single Source, All Destinations, and General Costs

- All edge costs (positive, negative, zero) are permitted
  - A more general (also more difficult) problem
  - Dijkstra's greedy strategy does not work here

- Offsetting all edge costs does not help
  - Paths consist of different number of edges
  - Different offset amounts change the length order of paths



Single Source, All Destinations, and General Costs



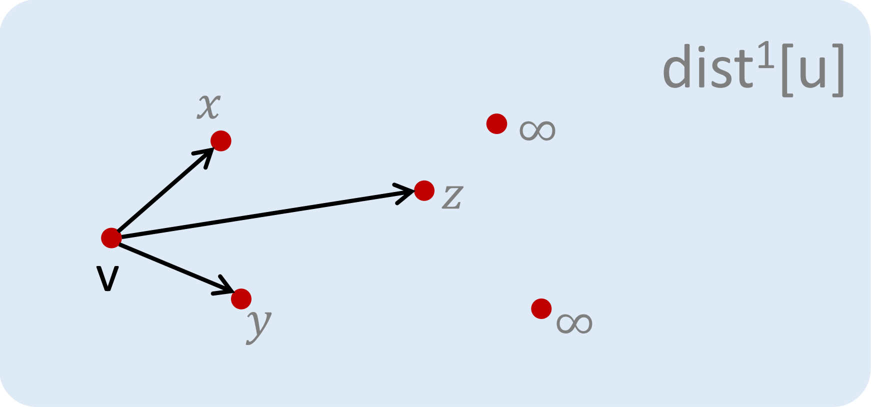
- Cycles with negative length are not permitted
  - Otherwise, a cycle produces a path with $-\infty$ cost
  - e.g., … a-b-c - a-b-c- …

- A shortest path must exist and has at most n-1 edges (i.e., n vertices)
  - Paths with more than n vertices must contain a cycle
    - Cycles do not lead to shorter paths
  - With at most n vertices, there are a finite number of possible paths
    - The shortest one must exist

Bellman-Ford Algorithm Concept

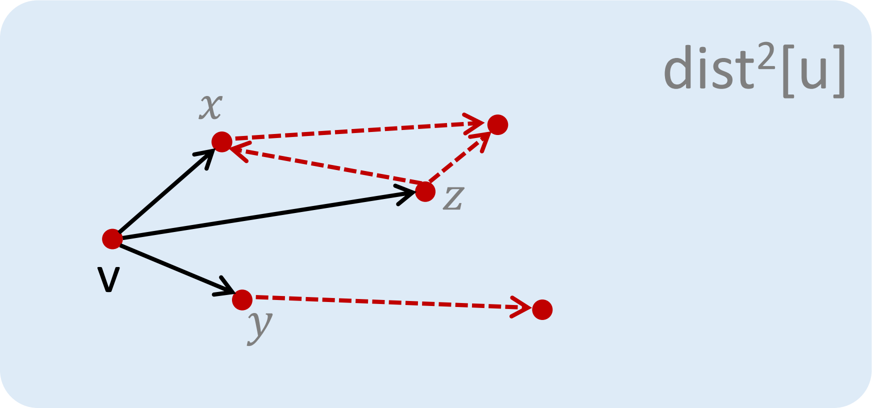
dist^l[u]	目標1	目標2	目標3	目標n-1
shortest path with #edge **=1**	這行可從edge cost直接得知			
shortest path with #edge ≦**2**				
shortest path with #edge ≦**3**				
…				
shortest path with #edge ≦**n-1**	這行是我們要的結果			
	#edge >n-1 的 path cost 並不會更低			

Bellman-Ford Algorithm



- **dist^l[u]**: Length of a shortest path from v to u with the number of edges ≦ $\boldsymbol{l}$
  - **dist**^{1}[u]
    - = edge weight if <v, i> exists
    - = ∞ if <v, i> doesn't exist
  - **dist^{n-1}[u] for all u** is our needed results

Bellman-Ford Algorithm

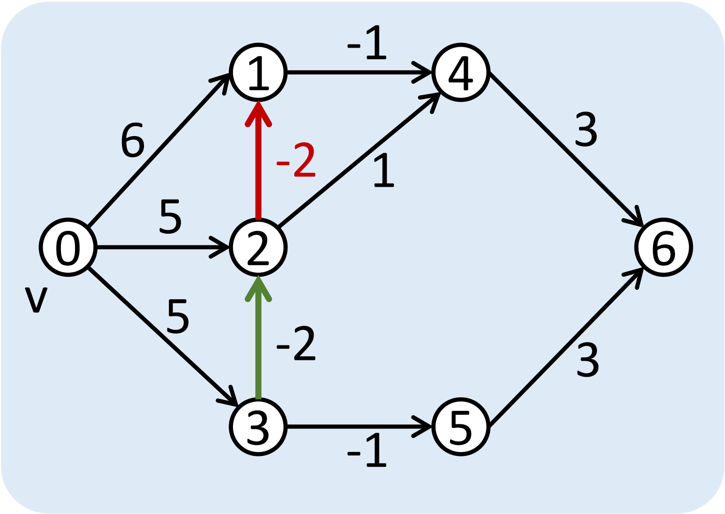


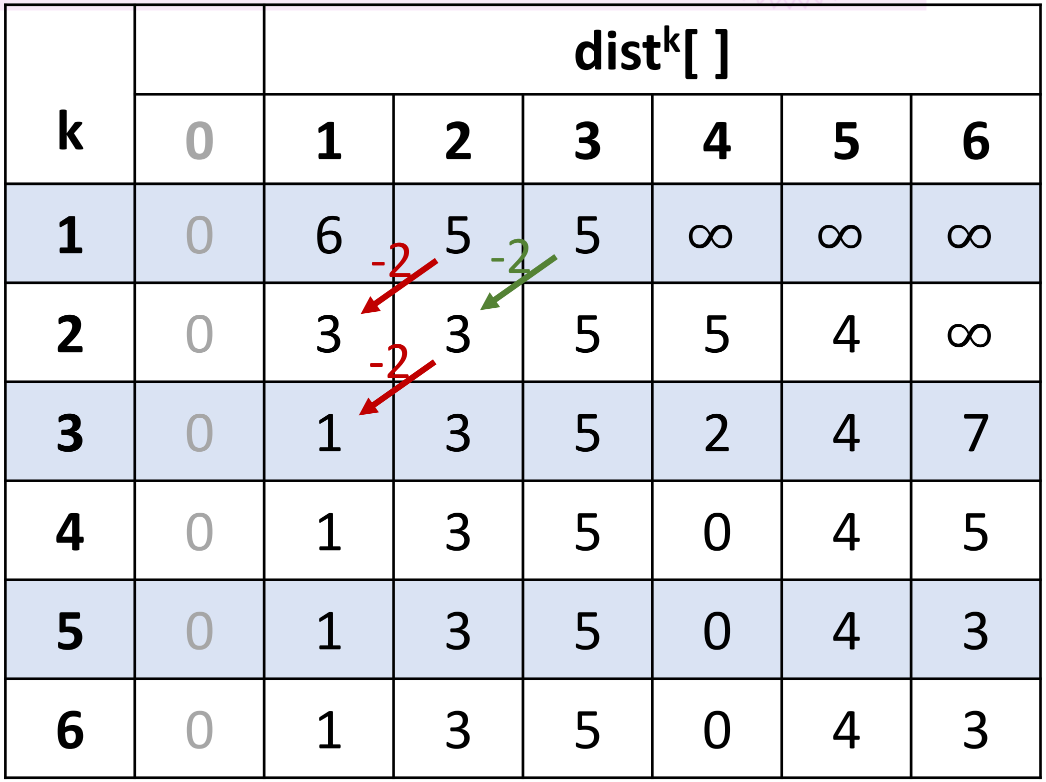
dist^2[x] = min of

- dist^1[x]
- dist^1[y] + cost(y, x)
- dist^1[z] + cost(z, x)
- dist^1[…] + cost(…, x)

- Calculate **dist**^{k}[u] from **dist**$^{k-1}$[u], k = 2~(n-1)
  - v-u shortest path with at most k edges, k>1, **dist**^{k}[u] is the minimum of
    - **dist**$^{k-1}$[u]
    - (**dist**$^{k-1}$[i] + length(<i, u>)) for all <i, u>

Bellman-Ford Example





k		dist^k[ ]					
	0	1	2	3	4	5	6
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

- Optimization
  - Updating **dist**[] in-place
    - Use only one array for **dist^1**[], **dist^2**[] …

Bellman-Ford Algorithm

```
void MatrixWDigraph::BellmanFord(const int v)
{ // n is the number of vertices
  // in-place update for dist[] is used
  for (int i = 0; i < n ; i++)
      dist[i] = length[v][i]; // dist¹[] initialization

  for (int k = 2; i <= n−1 ; k++) // dist² ~ dist⁽ⁿ⁻¹⁾
    for (each u, u != v)
      for (each <i, u> in the graph)
        if (dist[u] > dist[i] + length[i][u])
            dist[u] = dist[i] + length[i][u];
}
```

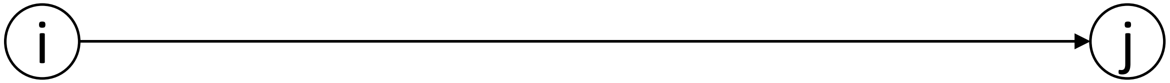
- "for(each u)" and "for (each <i, u>)" together is O(n^2) for an adjacency matrix and is O(e) for an adjacency list.
- The overall complexity is O(n^3) for an adjacency matrix and is O(ne) for an adjacency list.

All Pairs and General Costs

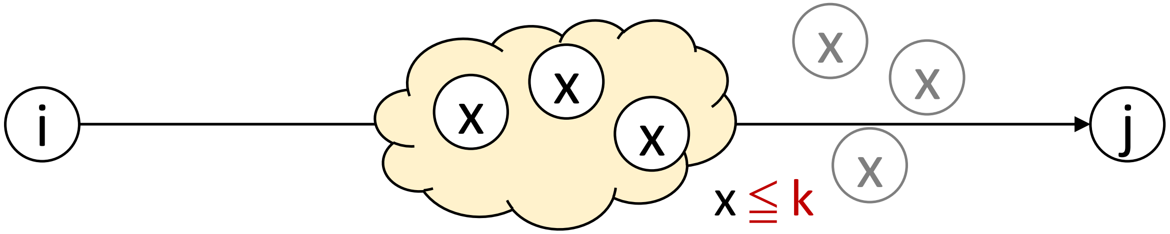
- Viable approaches
  - Perform n Bellman-Ford algorithms
    - $O(n^4)$ if an adjacency matrix is used
    - $O(n^2e)$ if an adjacency list is used
- There is an $O(n^3)$ all pair shortest path algorithm
  - Suitable for a dense graph with e being several folds of n

All-Pair Shortest Path Algorithm Concept

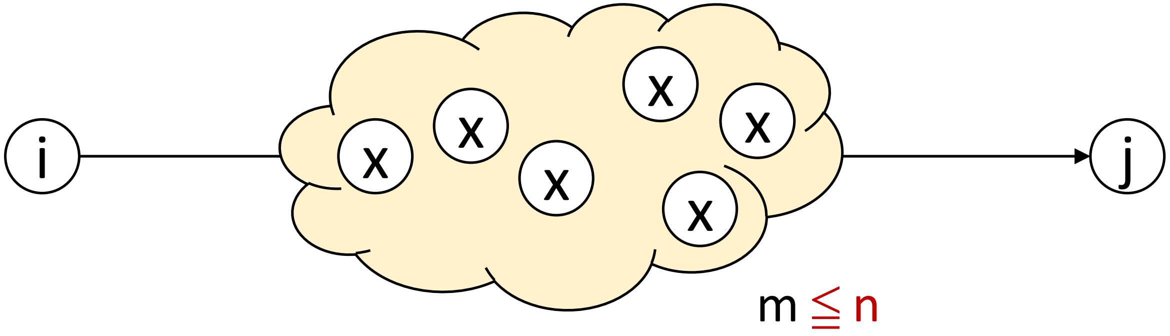
- For each (i, j) pair
  - Shortest path without an intermediate vertex



  - Shortest path with some restricted intermediate vertices

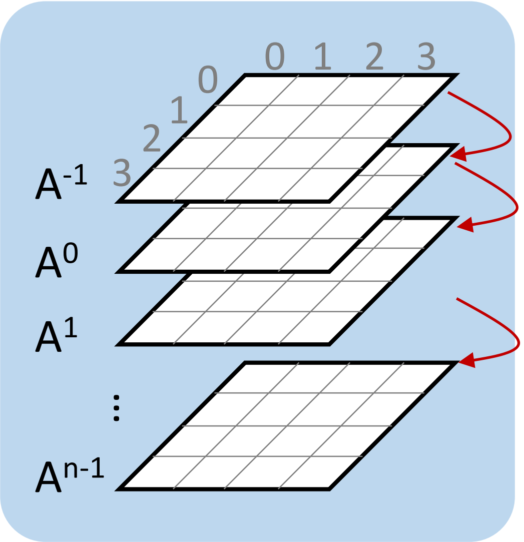


  - Shortest path with any intermediate vertices



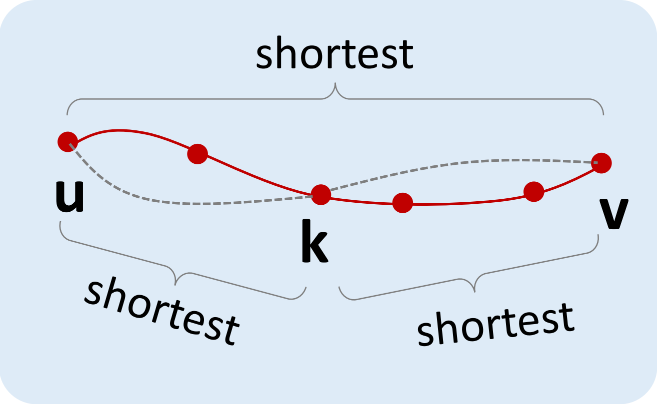
All-Pair Shortest Path Algorithm

- Define $A^k[i][j]$
  - length of the shortest path from i to j going through no intermediate vertex of index greater than k
  - $A^{-1}[i][j]$ is just the length of the edge <i, j>
  - $A^{n-1}[i][j]$ is our needed results



- Calculate $A^k[i][j]$ based on $A^{k-1}[i][j]$
  - $A^k[i][j]$ is the minimum of the following
    - $A^{k-1}[i][j]$
    - $A^{k-1}[i][k] + A^{k-1}[k][j]$

All-Pair Shortest Path Algorithm



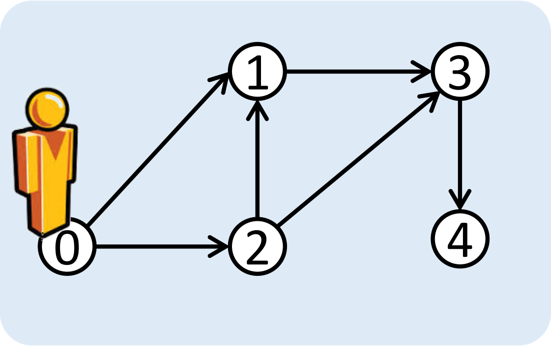
- Given
  - ShortestPathCost(u, k)
  - ShortestPathCost(k, v)
- If k is on a shortest path from u to v
  → ShortestPathCost(u, v)
  = ShortestPathCost(u, k) + ShortestPathCost(k, v)
  - Proof: If there were another path, (u, k, v)', with an even lower cost
    - Either Cost((u, k)') is < ShortestPath(u, k)
      or Cost((k, v)') is < ShortestPath(k, v),
      a contradiction

All-Pair Shortest Path Algorithm

```
void MatrixWDigraph::AllLengths(const int n)
{
  for (int i = 0; i<n; i++)
    for (int j = 0; j<n; j++)
      a[i][j]= length[i][j];

  for (int k= 0; k<n; k++)
    for (int i= 0; i<n; i++)
      for (int j= 0; j<n; j++)
        if(a[i][j] > (a[i][k] + a[k][j]))
           a[i][j] = a[i][k] + a[k][j];
}
```

Concept of Transitive Closure

- Transition matrix, **T**
  - State change after a step



$$\mathbf{T} = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 1 & 1 & & \\ 1 & & 1 & & 1 & \\ 2 & & 1 & 1 & 1 & \\ 3 & & & & 1 & 1 \\ 4 & & & & & 1 \\ \hline \end{array}$$

- Closure, **C**
  - The steady state after many steps
    - (**C** × **T**) = **C**
      - (Here we use AND for scalar multiplication and OR for scalar addition in the above example)

$$\mathbf{C} = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & 1 & & 1 & 1 \\ 2 & & 1 & 1 & 1 & 1 \\ 3 & & & & 1 & 1 \\ 4 & & & & & 1 \\ \hline \end{array}$$

Transitive Closure of a Graph

- Given a graph with unweighted edges
  - Transitive closure matrix, A^+
    - $A^+[i][j] = 1$ if there is a path of positive length from i to j
    - $A^+[i][j] = 0$ otherwise
  - Reflexive transitive closure matrix, A^*
    - $A^*[i][j] = 1$ if there is a path of non-negative length from i to j
    - $A^*[i][j] = 0$ otherwise
  - The only difference between two (given unweighted edges)
    - $A^+[i][i] = 0$
    - $A^*[i][i] = 1$, as the name "reflexive" suggests

- Meanings of '+' and '*'
  - '+' means "one or more" in regular expression
  - '*' means "zero or more" in regular expression

Transitive Closure Algorithm

- For a directed graph (with unweighted edges)
  - Perform all-pair shortest path algorithms
    - O(n^3) time complexity
  - Perform n independent Dijkstra algorithms
    - O(n^2e) time complexity

- For an undirected graph (with unweighted edges)
  - Perform connect component algorithm using searches (e.g., DFS or BFS)
    - O(n^2) time complexity

Outline

- 6.1 The graph abstract data type
- 6.2 Elementary graph operations
- 6.3 Minimum-cost spanning trees
- 6.4 Shortest paths and transitive closure
- **6.5 Activity networks**

Outline

- 6.1 The graph abstract data type
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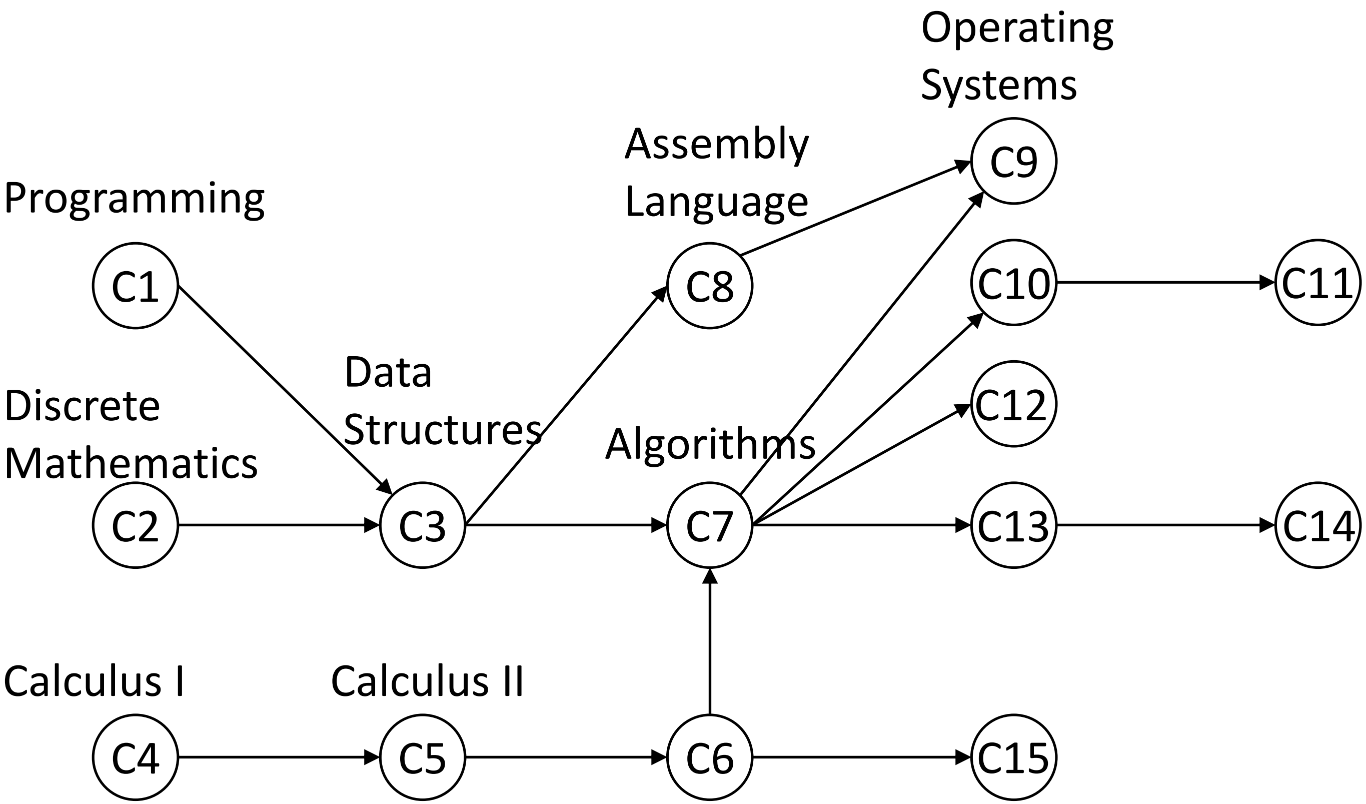
- **6.5 Activity networks**

Activity-on-Vertex (AoV) Networks

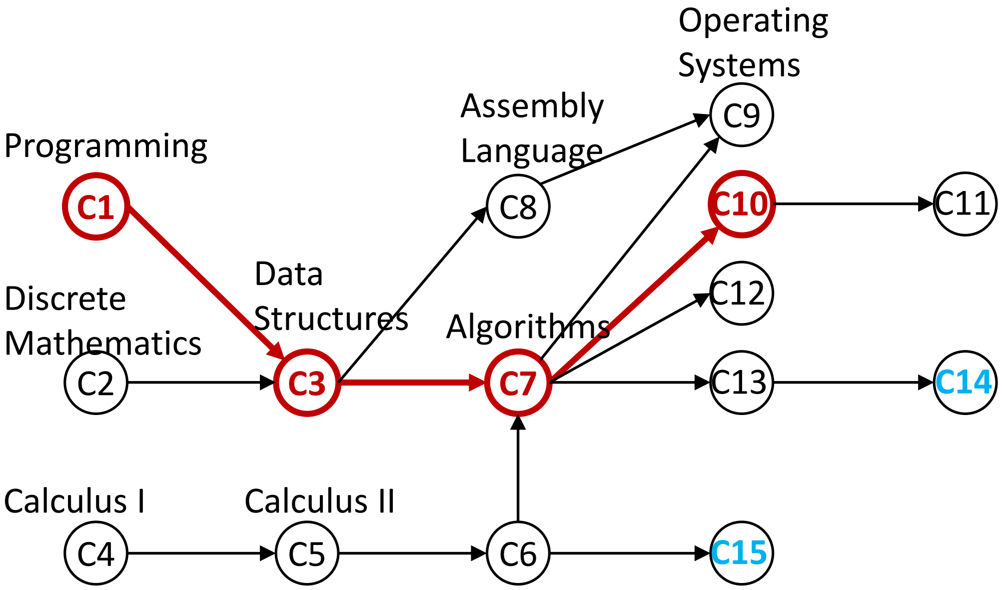
- Directed graphs
  - Vertices represent tasks (i.e., activities)
  - Edges represent precedence relations
  - Vertex i is a predecessor (successor) of vertex j *iff* there's a path from i to j (j to i)
  - Vertex i is an immediate predecessor (successor) of vertex j *iff* there's an edge from i to j (j to i)

Activity-on-Vertex (AoV) Networks



Topological Order

- A linear order of the vertices of a graph such that
  - for any two vertices i and j, if i is a predecessor of j in the graph, then i precedes j in the linear ordering



Note:

- Transitivity among >2 vertices
- Topology order between two vertices does not always imply their precedence in the graph

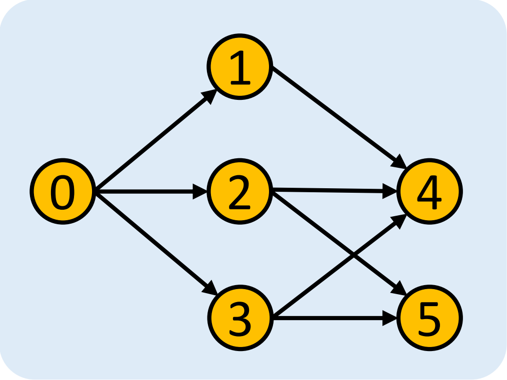
- Two valid topological orderings (there are many of them)
  - **C1**, C2, C4, C5, **C3**, C6, C8, **C7**, **C10**, C13, C12, **C14**, **C15**, C11, C9
  - C4, C5, C2, **C1**, C6, **C3**, C8, **C15**, **C7**, C9, **C10**, C11, C12, C13, **C14**

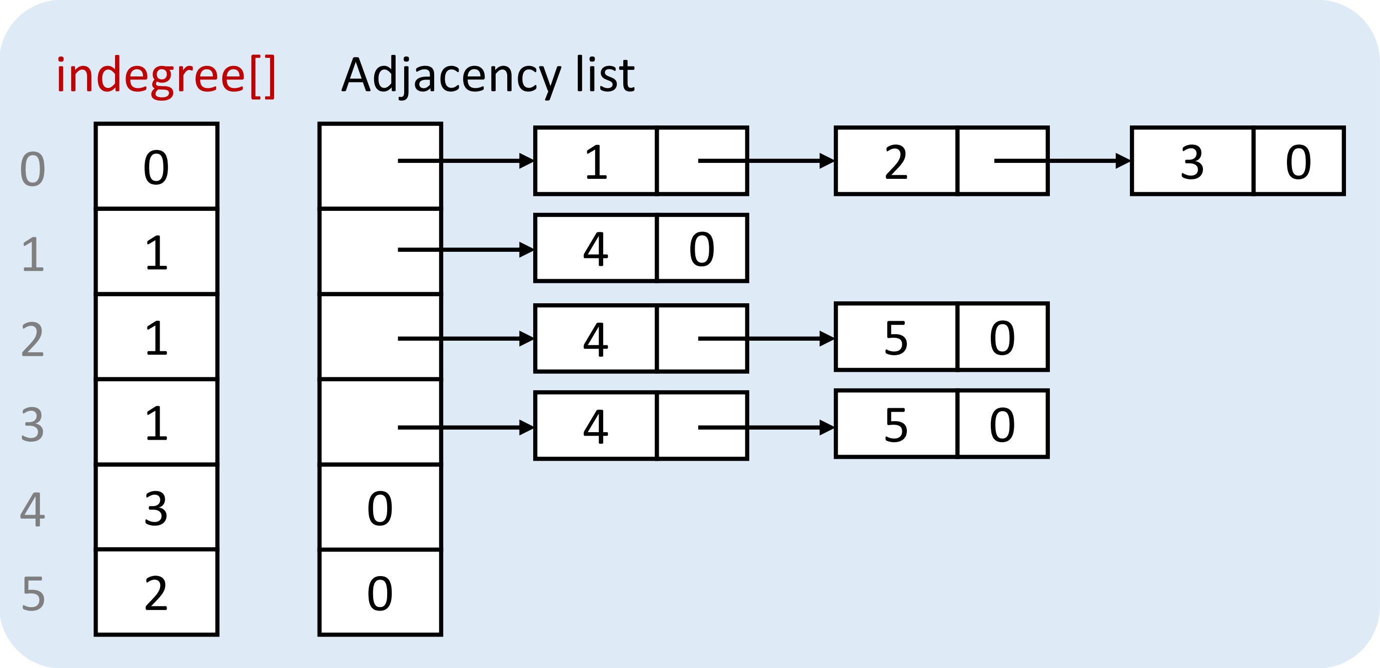
Topological Sorting Algorithm (Draft)

```
for (int i = 0; i<n; i++) {
  if (every vertex has a predecessor){
    // network has a cycle and thus is infeasible
    return;
  }
  if (vertex v has no predecessors) {
    cout << v;
    remove v and all edges leading out of v;
  }
}
```

- Graph representation considerations for the above algorithm
  - How can we remove all edges leading out of a vertex?
  - How can we determine whether a vertex has a predecessor?

Graph Representation Choice





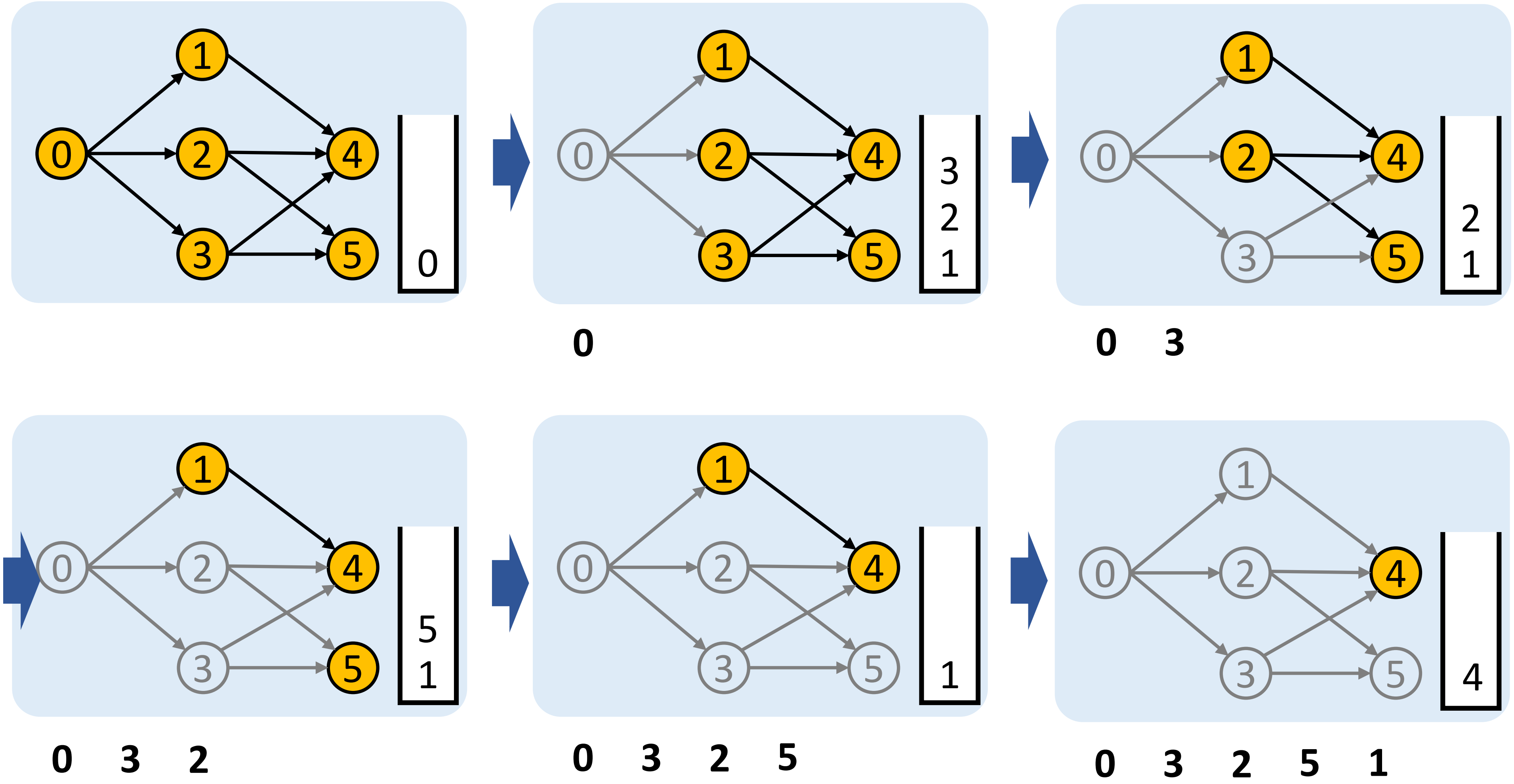
	indegree[]
0	0
1	1
2	1
3	1
4	3
5	2

Topological Sorting Algorithm

```
void LinkDigraph::TopologicalOrder()
{
    Stack s; // A stack holds 0-indegree vertices
             // Any container is good for this algorithm
    for (int i = 0; i<n; i++)
        if (indegree[i] == 0) s.push(i);

    for (i = 0; i< n; i++) {
        if (s.isEmpty() ) throw “Network has a cycle.”;
        int j = s.top(); s.pop();
        cout << j <<endl;
        Chain<int>::ChainIterator ji = adjLists[j].begin();
        while (ji) {
            indegree[*ji]--;
            if (indegree[*ji] == 0) s.push(*ji);
            ji++;
        }
    }
}
```

Topological Sorting Example



0

0 3

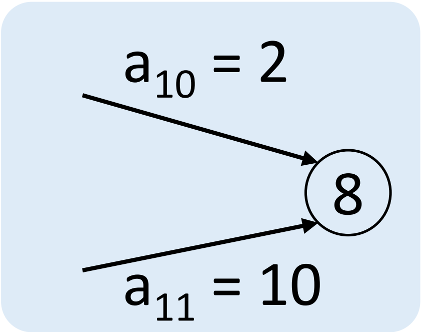
0 3 2

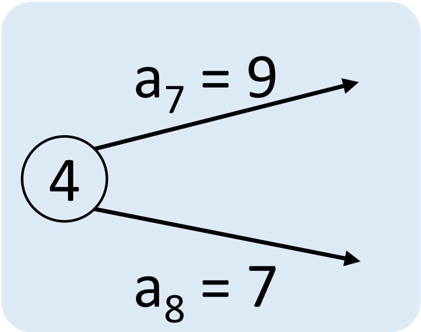
0 3 2 5

0 3 2 5 1

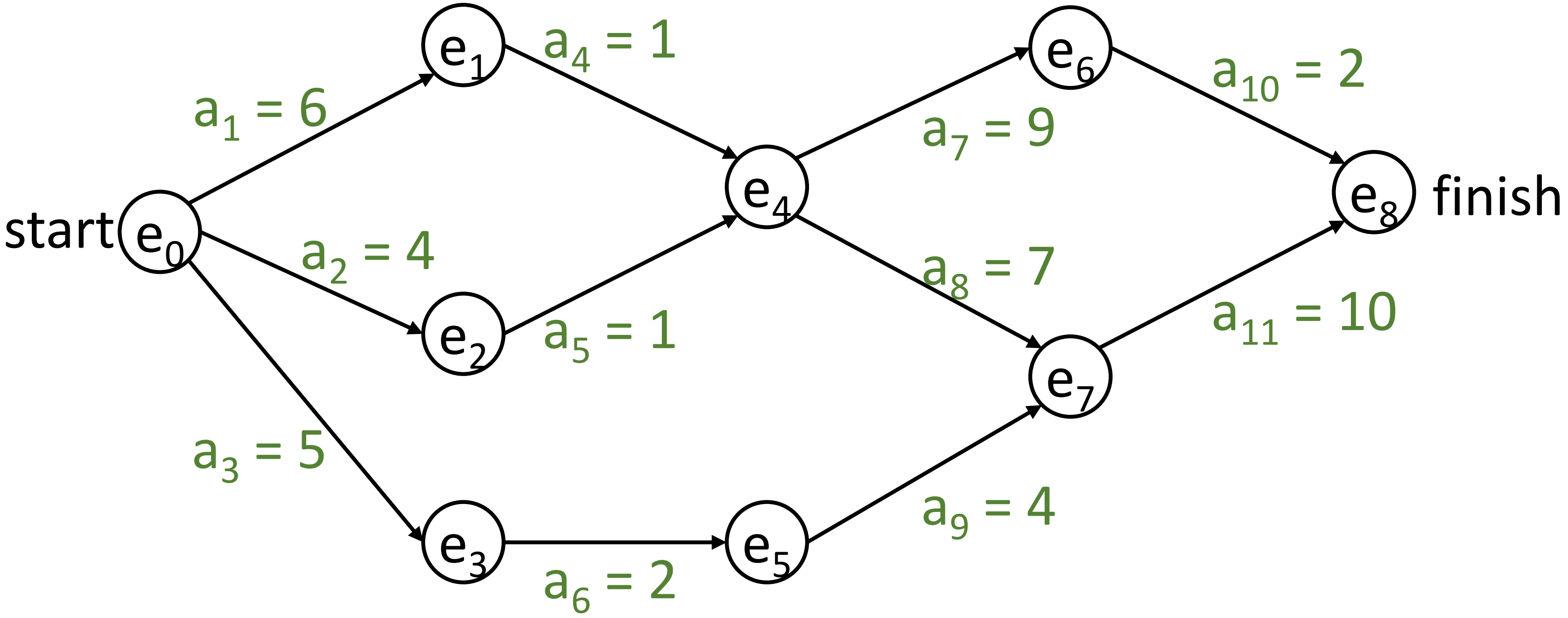
Activity-on-Edge (AoE) Networks

- Directed graph
  - Edges represent tasks (activities) to be performed
  - Vertices represent events
  - Edge cost of each activity is the time needed to perform the activity
  - Event vertex signals the completion of all activities edges entering the vertex
  - Edges leaving a vertex cannot be started until the event at the vertex has





Activity-on-Edge Network



- Events
  - e_0: start of the project
  - e_1: completion of activity a_1
  - e_4: completion of activities a_4 and a_5
  - …
  - e_8: finish of the project

Some Important Concepts

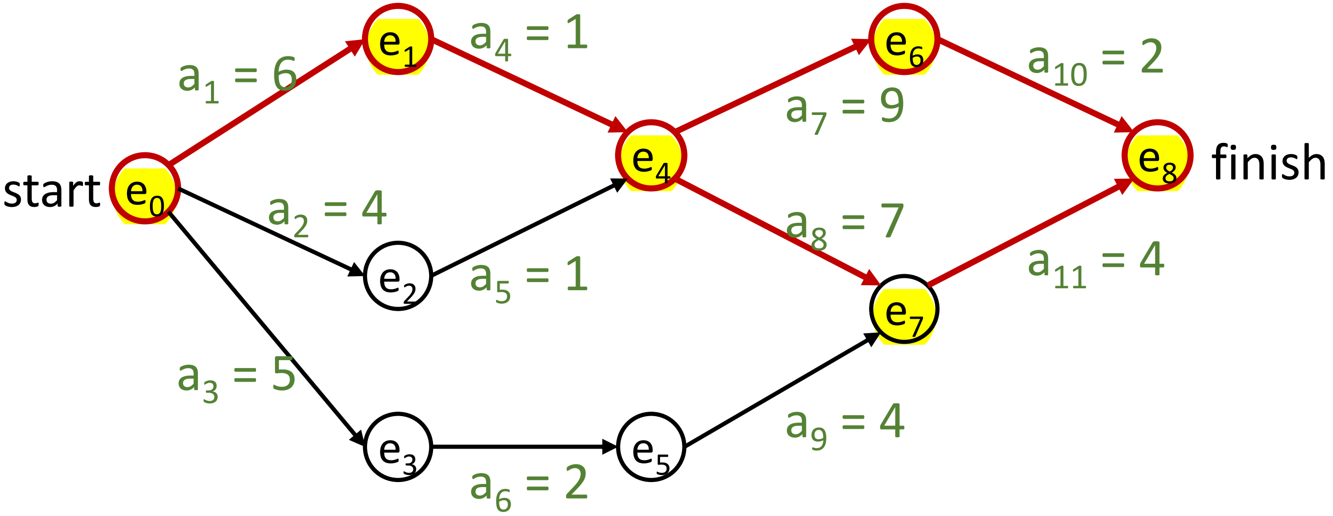
- Critical path
  - The longest path from the start vertex to the finish vertex
- Earliest time
  - The earliest time an activity (event) can start (occur)
- Latest time
  - The latest time an activity (event) must start (occur) so as not to delay the project
- Critical activities
  - All activities for which the earliest time equals the latest time

在critical path上，node的lastest time = earliest time

Critical Path

VoE

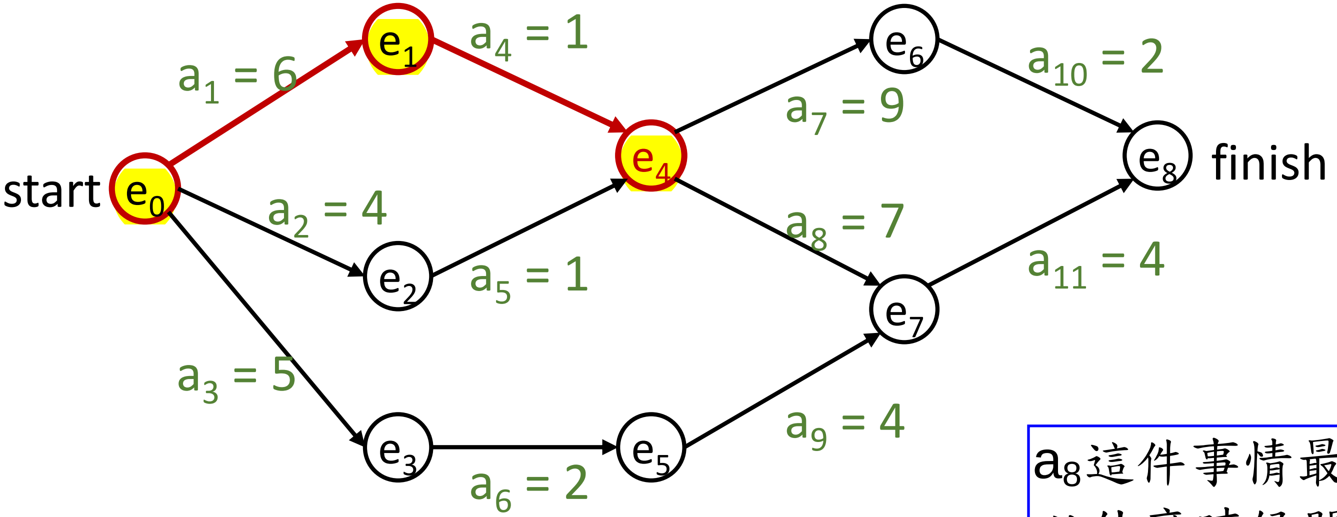
- The longest path from the start vertex to the finish vertex



- The above network has two critical paths
- Length(0, 1, 4, 6, 8) = 18
- Length(0, 1, 4, 7, 8) = 18

Earliest Event/Activity Time

- The length of the longest path from the start vertex to a vertex

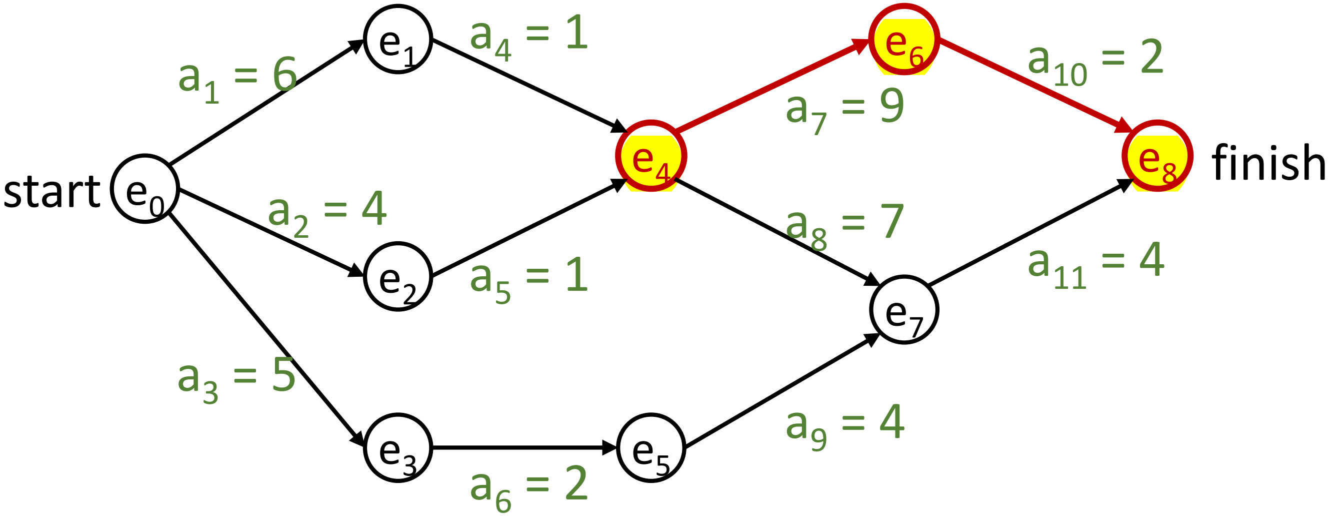


a_8這件事情最早可以什麼時候開始做

- Earliest event time(e_4) = 7 = 6 + 1
- Earliest activity time(a_7) = Earliest activity time(a_8) = 7
- Earliest event time(finish) = 18 = 6 + 1 + 11(9+2 or 7+4)

Latest Event/Activity Time

- Earliest time of the finish vertex - the length of the longest path a vertex to the finish



- Earliest event time(finish) = 18
- longest path length(e_4, finish) = 11
- latest event time(e_4) = 7

Critical Activities

- The difference between the earliest time and the latest time, i.e., the slack (寬裕), is a measure of the criticality of an activity
  - The time by which an activity may be delayed or slowed without delaying the finish of the project
- Activities having no slack are called critical activities

Critical Path Analysis

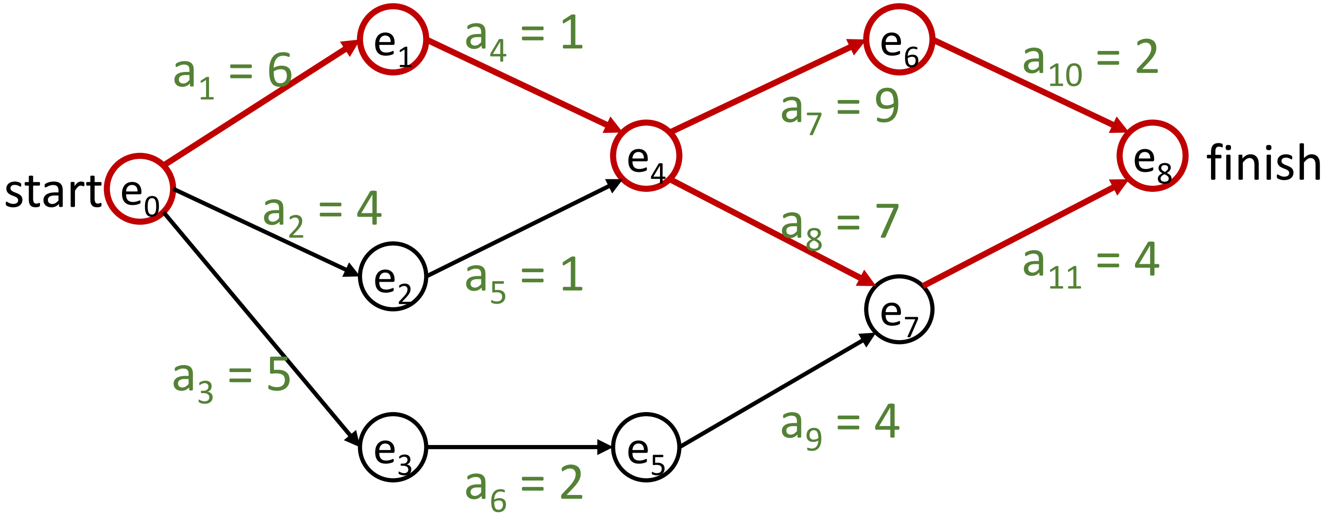
- Purpose
  - Speed up things, e.g., a project or a circuit

- Steps
  - Compute earliest time and latest time
  - Identify critical activities
  - Find paths in the graph with noncritical activities removed

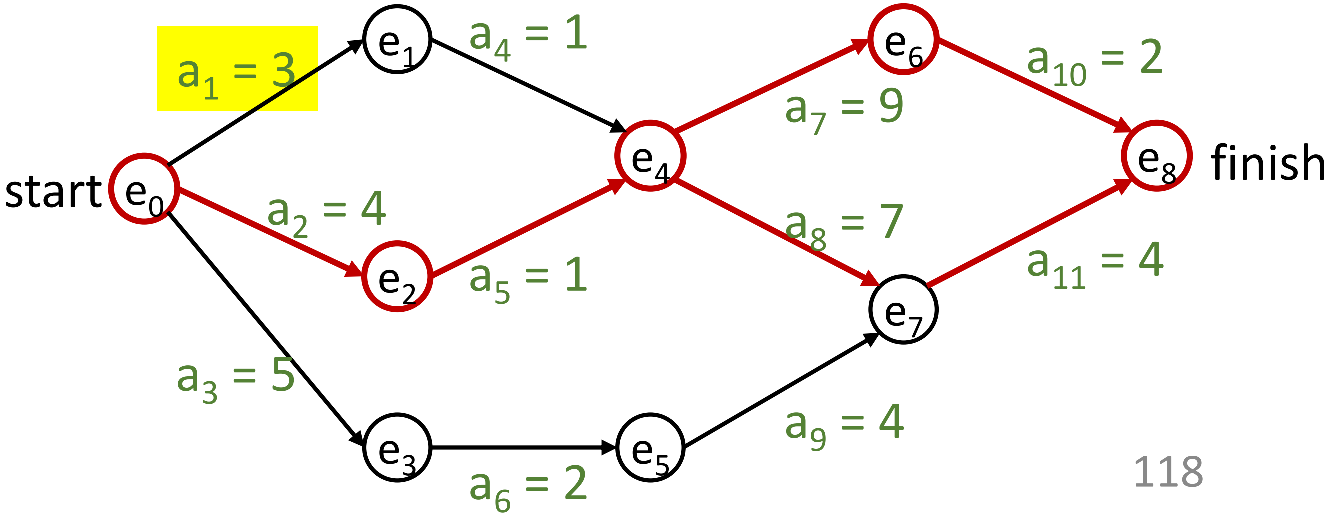
- Notes
  - Speeding up noncritical activities or single critical activity not on all critical paths will not reduce the overall duration
  - Critical paths can change after speeding up an activity

Critical Path Analysis

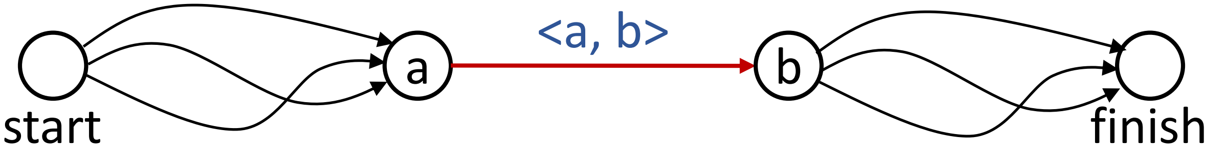
- a7, a8, a9, a10 are critical. However, speeding up one of them cannot reduce overall duration



- After a_1 is speeded up from 6 to 3 units, critical paths change



Calculating Earliest and Latest Times



- Earliest activity time(<a, b>)
  - = Earliest event time(a)
  - = Longest path(start, a)
- Latest activity time(<a, b>)
  - = Latest event time(b) - Edge cost(<a, b>)
- Latest event time(b)
  - = Earliest event time(finish) - Longest path(b, finish)

- The above calculation can be performed in two passes based on topological sorting
  - Detailed in the textbook

Critical Path Analysis in Circuit Design

- (Supplement materials)
- CAD (computer-aided design) algorithms
  - Identify critical paths in circuits
  - Push the limits for the paths to meet timing constraints
- The following circuit example is an adder
  - Add three bits and produce two resulting bits
  - Red parts are typical critical paths