Final Exam

June 24, 2009

Instructor: *Chin-Liang Wang*

1. Determine the bilateral Laplace transform and the corresponding region of convergence (ROC) for each of the following signals :

   (1) $x(t) = e^{-2|t|} + e^{-9|t|}$. (6%)

   (2) $x(t) = \frac{d^2}{dt^2}\left(e^{-2\left(t-\frac{1}{2}\right)}u\left(t-\frac{1}{2}\right)\right)$. (6%)

2. Consider the bilateral Laplace transform given by

$$X(s) = \frac{2s+16}{s^2-4}.$$

   (1) Determine the corresponding causal time-domain signal *x*(*t*). (4%)
   (2) Determine the corresponding anti-causal time-domain signal *x*(*t*). (4%)
   (3) Determine the corresponding stable time-domain signal *x*(*t*). (4%)
   (4) Is it possible to find a causal and stable solution? Why? (3%)

3. Consider a causal signal *x*(*t*) with Laplace transform $X(s) = \frac{4s}{s^2-4}$, Re{s}>2.

   Determine the Laplace transform of each of the following signals:

   (1) $x(t) * \frac{d^2}{dt^2}x(t)$. (4%)

   (2) $e^{-6t}x(t)$. (4%)

   (3) $\int_0^t x\left(\frac{1}{3}\tau\right)d\tau$. (4%)

4. Let $B(s)$ be the transfer function of a stable Butterworth filter of order 2 and

$$B(s)B(-s) = \frac{1}{1+(\frac{s}{2j})^4}.$$

   (1) Plot the poles and zeros of $B(s)B(-s)$. (6%)

   (2) Determine the transfer function $B(s)$. (6%)

5.

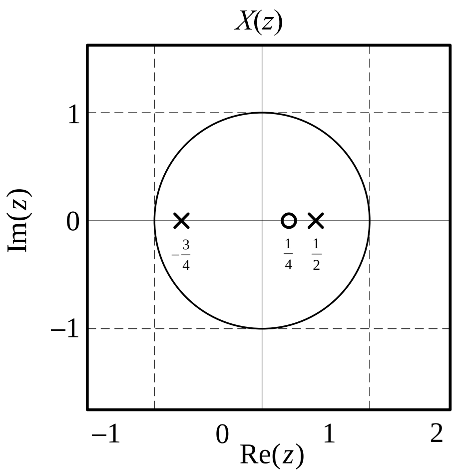
(1) Determine the z-transform and ROC of the time-domain signal

$$x[n] = (-\frac{1}{3})^n u[n] - (\frac{1}{2})^n u[-n-1]. \text{ (6\%)}$$

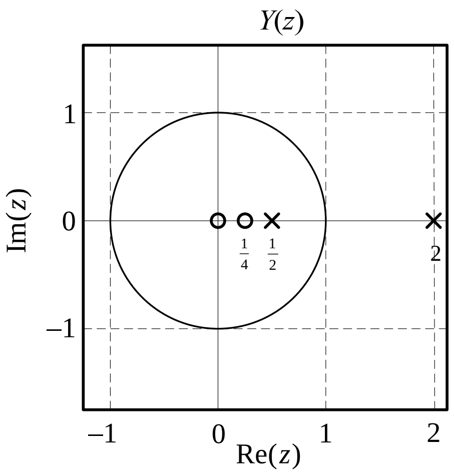
(2) Determine the inverse z-transform of

$$X(z) = \frac{1}{(1-rz^{-1})^2}, \ |z| > r > 0. \text{ (6\%)}$$

6. The signal $y[n]$ is the output of an LTI system with impulse response $h[n]$ for a given input $x[n]$. Assume that $x[n]$ is stable and has a z-transform $X(z)$ with the pole-zero plot shown in **Fig. 1**. Also assume that $y[n]$ is stable and has a z-transform $Y(z)$ with the pole-zero plot shown in **Fig. 2**.



(Fig. 1)



(Fig. 2)

(1) What is the ROC of $X(z)$? (2%)
(2) Is $x[n]$ a causal sequence? (2%)
(3) What is the ROC of $Y(z)$? (2%)
(4) Is $y[n]$ left-sided, right-sided, or two-sided? (2%)
(5) Draw the pole-zero plot of $H(z)$, and specify its ROC for a stable system. (2%)

7. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n-1].$$

Determine all possible ROCs of the transfer function $H(z)$, and find the corresponding impulse response $h[n]$ for each ROC. (12%)

8. Determine the signal $x[n]$ and the corresponding z-transforms $X(z)$ for each of the following cases:

   (1) $X(z)$ has two poles at $z = \frac{1}{2}$ and $z = -1$, a unknown single zero, $x[1]$=1, $x[-1]$=1, and the ROC includes $|z| = \frac{3}{4}$.(6%)

   (2) $x[n]$ is right sided with $x[0]$=2, and $x[2] = \frac{1}{2}$; $X(z)$ has a single pole and a single zero. (6%)

9.

   (1) Is the ideal low-pass filter (LPF) physically realizable? Why? (3%)

   (2) Indicate the frequency response type (LPF, HPF, BPF, or BSF), draw the pole-zero diagram, and estimate the 3-dB frequency or 3-dB bandwidth for the following filters:

      a. $H(s) = \frac{s}{s+1}$, $\mathrm{Re}\{s\} > -1$. (5%)

      b. $H(z) = \frac{1+z^{-1}}{1-0.95z^{-1}}$, $|z| > 0.95$. (5%)

Final Exam Reference Solutions

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1. (1) $X(s) = \dfrac{-4}{s^2-4} + \dfrac{-18}{s^2-81}$

   (2) $X(s) = s^2 \dfrac{1}{(s+2)} e^{-\frac{1}{2}s}, \mathrm{Re}\{s\} > -2$

2. $X(s) = \dfrac{-3}{s+2} + \dfrac{5}{s-2}$

   (1) Causal： $x(t) = -3e^{-2t}u(t) + 5e^{2t}u(t), \mathrm{Re}\{s\} > 2$

   (2) Anti-causal： $x(t) = 3e^{-2t}u(-t) - 5e^{2t}u(-t), \mathrm{Re}\{s\} < -2$

   (3) Stable： $x(t) = -3e^{-2t}u(t) - 5e^{2t}u(-t), -2 < \mathrm{Re}\{s\} < 2$

   (4) NO,since has a pole in the right-side,this system does not have a causal and stable solution.

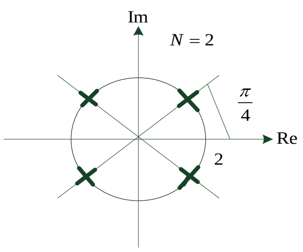
3. (1) $s^2 \dfrac{16s^2}{(s^2-4)^2}$

   (2) $\dfrac{4(s+6)}{(s+6)^2-4}$

   (3) $\dfrac{1}{s}\left[\dfrac{36s}{(3s)^2-4}\right]$

4. (1) The pole of $B(s)B(-s)$ are the solution of $1+\left(\dfrac{s}{2j}\right)^4 = 0$

   $\Rightarrow s_p = 2e^{j\left[\frac{\pi(2k+1)}{4}+\frac{\pi}{2}\right]}$,k is an integer.



(2) $B(s) = \frac{2^2}{\left(s+2e^{j\frac{\pi}{4}}\right)\left(s+2e^{-j\frac{\pi}{4}}\right)} = \frac{4}{s^2+2\sqrt{2}s+4}$

5.

(1) $X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \frac{1}{2} > |z| > \frac{1}{3}.$

(2) $(n+1)r^n u[n+1] \overset{z}{\longleftrightarrow} \frac{1}{(1-rz^{-1})^2}.$

6.

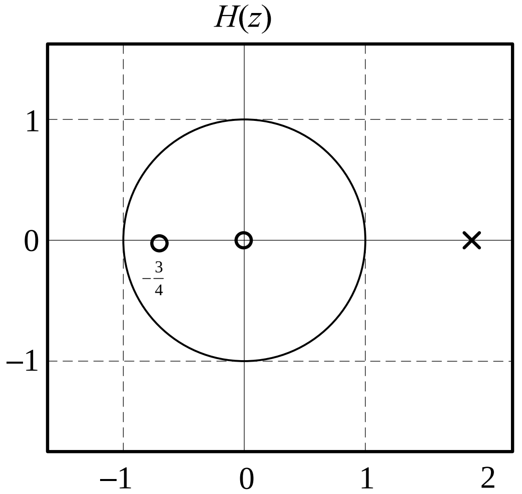
(1) $|z| > \frac{3}{4}.$

(2) Causal!

(3) $2 > |z| > \frac{1}{2}.$

(4) Two-sided!

(5) $|z| < 2.$



7.

$$X(z) = \frac{z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$= \frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{4}{1-\frac{1}{4}z^{-1}}$$

ROC: $|z| > \frac{1}{2} \Rightarrow x[n] = 4\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$

ROC: $\frac{1}{4} < |z| < \frac{1}{2} \Rightarrow x[n] = -4\left(\frac{1}{2}\right)^n u[-n-1] - 4\left(\frac{1}{4}\right)^n u[n]$

ROC: $\frac{1}{4} < |z| \Rightarrow x[n] = 4\left[-\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u[-n-1]$

8.

(1) $X(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+z^{-1}}$,

Since the ROC includes the point $z = 3/4$, the ROC is $\frac{1}{2} < |z| < 1$.

$$\begin{aligned}
X(z) &= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+z^{-1}} \\
x[n] &= A\left(\frac{1}{2}\right)^n u[n] - B(-1)^n u[-n-1] \\
x[1] = 1 &= A\left(\frac{1}{2}\right) \\
A &= 2 \\
x[-1] = 1 &= -1B(-1) \\
B &= 1 \\
x[n] &= 2\left(\frac{1}{2}\right)^n u[n] - (-1)^n u[-n-1]
\end{aligned}$$

(2)

$$\begin{aligned}
x[n] &= c(p)^n u[n] \quad \text{where } c \text{ and } p \text{ are unknown constants.} \\
x[0] = 2 &= c\,(p)^0
\end{aligned}$$

$$\begin{aligned}
c &= 2 \\
x[2] = \frac{1}{2} &= 2\,(p)^2 \\
p &= \frac{1}{2} \\
x[n] &= 2\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$