HW #02 108061112

Problem 1

(a) $x(t) = e^{-\pi|t|}$

(1) $X(f) = \int_{-\infty}^{+\infty} e^{-\pi|t|} e^{-j2\pi ft} dt$

$= \int_{-\infty}^{0} e^{-\pi(-t)} e^{-j2\pi ft} dt + \int_{0}^{+\infty} e^{-\pi(+t)} e^{-j2\pi ft} dt$

$= \int_{-\infty}^{0} e^{(\pi-j2\pi f)t} dt + \int_{0}^{+\infty} e^{(-\pi-j2\pi f)t} dt$

$= \frac{1-0}{\pi-j2\pi f} + \frac{0-1}{-\pi-j2\pi f}$

$= \frac{1}{\pi-j2\pi f} + \frac{1}{\pi+j2\pi f}$

$= \left(\frac{1}{1-j2f} + \frac{1}{1+j2f}\right)\frac{1}{\pi}$

$= \frac{2}{1+4f^2} \cdot \frac{1}{\pi}$

(b) $x(t) = e^{-\pi t^2}$

(1) $X(f) = \int_{-\infty}^{+\infty} e^{-\pi t^2} e^{-j2\pi ft} dt$

$= \int_{-\infty}^{+\infty} e^{-\pi t^2 - j2\pi ft} dt$

$= e^{-\pi f^2} \int_{-\infty}^{+\infty} e^{+\pi f^2} e^{-\pi t^2 - j2\pi ft} dt$

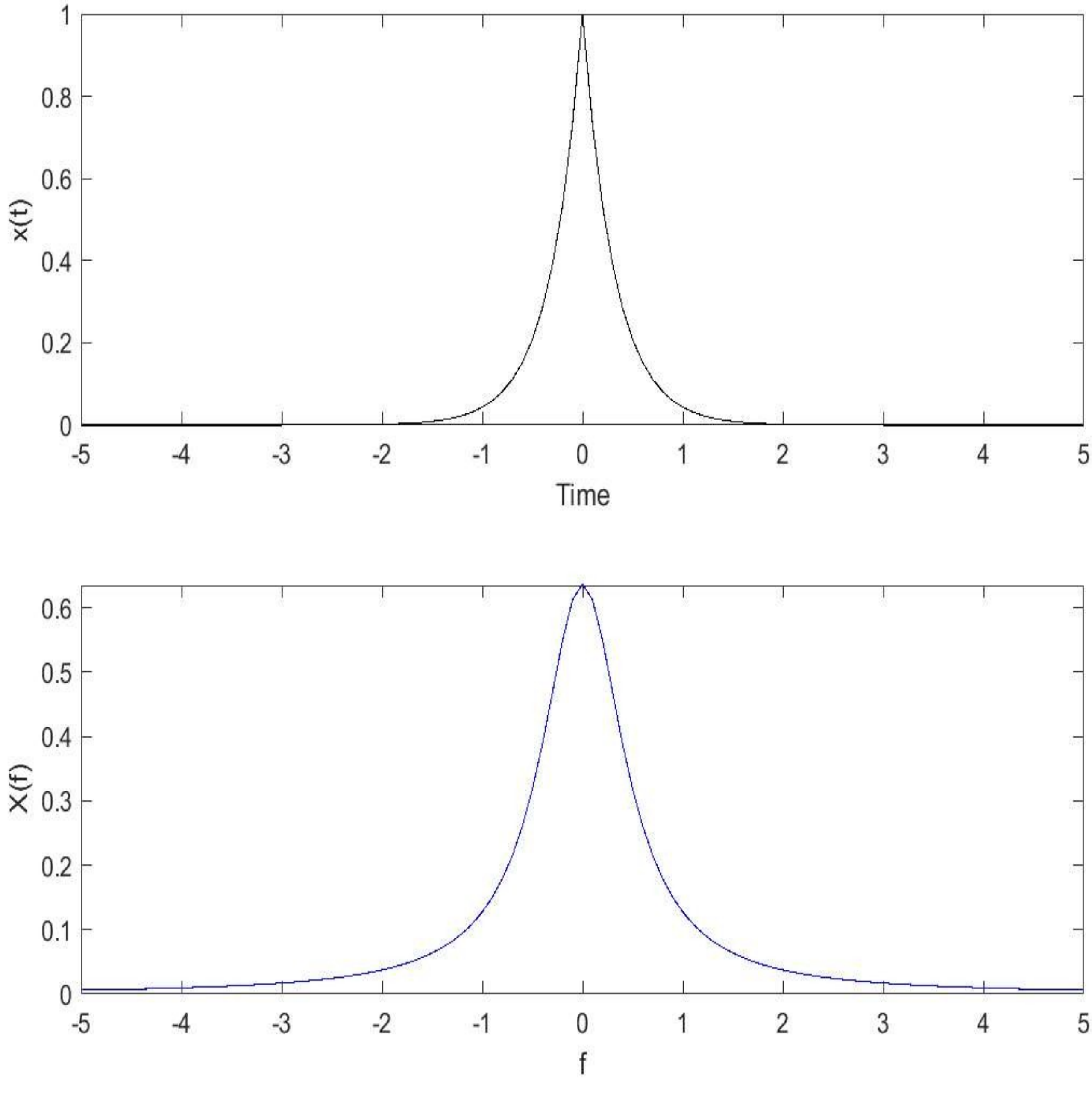
$= e^{-\pi f^2} \int_{-\infty}^{+\infty} e^{-\pi(t^2 + j2ft - f^2)} dt$

$= e^{-\pi f^2} \int_{-\infty}^{+\infty} e^{-\pi(t+jf)^2} dt$

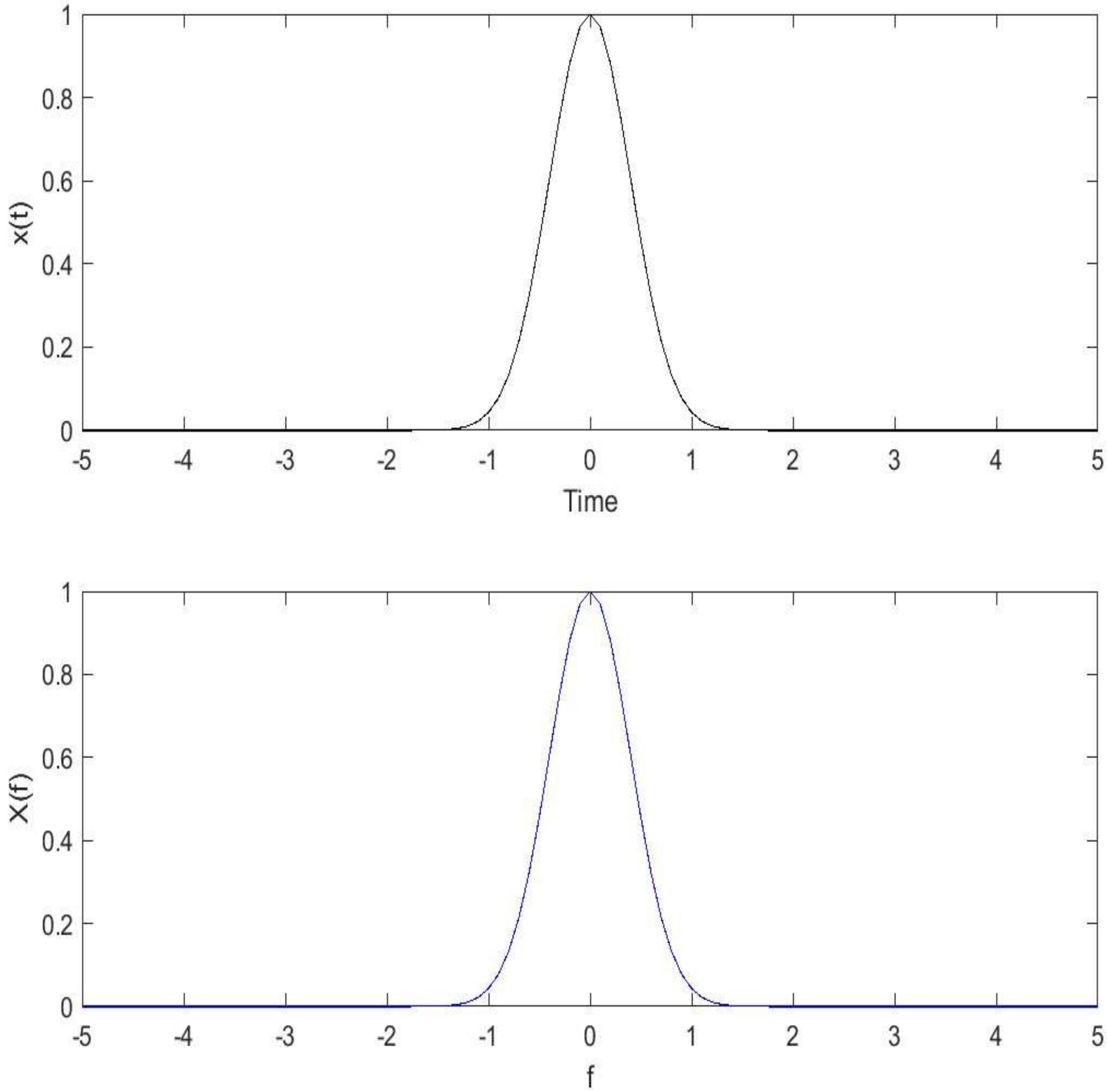
$= e^{-\pi f^2} \int_{-\infty}^{+\infty} e^{-\pi u^2} du$

$= e^{-\pi f^2}$

108061112 Homework #02 Problem 1 (a) (2) Figure 1



108061112 Homework #02 Problem 1 (b) (2) Figure 2



Problem 2

$$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

(a) $X[k]=\int_{0^-}^{T^-}\left[\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right]e^{-j2\pi\frac{k}{T}t}dt$

$=\int_{0^-}^{T^-}\delta(t)\cdot e^{-j2\pi\frac{k}{T}t}dt$

$=\int_{0^-}^{T^-}\delta(t)\cdot e^{-j2\pi\frac{k}{T}\cdot 0}dt$

$=\int_{0^-}^{T^-}\delta(t)\cdot 1\ dt$

$=1$

(b) $x(t)=\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{+j2\pi\frac{k}{T}\cdot t}$

(c) $=\sum_{k=-\infty}^{+\infty}\Delta f\, e^{+j2\pi(k\Delta f)t}$

$=\int_{-\infty}^{+\infty}e^{+j2\pi ft}df$

$X(f)=1$

(d) When $T=1$, $x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-n)$

and $X(f)=1$

$\blacksquare$

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Problem 3

$$x(t+T) = x(t) = \begin{cases} +1, & 0 < t < +\frac{T}{2} \\ -1, & -\frac{T}{2} < t < 0 \end{cases}$$

(a) $$X[k] = \int_{\frac{-T}{2}}^{\frac{+T}{2}} x(t) e^{-j2\pi\frac{k}{T}t} dt$$

$$= \int_{0}^{\frac{+T}{2}} (+1) e^{-j2\pi\frac{k}{T}t} dt + \int_{\frac{-T}{2}}^{0} (-1) e^{-j2\pi\frac{k}{T}t} dt$$

$$= \frac{1}{-j2\pi\frac{k}{T}}\left(e^{-j2\pi\frac{k}{T}\cdot(\frac{+T}{2})} - e^{-j2\pi\frac{k}{T}\cdot 0}\right) + \frac{-1}{-j2\pi\frac{k}{T}}\left(e^{-j2\pi\frac{k}{T}\cdot 0} - e^{-j2\pi\frac{k}{T}\cdot(\frac{-T}{2})}\right)$$

$$= \frac{1}{-j2\pi\frac{k}{T}}\left[\left(e^{-jk\pi} - 1\right) - \left(1 - e^{jk\pi}\right)\right]$$

$$= \frac{T}{-j2\pi k}\left[-2 + 2\cos(k\pi)\right]$$

$$= \frac{T(1-\cos(k\pi))}{j\pi k}$$

(b) $$x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} \frac{T(1-\cos(k\pi))}{j\pi k} e^{+j2\pi\frac{k}{T}t}$$

(c) $$= \sum_{k=-\infty}^{+\infty} \Delta f \frac{\frac{1}{\Delta f}(1-\cos(k\pi))}{j\pi k} e^{+j2\pi(k\Delta f)t}$$

$$= \sum_{k=-\infty}^{+\infty} \Delta f \left[\frac{1-\cos(k\pi)}{j\pi(k\Delta f)} e^{+j2\pi(k\Delta f)t}\right]$$

$$= \int_{-\infty}^{+\infty} \frac{1-\cos(k\pi)}{j\pi f} e^{+j2\pi f t} df$$

$$X(f) = \frac{1-\cos(k\pi)}{j\pi f}$$

Problem 3 (continued)

(d) When $T = 1$, $x(t) = \frac{1-\cos(k\pi)}{j\pi k}$

and $X(f) = \frac{1-\cos(k\pi)}{j\pi f}$ ▨

Problem 4

(a) $\phi_k(t) = e^{+j2\pi\frac{k}{T}t} \xrightarrow{\text{point sampling}} \phi_k[n] = e^{+j2\pi\frac{k}{N}n}$

$$\langle \phi_k[n], \phi_l[n] \rangle = \begin{cases} N, & k = l \\ 0, & k \neq l \end{cases} \qquad \text{(orthogonal)}$$

$$\phi_k[n+N] = \phi_k[n] \qquad \text{(periodic)}$$

$$x[n] = \sum_{k=-\infty}^{+\infty} c_k \phi_k[n] \qquad \text{(completed)}$$

$$\begin{aligned} &\quad\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad +\\ &= c_0 \cdot 1 + c_1 \cdot e^{+j2\pi\frac{1}{N}n} \cdots + c_{N-1} \cdot e^{+j2\pi\frac{N-1}{N}n} \quad + \\ &\quad\ c_N \cdot 1 + c_{N+1} e^{+j2\pi\frac{1}{N}n} \cdots + c_{2N-1} \cdot e^{+j2\pi\frac{N-1}{N}n} \quad + \\ &\quad\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

$$= \sum_{k=0}^{N-1} C_k \phi_k[n]$$

To find the C_k's :

$$\langle x[n], \phi_k[n] \rangle = C_k \langle \phi_k[n], \phi_k[n] \rangle$$

$$\Rightarrow \sum_{n=0}^{N-1} x[n] \phi_k^*[n] = C_k N$$

$$\Rightarrow \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{N}n} = C_k N$$

$$\Rightarrow C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{N}n}$$

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} \underbrace{\boxed{\sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{N}n}}}_{X[k]} e^{+j2\pi\frac{k}{N}n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j2\pi\frac{k}{N}n}$$

Problem 4 (continued)

(b) let $\frac{k}{N} = k\Delta f$ (as $N \to \infty$) $f = k\Delta f = \frac{k}{N}$

for odd numbers

$$x[n] = \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} \frac{1}{N}\left[\sum_{k=-\frac{N-1}{2}}^{+\frac{N-1}{2}} x[n]e^{-j2\pi\frac{k}{N}n}\right]e^{+j2\pi\frac{k}{N}n}$$

$$= \int_{f=-\frac{1}{2}}^{+\frac{1}{2}} \underbrace{\boxed{\sum_{k=-\infty}^{+\infty} x[n]e^{-j2\pi fn}}}_{X(f)} e^{+j2\pi fn}\,df$$

$$= \int_{f=-\frac{1}{2}}^{+\frac{1}{2}} X(f)e^{+j2\pi fn}\,df$$

since $\frac{+\frac{N-1}{2}}{N} \longrightarrow +\frac{1}{2}$ and $\frac{-\frac{N-1}{2}}{N} \longrightarrow -\frac{1}{2}$

similarly, for even numbers

$\frac{+\frac{N}{2}}{N} \longrightarrow +\frac{1}{2}$ and $\frac{-\frac{N}{2}}{N} \longrightarrow -\frac{1}{2}$

$\blacksquare$

Problem 5

(a) $x[n] = (\frac{3}{4})^n u[n]$

(1) $X(f) = \sum_{n=-\infty}^{+\infty} (\frac{3}{4})^n u[n] e^{-j2\pi fn}$

$= \sum_{n=0}^{+\infty} (\frac{3}{4})^n e^{-j2\pi fn}$

$= \sum_{n=0}^{+\infty} [\frac{3}{4} e^{-j2\pi f}]^n$

$= \frac{1}{1-[\frac{3}{4} e^{-j2\pi f}]}$

108061112 Homework #02 Problem 5 (a) (2) Figure 3