Final Exam

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Instructor: *Chin-Liang Wang*

1. Determine the bilateral Laplace transform or the inverse Laplace transform for the following signals:

(1) $x(t)=\dfrac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)$. (4%)

$$e^{-3t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+3},\ \text{with ROC } \operatorname{Re}\{s\}>-3$$

$$e^{-3(t-2)}u(t-2)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+3}e^{-2s},\ \text{with ROC } \operatorname{Re}\{s\}>-3$$

$$x(t)=\frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)\overset{\mathcal{L}}{\longleftrightarrow}X(s)=\frac{s^2}{s+3}e^{-2s},\ \text{with ROC } \operatorname{Re}\{s\}>-3$$

(2) $X(s)=s^{-1}\dfrac{d}{ds}\left(\dfrac{e^{-3s}}{s}\right)$ with ROC $\operatorname{Re}\{s\}>0$. (4%)

$$\underset{\text{right-sided}}{A(s)=\frac{1}{s}}\overset{\mathcal{L}}{\longleftrightarrow}a(t)=u(t)$$

$$B(s)=e^{-3s}A(s)\overset{\mathcal{L}}{\longleftrightarrow}b(t)=a(t-3)=u(t-3)$$

$$C(s)=\frac{d}{ds}B(s)\overset{\mathcal{L}}{\longleftrightarrow}c(t)=-tb(t)=-tu(t-3)$$

$$X(s)=\frac{1}{s}C(s)\overset{\mathcal{L}}{\longleftrightarrow}x(t)=\int_{-\infty}^{t}c(\tau)\,d\tau=-\int_{3}^{t}\tau d\tau=-\frac{1}{2}\left(t^2-9\right)u(t-3)$$

2. Use the method of partial fractions to determine the time-domain signals corresponding to the following bilateral Laplace transform:

$$X(s)=\frac{-s-4}{s^2+3s+2}=\frac{-3}{s+1}+\frac{2}{s+2}$$

(1) With ROC $\operatorname{Re}\{s\}<-2$ (3%)

Left-sided: $x(t)=\left(3e^{-t}-2e^{-2t}\right)u(-t)$

(2) With ROC $\operatorname{Re}\{s\}>-1$ (3%)

Right-sided: $x(t)=\left(-3e^{-t}+2e^{-2t}\right)u(t)$

(3) With ROC $-2<\operatorname{Re}\{s\}<-1$ (3%)

Two-sided: $x(t)=3e^{-t}u(-t)+2e^{-2t}u(t)$

3. Let $x(t)=0$ for $t<0$ and $x(t)$ contains no impulses or higher-order

singularities at the origin. Show that $x\left(0^+\right)=\lim_{s\to 0} sX(s)$. (7%)

Expanding $x(t)$ as a Taylor series at $t=0^+$,

$$x(t)=\left[x\left(0^+\right)+x^{(1)}\left(0^+\right)t+\cdots+x^{(n)}\left(0^+\right)\frac{t^n}{n!}+\cdots\right]u(t)$$

where $x^{(n)}\left(0^+\right)$ denotes the *n*th derivative of $x(t)$ evaluated at $t=0^+$.

$$u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s}$$

$$tu(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s^2}$$

$$\vdots$$

$$\frac{t^n}{n!}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s^n}$$

$$\Rightarrow \mathcal{L}\{x(t)\}=\frac{1}{s}x\left(0^+\right)+\frac{1}{s^2}x^{(1)}\left(0^+\right)+\cdots+\frac{1}{s^n}x^{(n)}\left(0^+\right)+\cdots=X(s)$$

$$\Rightarrow sX(s)=x\left(0^+\right)+\frac{1}{s}x^{(1)}\left(0^+\right)+\cdots+\frac{1}{s^{n-1}}x^{(n)}\left(0^+\right)+\cdots$$

$\Rightarrow \lim_{s\to\infty} sX(s)=x\left(0^+\right)\cdots\cdots\cdots$ The initial value theorem

4. A stable system has the indicated input $x(t)$ and output $y(t)$. Use Laplace transforms to determine the transfer function and impulse response of the system. (8%)

$$x(t)=e^{-2t}u(t),\ y(t)=-2e^{-t}u(t)+2e^{-3t}u(t)$$

$$X(s)=\frac{1}{s+2}$$

$$Y(s)=\frac{-2}{s+1}+\frac{2}{s+3}$$

$$H(s)=Y(s)/X(s)=\frac{-4(s+2)}{(s+1)(s+3)}=\frac{-2}{s+1}+\frac{-2}{s+3}\text{ with ROC }\mathrm{Re}\{s\}>-1$$

$$h(t)=\left(-2e^{-t}-2e^{-3t}\right)u(t)$$

5. Determine (1) whether the system described by the following transfer function is both stable and causal and (2) whether a stable and causal inverse system exists:

$$H(s)=\frac{s+5}{2s^2+4s+4}$$

You need to justify your answers. (8%)

$$H(s)=\frac{s+5}{2s^2+4s+4}=\frac{\frac{4j+1}{4}}{S+1+j}-\frac{\frac{4j-1}{4}}{S+1-j}$$

zero at: -5
poles at: $-1\pm 3j$

$$\left(H^{-1}(s)=\frac{2s^2+4s+4}{s+5}=2S-6+\frac{34}{S+5}\right)$$

zero at: $-1\pm 3j$
poles at: -5

(1) All poles are in the left half of s-plane, and with ROC: $\mathrm{Re}\{s\}>-1$, the system is both stable and causal. (4%)

(2) All zeros are in the left half of s-plane, and with ROC: $\mathrm{Re}\{s\}>-5$, so a stable and causal inverse system exists. (4%)

6. Indicate whether each of the following statements is true or false. If true, give a brief explanation, and if false, give a counterexample.

(1) The ROC for the Laplace transform of a linear combination of signals cannot extend beyond the intersection of the ROCs for the individual terms. (4%)

False. For example:

$$X_1(s)=\frac{1}{s+1},\ \mathrm{Re}\{s\}>-1,\quad X_2(s)=\frac{1}{(s+1)(s+2)},\ \mathrm{Re}\{s\}>-1$$

$$x(t)=x_1(t)-x_2(t)$$

$$X(s)=X_1(s)-X_2(s)=\frac{1}{s+1}-\frac{1}{(s+1)(s+2)}$$

$$=\frac{s+1}{(s+1)(s+2)}=\frac{1}{s+2},\ \mathrm{Re}\{s\}>-2$$

(2) An ROC to the right of the rightmost pole does not guarantee that the system is causal; rather, it guarantees only that the impulse response is right sided. (4%)

True.

Because the effect on the ROC of shifting in the S-domain may make the system uncausal ($h(t)\neq 0$ for $t<0$).

7. Given the transform pair $x(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{2s}{s^2+2}$, where $x(t)=0$ for $t<0$, determine the Laplace transform of the following time signals:

(1) $e^{-3t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s+3) = \dfrac{2(s+3)}{(s+3)^2+2}$ (3%)

(2) $x(t) * \dfrac{d}{dt}x(t)$

$$b(t) = \frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} B(s) = sX(s)$$

$$y(t) = x(t) * b(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = X(s)B(s) = sX^2(s) = s\left(\frac{2s}{s^2+2}\right)^2$$ (4%)

8. Determine the z-transform and sketch the poles, zeros, and ROC in the z-plane for the following signal: (10%)

$$X(z) = \sum_{n=0}^{4} \frac{1}{5}\left(\frac{1}{5}z^{-1}\right)^n = \frac{1}{5} \cdot \frac{1-\left(\frac{1}{5}z^{-1}\right)^5}{1-\frac{1}{5}z^{-1}}$$

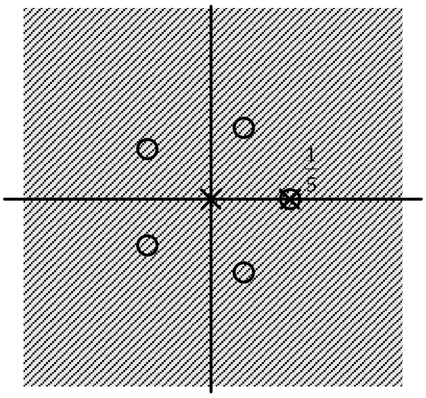
$$= \frac{1}{5} \cdot \frac{\left[1-\left(\frac{1}{5}\right)^5 z^{-5}\right]}{1-\frac{1}{5}z^{-1}} = \frac{1}{5} \cdot \frac{\left[z^5-\left(\frac{1}{5}\right)^5\right]}{z^4\left(z-\frac{1}{5}\right)}$$

4 poles at $z=0$, 1 poles at $z=\frac{1}{5}$ (5%)

5 poles at $z=\frac{1}{5}e^{jk\frac{2\pi}{5}}$, $k=0, 1, 2, 3, 4$

Note the zero at $z=\frac{1}{5}$ is cancelled by the pole at $z=\frac{1}{5}$.

ROC: $0<|z|<\infty$



(5%)

9. Find the time-domain signals for the following z-transforms:

(1) $X(z)=\dfrac{-1.25z^{-1}}{1-2.75z^{-1}+1.5z^{-2}}=\dfrac{1}{1-0.75z^{-1}}-\dfrac{1}{1-2z^{-1}},\quad 0.75<|z|<2$ (4%)

$$x[n]=(0.75)^n u[n]+2^n u[-n-1] \qquad (3\%)$$

(2) $X(z)=\ln\left(1+2z^{-1}\right)\ ,\ |z|>2$ (You may need $nx[n]\overset{z}{\longleftrightarrow}-z\dfrac{d}{dz}X(z)$.)

$$nx[n]\overset{z}{\longleftrightarrow}-z\frac{d}{dz}X(z)=\frac{2z^{-1}}{1+2z^{-1}}\quad ,\quad |z|>2 \qquad (2\%)$$

$$\because (-2)^n u[n]\overset{z}{\longleftrightarrow}\frac{1}{1+2z^{-1}}\quad ,\quad |z|>2$$

$$\therefore 2(-2)^n u[n]\overset{z}{\longleftrightarrow}\frac{2}{1+2z^{-1}}\quad ,\quad |z|>2 \qquad (3\%)$$

$$\Rightarrow 2(-2)^{n-1} u[n-1]\overset{z}{\longleftrightarrow}\frac{2z^{-1}}{1+2z^{-1}}\quad ,\quad |z|>2$$

$$\Rightarrow nx[n]=-(-2)^n u[n-1]$$

$$\Rightarrow x[n]=\frac{-(-2)^n}{n}u[n-1] \qquad (2\%)$$

10. Indicate the frequency type (LPF, HPF, BPF, or BSF), draw the pole/zero plot, and estimate the 3-dB bandwidth for each of the following filter:

$$H(z)=\frac{(1-a)\left(1+z^{-1}\right)}{2\left(1-az^{-1}\right)},\quad |z|>a,\ \text{where}\ 0\ll a<1.$$

(1) $H\left(e^{j\Omega}\right)=H(z)\big|_{z=e^{j\Omega}}=\dfrac{(1-a)\left(1+e^{-j\Omega}\right)}{2\left(1-ae^{-j\Omega}\right)}$

when $\Omega=0$ (DC value):

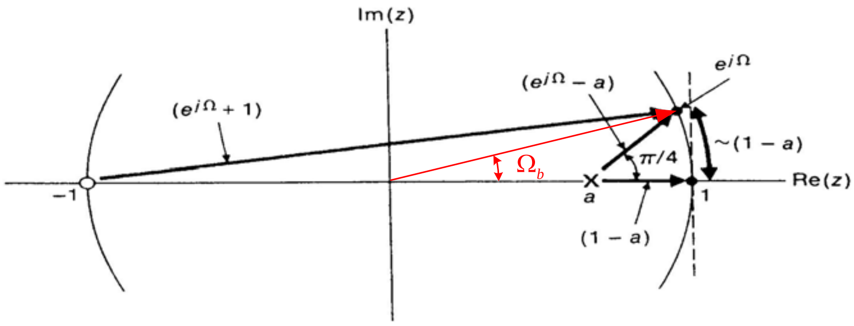
$$\left|H(e^{j0})\right|=\left|\frac{(1-a)(1+1)}{2(1-a)}\right|=1$$

when $\Omega=\pi$ (high band): (2%)

$$\left|H(e^{j\pi})\right|=\left|\frac{(1-a)(1-1)}{2(1+a)}\right|=0$$

$\therefore$ It is a LPF.

(2)



(3%)

(3) $H\left(e^{j\Omega}\right)=\dfrac{(1-a)\left(1+e^{-j\Omega}\right)}{2\left(1-ae^{-j\Omega}\right)}=\dfrac{(1-a)\left(e^{j\Omega}+1\right)}{2\left(e^{j\Omega}-a\right)}$

As the pole/zero plot shows:

$$\left|H\left(e^{j\Omega_b}\right)\right|=\left|\frac{1-a}{2}\right|\frac{\left|e^{j\Omega_b}+1\right|}{\left|e^{j\Omega_b}-a\right|}\approx\frac{1-a}{2}\frac{2}{\sqrt{2}(1-a)}=\frac{1}{\sqrt{2}} \quad (3\%)$$

$\therefore \Omega_b$ is the 3-dB bandwidth

$\because\ 1-a\approx r\Omega_b$ and $r=1$

$\therefore\ \Omega_b\approx 1-a$

11. Consider a causal FIR system whose impulse response $h[n]$ is real and has the following property:

$$h[n]=h[M-n],\ n=0,\ 1,\ 2,\ \cdots,\ M,\ \ M:\text{odd}$$

where $M+1$ is the length of the impulse response sequence.

(1) $H(z)=\sum_{n=0}^{M}h[n]z^{-n}=\sum_{n=0}^{\frac{M-1}{2}}h[n]z^{-n}+\sum_{n=\frac{M+1}{2}}^{M}h[n]z^{-n}=\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(z^{-n}+z^{-(M-n)}\right)$ (6%)

$$H\left(e^{j\Omega}\right)=H\left(z\right)\Big|_{z=e^{j\Omega}}=\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(e^{-j\Omega n}+e^{-j\Omega(M-n)}\right)$$
$$=e^{-j\Omega M/2}\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(e^{j\Omega(M/2-n)}+e^{-j\Omega(M/2-n)}\right)$$
$$=e^{-j\Omega M/2}\underbrace{\sum_{n=0}^{\frac{M-1}{2}}2h[n]\cos\left[\Omega\left(\frac{M}{2}-n\right)\right]}_{real\ \ function\ \ R(\Omega)}$$

$$\angle H\left(e^{j\Omega}\right)=-\frac{M}{2}\Omega+\angle R\left(\Omega\right)$$

Since $\angle R(\Omega)$ is either 0 if $R(\Omega)\geq 0$ or $\pm\pi$ if $R(\Omega)<0$, $\angle H\left(e^{j\Omega}\right)$ can be seen a linear function of Ω. $\Rightarrow$ This is a linear phase system.

(2) $H(z)=\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(z^{-n}+z^{-(M-n)}\right)$

$$H\left(z^{-1}\right)=\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(z^{n}+z^{M-n}\right)$$
$$z^{-M}H\left(z^{-1}\right)=\sum_{n=0}^{\frac{M-1}{2}}h[n]\left(z^{-(M-n)}+z^{-n}\right)=H(z)$$
(3%)

(3) $H(z_0)=0$ (4%)

$$H(z)=\sum h[n]z^{-n}$$

$$H\left(z^*\right)=\sum h[n]\left(z^*\right)^{-n}=\sum h[n]\left(z^{-n}\right)^*=\sum h^*[n]\left(z^{-n}\right)^*$$
$$=\left(\sum h[n]z^{-n}\right)^*=H^*(z)$$

$H\left(z_0^*\right)=H^*\left(z_0\right)=0$ --(I)

$H\left(z_0\right)=z_0^{-M}H\left(z_0^{-1}\right)=0\Rightarrow H\left(z_0^{-1}\right)=0$ -----------------(II)

Combining (I) and (II), $H\left(\frac{1}{z_0^*}\right)=0$

12. Consider a discrete-time LTI system that has the following properties:

(i) If the input is $x[n]=(-2)^n$ for all n, then the output is $y[n]=0$ for all n.

(ii) If the input is $x[n]=(1/2)^n u[n]$ for all n, then the output is

$y[n]=\delta[n]+\alpha(1/4)^n u[n]$ for all n, where α is a constant.

(1) $x[n]=z^n$ is an eigenfunction of discrete-time LTI systems

using the property (i): (3%)

$$\begin{aligned} y[n] &= h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n\sum_{k=-\infty}^{\infty} h[k]z^{-k} \\ &= z^n H(z) = H(z)z^n \end{aligned}$$

$$\begin{aligned} y[n] &= h[n]*(-2)^n = \sum_{k=-\infty}^{\infty} h[k](-2)^{n-k} \\ &= (-2)^n \sum_{k=-\infty}^{\infty} h[k](-2)^{-k} \\ &= (-2)^n H(-2) = 0 \Rightarrow H(-2)=0 \end{aligned}$$

using the property (ii): (3%)

$$x[n]=(1/2)^n u[n] \xleftrightarrow{z} X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$y[n]=\delta[n]+\alpha(1/4)^n u[n] \xleftrightarrow{z} Y(z)=1+\frac{\alpha}{1-\frac{1}{4}z^{-1}}$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\dfrac{\alpha}{1-\frac{1}{4}z^{-1}}}{\dfrac{1}{1-\frac{1}{2}z^{-1}}}=\left(1+\frac{\alpha}{1-\frac{1}{4}z^{-1}}\right)\left(1-\frac{1}{2}z^{-1}\right)$$

$\because H(-2)=0$

$$\therefore H(-2)=\left(1+\frac{\alpha}{\frac{9}{8}}\right)\left(1+\frac{1}{4}\right)=\left(1+\frac{8}{9}\alpha\right)\left(1+\frac{1}{4}\right)=0$$

$$\alpha=-\frac{9}{8}$$

(2) Apply $x[n]=\left(\frac{1}{3}\right)^n$ as an eigenfunction.

$$y[n]=\left(\frac{1}{3}\right)^n H\left(\frac{1}{3}\right)=\frac{7}{4}\left(\frac{1}{3}\right)^n \quad (4\%)$$