訊號與系統
Signals and Systems
EECS2020
Final Exam
16:30 – 18:45, 06/24/2021

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National Tsing Hua University
Spring 2021

- Remember to write down your name, student ID number, date, and page number (page number/total page number) at the **top-right corner** of **each page** of your solution paper,

  e.g., Calvin Li, 1080610111, 06/24/2021, 2/7

- **10 problems** in total (100 points)

- Total pages of this exam paper: **19** (including 7 pages of the FT, LT, and z-T pair tables)

- If any definition is not clear, please feel free to make your own assumptions or ask the TAs and instructor. If you make your own assumptions, you will need to clarify what your assumptions are, nonetheless.

- Tables of Fourier transform, Laplace transform, and z-transform pairs are provided at the end of the exam paper.

Problem 1 (12%)

Consider a continuous-time LTI system whose impulse response $h(t)$ is real and even and its frequency response is band limited. Tell each of the following statements is **true** or **false**.

(a) The system is causal.

(b) The system is invertible.

(c) The frequency response is even.

(d) The frequency response is real.

(e) The continuous-time Fourier transform of $\frac{dh(t)}{dt}$ is imaginary.

(f) The continuous-time Fourier transform of $t \cdot h(t)$ is real.

Problem 2 (8%)

Let $x(t)$ be a real-value continuous-time signal and its maximum frequency (in Fourier transform) is 100 Hz. Determine the minimum sampling rate for each of the following signals, which results in no aliasing in frequency domain. (You do not need to show the calculation procedure. Just write down your answer)

(1) $x(t) + x(t\text{-}1)$

(2) $\frac{dx(t)}{dt}$

(3) $x^2(t)$

(4) $x(t)\cos(400\pi t)$

Problem 3 (5%)

Find the Fourier transform of $x(t) = \frac{1}{1+jt}$. (*Hint*: Duality property.)

Remember to write down your derivation.

Problem 4 (10%)

Derive the transfer functions of the following impulse responses. Remember to write down your derivation and include the region of convergence (ROC) in your solution.

(1) $h(t) = -te^{-0.5t}u(-t)$ (Given transform pairs 6&7 of Table 9.2 in page 17) (5%)

(2) $h[n] = -(n+1)0.1^n u[-n-1]$ (Given transform pairs 5&6 of Table 10.2 in page 19) (5%)

Problem 5 (10%)

A CT LTI system is described by the following transfer function.

$$H(s) = \frac{1}{s^2 + 7s + 6}$$

(1) Plot the pole-zero plot. (1%)

(2) Discuss the causality and stability of different ROCs. (3%)

(3) Find the impulse responses of different ROCs. (6%)

Problem 6 (10%)

A DT LTI system is described by the following transfection.

$$H(z)=\frac{2-0.75z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

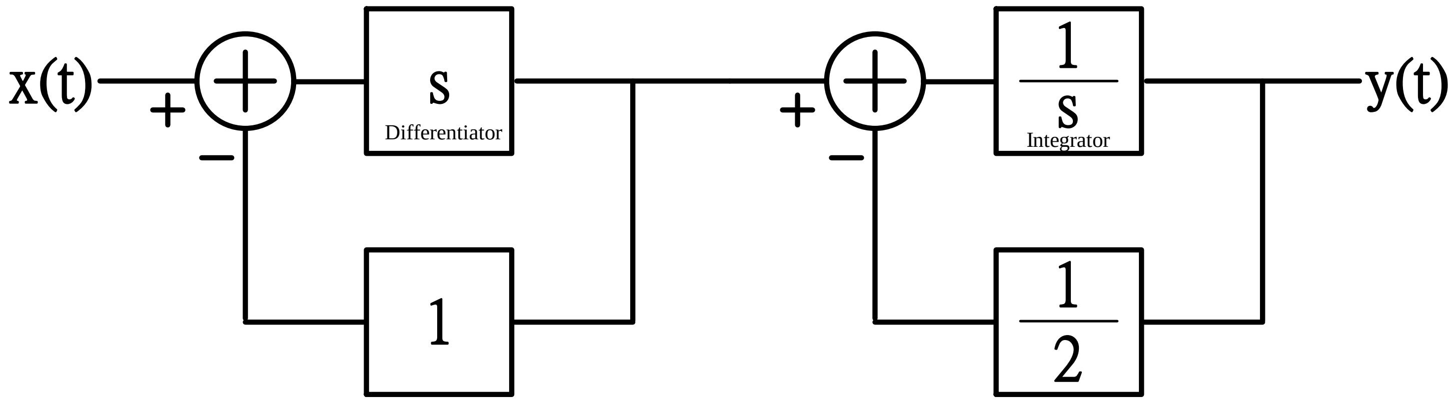
(1) Plot the pole-zero plot. (1%)

(2) Discuss the causality and stability of different ROCs. (3%)

(3) Find the impulse response of different ROCs. (6%)

Problem 7 (15%)

A CT "**causal**" LTI system is represented by the following block diagram.



(1) Find the transfer function of the system H(s). Note that the transfer function should include the region of convergence (ROC). (4%)

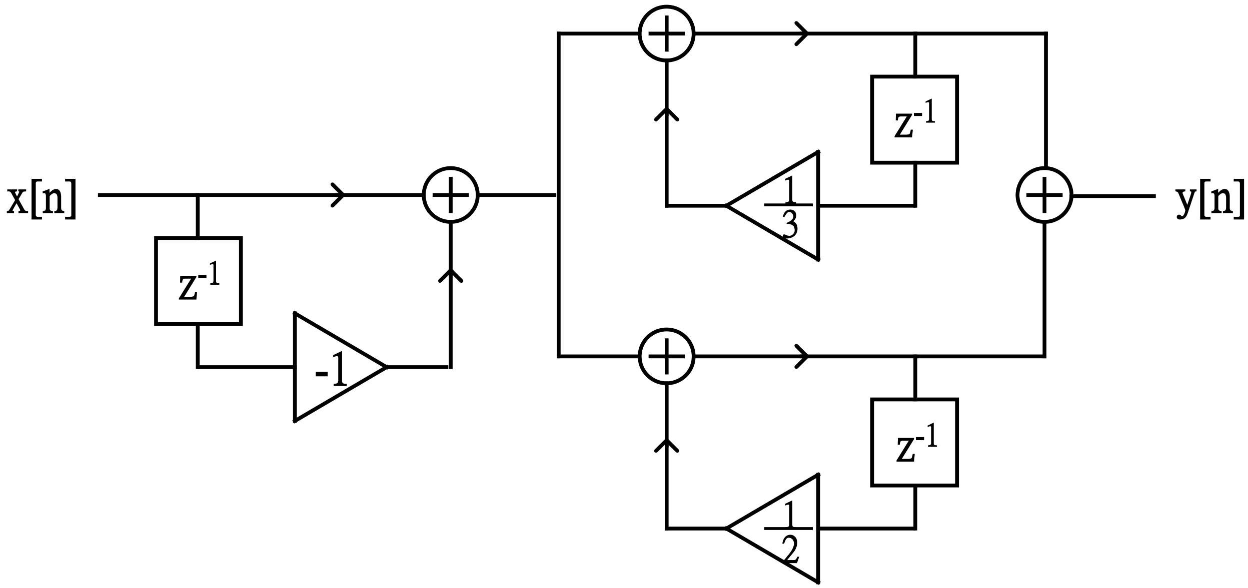
(2) Give a linear constant coefficient differential equation describing this system. (3%)

(3) Is the system stable? (3%)

(4) Assume that a computer-aided design (CAD) package will be used to design a circuit implementing this LTI system, and this CAD package only has built-in integrator modules and has no differentiator modules built in, re-draw a new block diagram which allows you to use this CAD package to design the circuit. (5%)

Problem 8 (15%)

A DT "**causal**" LTI system is represented by the following block diagram.



(1) Find the transfer function of the system H(z). Note that the transfer function should include the region of convergence (ROC). (4%)

(2) Give a linear constant coefficient difference equation describing this system. (3%)

(3) Is the system stable? (3%)

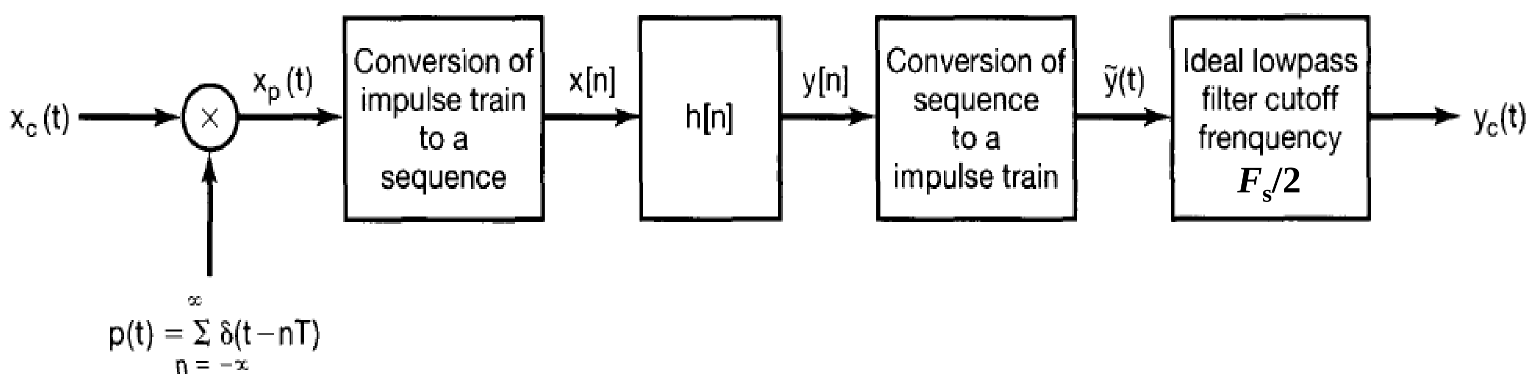
(4) Re-draw the block diagram so that at most two unit delayers are required to implement this DT system. (5%)

Problem 9 (10%)

The figure shown below is a system that processes continuous-time signals using a discrete time (DT) linear time-invariant (LTI) filter with impulse response *h*[*n*]. For input signals which are band limited within Nyquist rate (i.e., after sampling, there will be no aliasing in frequency domain), the overall system in the figure is equivalent to a continuous-time (CT) LTI system. Note that T is the sampling interval, F_s = 1/T = 1000 Hz is the sampling rate, and *h*[*n*] is real. If this effective CT LTI system we would like to design is a stable, causal and non-ideal bandpass filter and has its maximum magnitude response at 125 Hz,

(1) At what (normalized) frequency should the DT LTI system *h*[*n*] have the maximum magnitude response? Justify your answer. (5%)

(2) Provide the possible pole-zero placement (i.e., pole-zero plot of the transfer function of the system *h*[*n*]) which you will use to design this DT LTI system *h*[*n*] with the assumption that the transfer function of the system *h*[*n*] only has one zero at 0. Justify your answer. (5%)



Problem 10 (5%)

There is an abnormal person whose heart rate (i.e., the heart beat frequency) is constant. His every heart beat is exactly the same. That is, if we listen to the beating of his heart, we will hear the sound $x(t)$ in time domain as shown in the following figure. The heart rate is defined as the inverse of the peak-to-peak time duration in the heartbeat sound signal. For example, $1/(t_2\text{-}t_1)$ is defined as the heart rate at t_2. In this case, can Fourier transform of $x(t)$ be used to estimate the heart rate?

No matter your answer is yes or no, tell the reason why you give such an answer.

Remember to justify your answer.

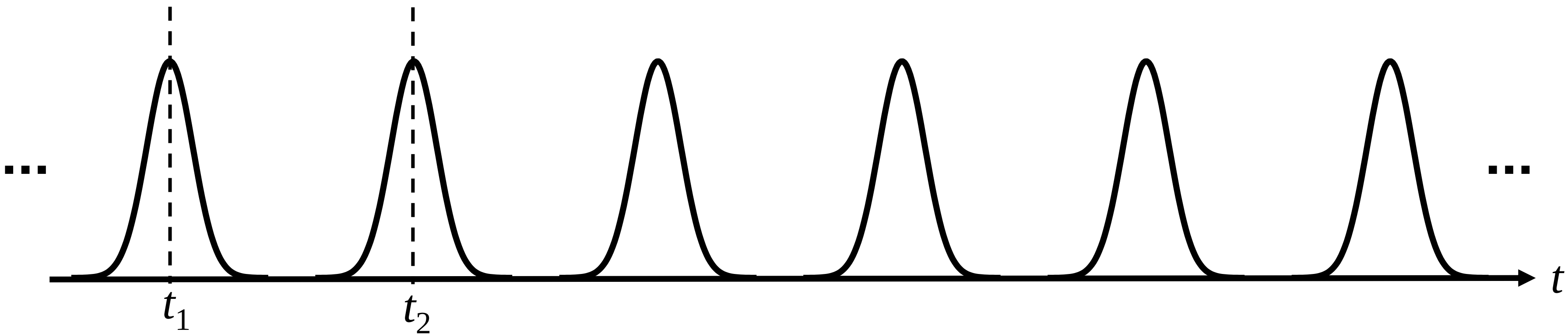


TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$

$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$

Periodic square wave $x[n] = \begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & N_1 < \lvert n \rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & \lvert n \rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert \omega \rvert \le W \\ 0, & W < \lvert \omega \rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$

9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1 - [\cos\omega_0]z^{-1}}{1 - [2\cos\omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1 - [2\cos\omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos\omega_0 n]u[n]$	$\frac{1 - [r\cos\omega_0]z^{-1}}{1 - [2r\cos\omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1 - [2r\cos\omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$