Problem 1

$$\boxed{rect(t) \xleftrightarrow{CTFT} sinc(f)}$$

$$\int_{-\infty}^{+\infty} rect(t) e^{-j2\pi ft} dt$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} (1)\, e^{-j2\pi ft}\, dt$$

$$= \frac{1}{-j2\pi f}\left\{ e^{-j2\pi f(+\frac{1}{2})} - e^{-j2\pi f(-\frac{1}{2})} \right\}$$

$$= \frac{1}{-j2\pi f}\left\{ \left[\cos(-\pi f) + j\sin(-\pi f)\right] - \left[\cos(\pi f) + j\sin(\pi f)\right] \right\}$$

$$= \frac{1}{-j2\pi f}\left\{ -j2\sin(\pi f) \right\}$$

$$= \frac{\sin(\pi f)}{\pi f}$$

$$= sinc(f) \quad \blacksquare$$

$$\boxed{\Lambda(t) \xleftrightarrow{CTFT} \mathrm{sinc}^2(f)}$$

Problem 1 (continued)

$$\mathrm{rect}(t) * \mathrm{rect}(t) = \int_{-\infty}^{+\infty} \mathrm{rect}(\tau)\,\mathrm{rect}(t-\tau)\,d\tau$$

$$= \int_{+\frac{1}{2}}^{+\infty} 0 \cdot \mathrm{rect}(t-\tau)\,d\tau$$

$$+ \int_{-\frac{1}{2}}^{+\frac{1}{2}} 1 \cdot \mathrm{rect}(t-\tau)\,d\tau$$

$$+ \int_{-\infty}^{-\frac{1}{2}} 0 \quad \mathrm{rect}(t-\tau)\,d\tau$$

$$= \int_{t+\frac{1}{2}}^{t-\frac{1}{2}} \mathrm{rect}(u)\,(-du)$$

$$= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \mathrm{rect}(u)\,du$$

$$= \Lambda(t)$$

apply the convolution property. ▨

Problem 1 (continued)

$$\int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j2\pi f(t-t_0)}\, e^{-j2\pi f t_0}\, d(t-t_0)$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{+\infty} x(u)\, e^{-j2\pi f u}\, du$$

$$= e^{-j2\pi f t_0}\, X(f)$$

$$\boxed{x(t-t_0) \xleftrightarrow{CTFT} e^{-j2\pi f t_0} X(f)}$$

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$$\int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi f t}\, dt$$

$$\boxed{x(at) \xleftrightarrow{CTFT} \frac{1}{|a|} X\left(\frac{f}{a}\right)}$$

$$= \int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi \frac{f}{a}(at)}\, \frac{1}{a}\, d(at) \qquad (a > 0)$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(u)\, e^{-j2\pi \frac{f}{a} u}\, du$$

$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

$$\int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi f t}\, dt$$

$$= \int_{+\infty}^{-\infty} x(at)\, e^{-j2\pi \frac{f}{a}(at)}\, \frac{1}{a}\, d(at) \qquad (a < 0)$$

$$= \int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi \frac{f}{a}(at)}\, \frac{-1}{a}\, d(at)$$

$$= \frac{-1}{a} \int_{-\infty}^{+\infty} x(u)\, e^{-j2\pi \frac{f}{a} u}\, du$$

$$= \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

■

Problem 1 (continued)

$$\int_{-\infty}^{+\infty} e^{+j2\pi f_0 t} x(t) e^{-j2\pi f t} dt \qquad \boxed{e^{+j2\pi f_0 t} x(t) \xleftrightarrow{CTFT} X(f-f_0)}$$

$$= \int_{-\infty}^{+\infty} x(t) e^{-j2\pi (f-f_0) t} dt$$

$$= X(f-f_0) \qquad \blacksquare$$

Problem 2

$$\begin{cases} 1 & , |n| \leq M \\ 0 & , \text{else} \end{cases} \xleftrightarrow{\text{DTFT}} \frac{\sin((2M+1)\pi f)}{\sin(\pi f)}$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f n}$$

$$= \sum_{n=-M}^{+M} (1)\, e^{-j2\pi f n}$$

$$= \sum_{n=-M}^{+M} \left(e^{j(-\pi f 2)}\right)^n$$

$$= \left(e^{j(-\pi f 2)}\right)^{-M} \frac{1 - \left(e^{j(-\pi f 2)}\right)^{2M+1}}{1 - \left(e^{j(-\pi f 2)}\right)}$$

$$= \left(e^{j(-\pi f 2)}\right)^{-M} \frac{\left(e^{j(-\pi f)}\right)^{2M+1} \left[\left(e^{j(+\pi f)}\right)^{2M+1} - \left(e^{j(-\pi f)}\right)^{2M+1}\right]}{\left(e^{j(-\pi f)}\right) \left[\left(e^{j(+\pi f)}\right) - \left(e^{j(-\pi f)}\right)\right]}$$

$$= \frac{-2j \sin((2M+1)\pi f)}{-2j \sin(\pi f)}$$

$$= \frac{\sin((2M+1)\pi f)}{\sin(\pi f)}$$

■

Problem 2 (continued)

$$x[n] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} X(f) e^{+j2\pi f n} df$$

$$\boxed{\frac{\sin(K\pi n)}{\pi n} \xleftrightarrow{\text{DTFT}} \begin{cases} 1, & |f| \le \frac{K}{2} \\ 0, & \text{else} \end{cases}}$$

$$= \int_{-\frac{K}{2}}^{+\frac{K}{2}} (1)\, e^{+j2\pi f n} df$$

$$= \frac{1}{j2\pi n}\left[e^{+j2\pi(+\frac{K}{2})n} - e^{+j2\pi(-\frac{K}{2})n}\right]$$

$$= \frac{1}{j2\pi n}\left[2j\sin(K\pi n)\right]$$

$$= \frac{\sin(K\pi n)}{\pi n}$$ ▨

Problem 2 (continued)

$$\sum_{n=-\infty}^{+\infty} \delta[n] e^{-j2\pi f n}$$

$$\boxed{\delta[n] \xleftrightarrow{\text{DTFT}} 1}$$

$$= \delta[0]\, e^{-j2\pi f\cdot(0)}$$

$$= 1 \qquad \blacksquare$$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \delta(f)\, e^{+j2\pi f n}\, df$$

$$\boxed{1 \xleftrightarrow{\text{DTFT}} \delta(f)}$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \delta(f)\, e^{+j2\pi f\cdot 0}\, df$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \delta(f)\, df$$

$$= 1 \qquad \blacksquare$$

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Problem 3

$$X[k] = \int_{0^-}^{T^-} \sum_{n=-\infty}^{+\infty} \delta(t-nT) e^{-j2\pi\frac{k}{T}t} dt$$

$$\boxed{\sum_{n=-\infty}^{+\infty} \delta(t-nT) \xleftrightarrow{CTFS} 1}$$

$$= \int_{0^-}^{T^-} \delta(t) e^{-j2\pi\frac{k}{T}t} dt$$

$$= \int_{0^-}^{T^-} \delta(t) e^{-j2\pi\frac{k}{T}\cdot 0} dt$$

$$= \int_{0^-}^{T^-} \delta(t) \cdot dt$$

$$= 1$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} X[k] e^{+j2\pi\frac{k}{T}t}$$

$$\boxed{e^{+j2\pi\frac{m}{T}t} \xleftrightarrow{CTFS} \begin{cases} T, & k = m \\ 0, & else \end{cases}}$$

$$= (1) \quad e^{+j2\pi\frac{m}{T}t}$$

$$\Rightarrow \frac{1}{T} X[k] = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

$$\Rightarrow X[k] = \begin{cases} T, & k = m \\ 0, & else \end{cases}$$

Problem 4

$$X[k] = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{+\infty} \delta[n-lN] e^{-j2\pi\frac{k}{N}n}$$

$$\boxed{\sum_{l=-\infty}^{+\infty} \delta[n-lN] \xleftrightarrow{DTFS} 1}$$

$$= \sum_{n=0}^{N-1} \delta[n] e^{-j2\pi\frac{k}{N}n}$$

$$= \delta[0] e^{-j2\pi\frac{k}{N}\cdot 0}$$

$$= 1$$

$$\boxed{e^{+j2\pi\frac{m}{N}n} \xleftrightarrow{DTFS} \begin{cases} N, & k=m \\ 0, & \text{else} \end{cases}}$$

$$x[n] = \sum_{n=0}^{N-1} \frac{1}{N} X[k] e^{+j2\pi\frac{k}{N}n}$$

$$= (1) \quad e^{+j2\pi\frac{m}{N}n}$$

$$\Rightarrow \frac{1}{N} X[k] = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases}$$

$$\Rightarrow X[k] = \begin{cases} N, & k=m \\ 0, & \text{else} \end{cases}$$