Reference Solution of Final Exam

1.

(1) $X(s)=\frac{1-s}{s+2},\quad \mathrm{Re}\{s\}<-2.$ (5%)

(2) $X(z)=\frac{1}{2}\left[\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}}+\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}\right],\quad |z|>\frac{1}{3}.$ (5%)

2.

(1) $x(t)=e^{-2(t+1)}u(t+1)$ (5%)

(2) $X(z)=\ln\left(1-\frac{1}{2}z^{-1}\right), |z|>\frac{1}{2}$

$$nx[n]\overset{z}{\leftrightarrow}-z\frac{d}{dz}X(z)=\frac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, |z|>\frac{1}{2}$$ (2%)

$$\because \left(\frac{1}{2}\right)^n u[n]\overset{z}{\leftrightarrow}\frac{1}{1-\frac{1}{2}z^{-1}}, |z|>\frac{1}{2}$$

$$\therefore -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n]\overset{z}{\leftrightarrow}\frac{-\frac{1}{2}}{1-\frac{1}{2}z^{-1}}, |z|>\frac{1}{2}$$

$$\Rightarrow -\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]\overset{z}{\leftrightarrow}\frac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, |z|>\frac{1}{2}$$

$$\Rightarrow nx[n]=-\left(\frac{1}{2}\right)^n u[n-1]$$

$$\Rightarrow x[n]=-\frac{1}{n}\left(\frac{1}{2}\right)^n u[n-1]$$

(3%)

3.

(1) *False.*

We set

$$H_1(s)=\frac{1}{s+1}, \mathrm{ROC}_1=\mathrm{Re}\{s\}>-1,\ H_2(s)=\frac{1}{(s+1)(s+2)}, \mathrm{ROC}_2=\mathrm{Re}\{s\}>-1,$$

$$\text{ROC}_{\cap} = \text{ROC}_1 \bigcap \text{ROC}_2 = \text{Re}\{s\} > -1.$$

$$x(t) = x_1(t) - x_2(t) \Rightarrow X(s) = X_1(s) - X_2(s),$$

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{1}{s+2}, \text{ ROC=Re\{s\}>-2.}$$

$$\therefore \text{ROC} \neq \text{ROC}_{\cap} \quad (5\%)$$

(2) *True*.

This is because the poles of the original system will be the zeros of its inverse system.and the zeros of the original system will be the poles of the inverse system. For both of them being causal and stable, their poles and zeros must be inside the unit circle. (5%)

4. According to the clue 1 and 2, we have

$$X(s) = \frac{A}{(s-s_{p1})(s-s_{p2})}. \quad (2\%)$$

By clue 3 we know $X(s)$ has a pole at s = -2+j. Since $x(t)$ is real, the poles of $X(s)$ will occur in conjugate reciprocal pairs. Therefore $p1$= -1+j and $p2$ = -1 - j.

$$\Rightarrow X(s) = \frac{A}{(s+1-j)(s+1+j)} = \frac{A}{s^2+2s+2} \quad (2\%)$$

From clue 5, we know that $X(0)$ = 8. That is,

$$X(0) = \frac{A}{2} = 8 \Rightarrow A = 16 \Rightarrow X(s) = \frac{16}{s^2+2s+2}. \quad (2\%)$$

The ROC of $X(s)$ are Re{s}> -1 or Re{s}< -1. Finally, we use clue 4.

$$y(t) = e^{-2t}x(t) \xleftrightarrow{L} Y(s) = X(s+2).$$

The ROC of $Y(s)$ is the POC of $X(s)$ shifted by 2 to the left. Since $Y(s)$ includes $j\omega$-axis, the possible ROC of $X(s)$ is Re{s}> -1. (2%)

5.

$$X(z) = -z^{-1} + 2 + \frac{0.5z^{-1}}{(1-z^{-1})(1-0.5z^{-1})} \quad (2\%)$$

$$= -z^{-1} + 2 + \frac{1}{1-z^{-1}} + \frac{-1}{1-0.5z^{-1}} \quad (3\%)$$

$$x[n] = -\delta[n-1] + 2\delta[n] - u[-n-1] + \left(\frac{1}{2}\right)^n u[-n-1] \quad (3\%)$$

6.

$$H(s)=\frac{2s+16}{s^2-4}=\frac{2s+16}{(s-2)(s+2)}=\frac{A}{s-2}+\frac{B}{s+2}$$

$$A=\left.\frac{2s+16}{s+2}\right|_{s=2}=\frac{20}{4}=5$$

$$B=\left.\frac{2s+16}{s-2}\right|_{s=-2}=\frac{12}{-4}=-3$$

$$\Rightarrow H(s)=\frac{5}{s-2}+\frac{-3}{s+2}$$

(1) The system is causal => right-sided => Re{s} > 2

$\Rightarrow h(t)=5e^{-2t}u(t)-3e^{2t}u(t)$ (5%)

(2) The system is stable => ROC includes $j\omega$-axis => -2 < Re{s} < 2

$\Rightarrow h(t)=-5e^{-2t}u(-t)-3e^{2t}u(t)$ (5%)

(3) This system is not possible to be both causal and stable. (3%)

7.

(1)

$$H_A(s)=\frac{1}{s+2},\ \mathrm{Re}\{s\}>-2$$

$$sY(s)+Y(s)=sW(s)+\alpha W(s)\Rightarrow H_B(s)=\frac{s+\alpha}{s+1},\ \mathrm{Re}\{s\}>-1.$$ (1%)

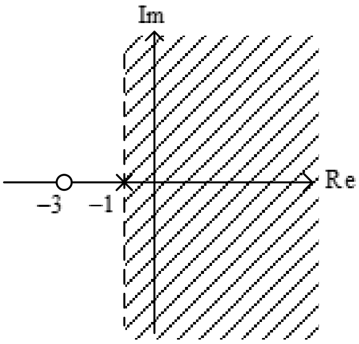
$$H(s)=H_A(s)H_B(s)=\frac{s+\alpha}{(s+1)(s+2)}$$

Input signal $x(t)=e^{-3t}$ is an eigenfunction of an LTI system, then $y(t)$=0. Since ROC of $H(s)$ is Re{s}> -1, H(-3) dose not exist. Hence we do not have enough information to solve H(s).

We assume H(-3) exist and solve $H(s)$, and we have $H_B(-3)=0$. (2%)

$$\Rightarrow H_B(-3)=\frac{-3+\alpha}{-3+1}=0\Rightarrow\alpha=3.$$

$$\Rightarrow H_B(s)=\frac{s+3}{s+1},\ \mathrm{Re}\{s\}>-1.$$ (2%)



ROC of $H_B(s)$ (2%)

(2) After we solving $H_B(s)$, the $H(s)$ can now be represented as follows:

$$H(s) = \frac{s+3}{(s+1)(s+2)}, \ \mathrm{Re}\{s\} > -1.$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+3s+2}. \quad (2\%)$$

$$\Rightarrow Y(s)(s^2+3s+2) = X(s)(s+3)$$

$$\Rightarrow \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = \frac{d}{dt}x(t) + 2x(t). \quad (2\%)$$

8.

$$B(s)B(-s) = \frac{1}{1+(s/3j)^6}, \ \omega_c = 3.$$

(1) The poles are

$$s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}\right]} = 3e^{j\left[\frac{\pi(2k+1)}{6}+\frac{\pi}{2}\right]}, k = 1, 2, ..., 6. \quad (5\%)$$

(2)

$$B(s) = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} = \frac{27}{s^3+6s^2+18s+27} \quad (5\%)$$

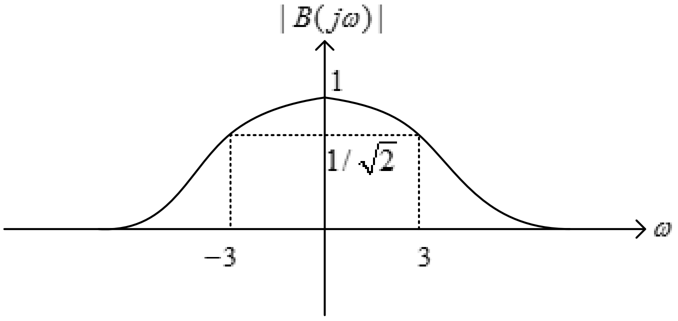
(3)

$$B(s)\big|_{s=j\omega} = B(j\omega) = \frac{27}{-j\omega^3 + 6\omega^2 + j18\omega + 27}$$

For $\omega = 0$, $|B(j\omega)| = 1$

For $\omega = 3$, $|B(j\omega)| = 1/\sqrt{2}$

For $\omega \to \infty$, $|B(j\omega)| = 0$



(3%)

9.

$$H(z)=\frac{1-z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{2}{1+\frac{1}{2}z^{-1}}+\frac{-1}{1-\frac{1}{4}z^{-1}}$$

Since the system is causal and stable, the ROC is $\frac{1}{2}<|z|$.

(1) $h[n]=2(\frac{-1}{2})^n u[n]-(\frac{1}{4})^n u[n]$. (5%)

(2) $x[n]=\cos(\pi n)=(-1)^n$

$y[n]=x[n]\times H(-1)=\frac{16}{5}x[n]=\frac{16}{5}\cos(\pi n)$. (4%)

(3) NO!

$$H_{inv}=\frac{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}{1-z^{-1}}$$

Since there is a pole on the unit circle, the inverse system of $H(z)$ is not stable. (3%)

10.

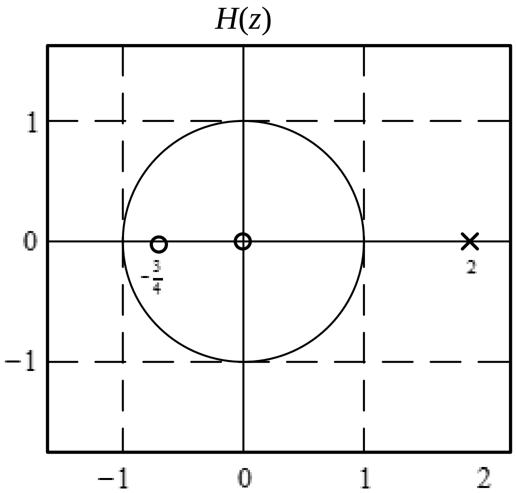
(1) $|z|>\frac{3}{4}$. (2%)

(2) Causal! (2%)

(3) $2>|z|>\frac{1}{2}$. (2%)

(4) Two-sided! (2%)

(5) $|z|<2$.



(4%)

11.

$$X(z)=\frac{\alpha+\beta z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}=\frac{(\alpha+2\beta)/3}{1-\frac{1}{2}z^{-1}}+\frac{2(\alpha-\beta)/3}{1+z^{-1}}$$

Since $x[n]$ is a two-sided signal, the ROC is $\frac{1}{2}<|z|<1$.

$$x[n]=\frac{\alpha+2\beta}{3}(\frac{1}{2})^n u[n]-\frac{2(\alpha-\beta)}{3}(-1)^n u[-n-1]=A(\frac{1}{2})^n u[n]-B(-1)^n u[-n-1]$$

$$x[1]=1=A(\frac{1}{2})\Rightarrow A=2$$

$$x[-1]=1=-B(-1)\Rightarrow B=1$$

$$x[n]=2(\frac{1}{2})^n u[n]-(-1)^n u[-n-1].$$ (6%)

12.

(1) $y[n]-\frac{2}{3}y[n-1]+\frac{1}{9}y[n-2]=x[n]-6x[n-1]+8x[n-2]$. (4%)

(2) $Y(z)[1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}]=X(z)[1-6z^{-1}+8z^{-2}]$

$$H(z)=\frac{1-6z^{-1}+8z^{-2}}{1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}}$$

$H(z)$ has only two poles. These are both at $z=\frac{1}{3}$. Since the system is stable, the ROC includes the unit circle. Since the ROC will be of the form $|z|>\frac{1}{3}$, the system is causal. (3%)