Midterm Exam II Reference Solutions

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1.

(1) $\int_{-\infty}^{\infty} x(t)\,dt = X(0) = 2$ (2%)

(2) $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega = \frac{10}{\pi}$ (3%)

(3) $\int_{-\infty}^{\infty} x(t)e^{j2t}\,dt = X(-2) = 1$ (2%)

(4) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega\cdot 0}\,d\omega = \frac{6}{\pi}$ (2%)

(5) $\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{-j\omega\cdot 0}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega}_{\text{odd function}} \cdot \underbrace{X(j\omega)}_{\text{even function}}\,d\omega = 0$ (3%)

2.

(1) (4%) Since $x[n]$ is real and even, $Y(e^{j\Omega}) = \mathrm{Im}\{X(e^{j\Omega})\} = 0$.

(2) (4%) Using time shifting property.

$$y[n] = x[n-4] = |n-4|\left(\frac{1}{3}\right)^{|n-4|}.$$

(3) (4%) Using frequency shifting property.

$$y[n] = x[n]\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) = 2|n|\left(\frac{1}{3}\right)^{|n|}\cos\left(\frac{\pi}{4}n\right).$$

3.

(1) (6%)

$$Y(e^{j\Omega}) - \frac{3}{4}Y(e^{j\Omega})e^{-j\Omega} + \frac{1}{8}Y(e^{j\Omega})e^{-j2\Omega} = X(e^{j\Omega})2e^{-j\Omega}$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{2e^{-j\Omega}}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

(2) (6%)

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},\ \mathrm{Y}(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega}).$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{1}{1-\frac{1}{2}e^{-j\Omega}}\cdot\frac{2e^{-j\Omega}}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-j2\Omega}}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-16}{1-\frac{1}{2}e^{-j\Omega}}+\frac{8}{\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}+\frac{8}{1-\frac{1}{4}e^{-j\Omega}}$$

$$y[n]=-8\left(\frac{1}{2}\right)^n u[n]+8\left(\frac{1}{4}\right)^n u[n]+8n\left(\frac{1}{2}\right)^n u[n].$$

4.

(1) (5%) From Euler's formula, we have

$$x(t)=\sum_{k=0}^{2}(-1)^k\sin\left(\frac{2\pi k}{3}t\right)=\frac{1}{2j}\sum_{k=0}^{2}(-1)^k\left(e^{j\frac{2\pi k}{3}t}-e^{-j\frac{2\pi k}{3}t}\right)$$

$$\Rightarrow X(j\omega)=\frac{\pi}{j}\sum_{k=0}^{2}(-1)^k\left[\delta\left(\omega-\frac{2\pi k}{3}\right)-\delta\left(\omega+\frac{2\pi k}{3}\right)\right]$$

(2) (5%) From $te^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{(a+j\omega)^2}$ and the property mentioned in lecture note, We have

$$-jtx(t)=-jt^2e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{d}{d\omega}\left(X(\omega)\right)=\frac{d}{d\omega}\left(\frac{1}{(a+j\omega)^2}\right)=\frac{-2j}{(a+j\omega)^3}$$

$$\Rightarrow\frac{t^2}{2}e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{(a+j\omega)^3}$$

$$\Rightarrow x(t)=\frac{t^2}{2}e^{-at}u(t),\ a>0.$$

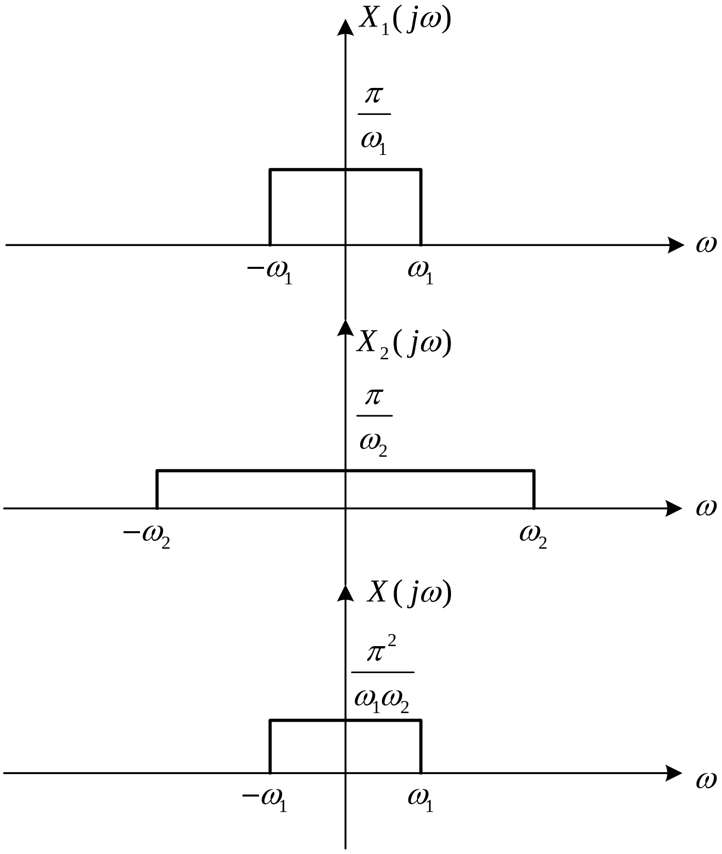
5.

(1) (4%)

$$x(t)=x_1(t)*x_2(t)\overset{\mathcal{F}}{\longleftrightarrow}X(j\omega)=X_1(j\omega)X_2(j\omega)$$

$$x_1(t)=\frac{\pi}{\omega_1}\cdot\frac{\sin(\omega_1 t)}{\pi t}\overset{\mathcal{F}}{\longleftrightarrow}X_1(j\omega)=\begin{cases}\frac{\pi}{\omega_1},\ |\omega|<\omega_1\\ 0,\ |\omega|>\omega_1\end{cases}$$

$$x_2(t)=\frac{\pi}{\omega_2}\cdot\frac{\sin(\omega_2 t)}{\pi t}\overset{\mathcal{F}}{\longleftrightarrow}X_2(j\omega)=\begin{cases}\frac{\pi}{\omega_2},\ |\omega|<\omega_2\\ 0,\ |\omega|>\omega_2\end{cases}$$

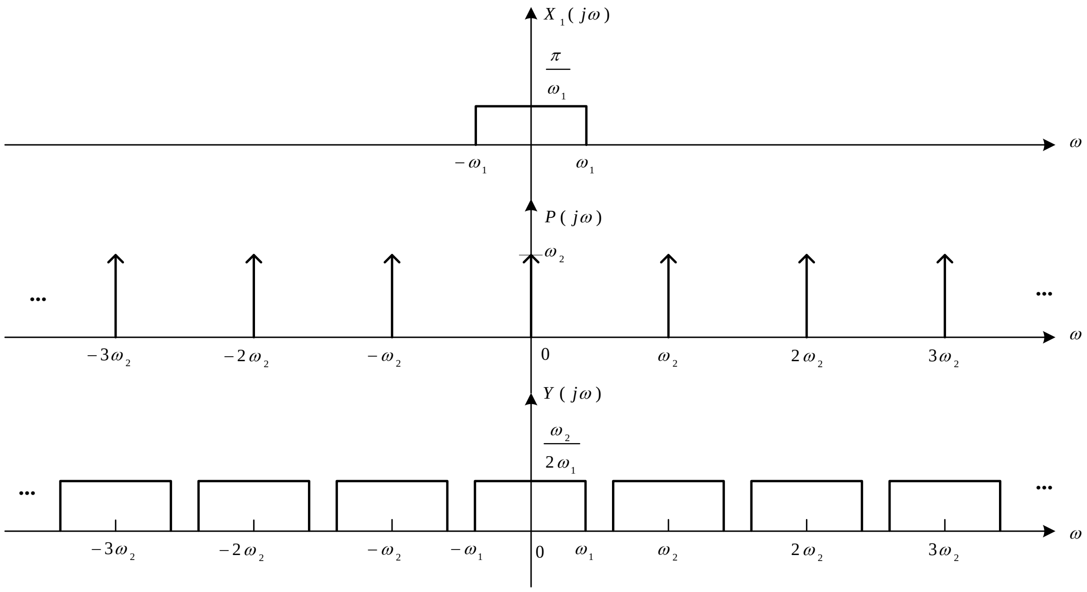


(2) (4%)

$$y(t) = x_1(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi}\left[X_1(j\omega) * P(j\omega)\right]$$

$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \Rightarrow P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}k\right)$, where

$\omega_s = \frac{2\pi}{T} = \omega_2$.



(3) Yes. Since the sampling frequency $\omega_s = \omega_2 > 2\omega_1$, no aliasing in $Y(j\omega)$.

$x_1(t)$ can be reconstructed from $y(t)$. (4%)

6.

(1) (7%)

$$(j\omega)^2 Z(j\omega) - (j\omega)Z(j\omega) - 6Z(j\omega) = X(j\omega)$$

$$\Rightarrow H_A(j\omega) = \frac{Z(j\omega)}{X(j\omega)} = \frac{1}{(j\omega)^2 - j\omega - 6} = \frac{1}{(-3+j\omega)(2+j\omega)} = \frac{0.5}{(-3+j\omega)} + \frac{(-0.5)}{(2+j\omega)}$$

$$\Rightarrow h_A(t) = \frac{1}{2}\left[e^{3t}u(-t) - e^{-2t}u(t)\right]$$

(2) 題目有誤，送分 (5%)

$$\frac{dy(t)}{dt} + 6y(t) = \frac{dz(t)}{dt} + bz(t) \qquad (corrected\ version)$$

$$\Rightarrow H_B(j\omega) = \frac{Y(j\omega)}{Z(j\omega)} = \frac{(b+j\omega)}{(6+j\omega)}$$

In order to make the system causal, we have to cancel out the non-causal term, i.e. the first term of $H_A(j\omega)$, yields $b = -3$.

7.

(1) (4%)

$$H(e^{j\Omega}) = -e^{-j\Omega} + 2e^{-j2\Omega} - 2e^{-4j\Omega} + e^{-5j\Omega} = e^{-3j\Omega}(2e^{j\Omega} - 2e^{j\Omega} - e^{2j\Omega} + e^{-j2\Omega})$$

$$= e^{-3j\Omega} j(4\sin(\Omega) - 2\sin(2\Omega)) = e^{-j(3\Omega - \frac{\pi}{2})}(4\sin(\Omega) - 2\sin(2\Omega)).$$

(2) (4%)

Yes! $\because \dfrac{d(-3\Omega - \frac{\pi}{2})}{d\Omega} = -3 = \text{constant.}$

8.

(1) (5%)

The F.S. coefficients of $x[n]$ are $a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi kn/4} = \frac{1}{4}$ for all k.

Thus, the DTFS representation of $x[n]$ is $x[n] = \sum_{k=<4>} \frac{1}{4} e^{j2\pi kn/4}$.

(2) (6%)

$$H(e^{j\Omega}) = 1 + e^{-j\Omega} + e^{-2j\Omega} - e^{j\Omega} - e^{2j\Omega}$$

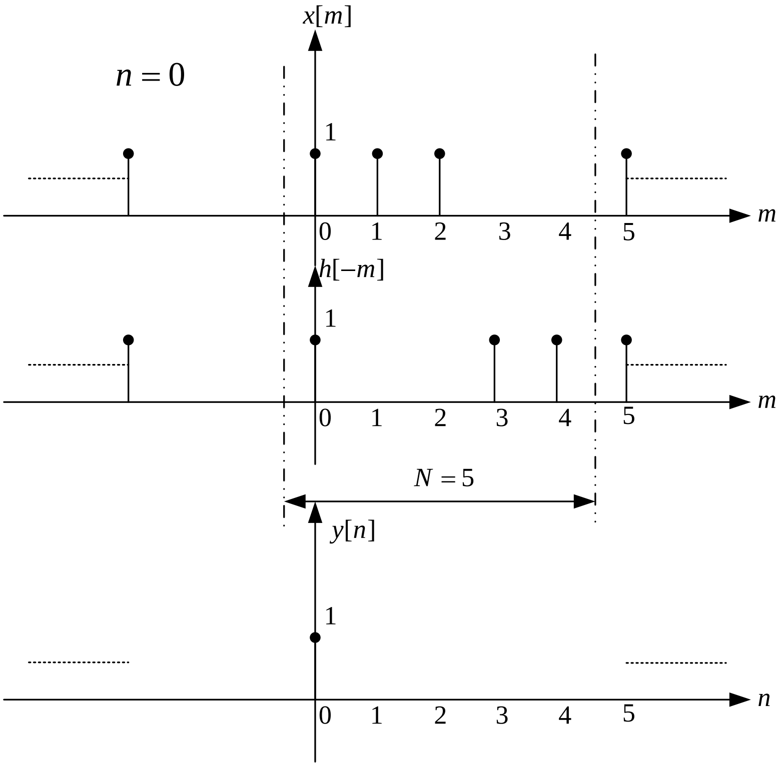
$$Y(e^{j\Omega}) = \frac{1}{4}(1 + e^{-j\Omega} + e^{-2j\Omega} - e^{j\Omega} - e^{2j\Omega})$$

$$\Rightarrow b_k = \frac{1}{4}(1 + e^{jk\pi/2} + e^{-jk\pi/2})$$

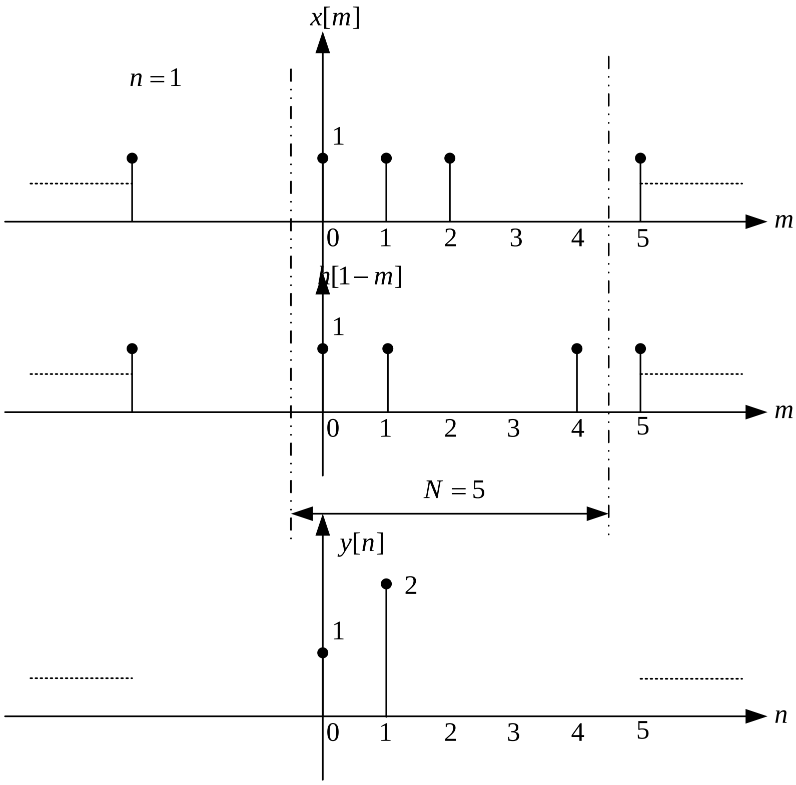
9.

(1) (3%) $N \geq 3 + 3 - 1 = 5$.

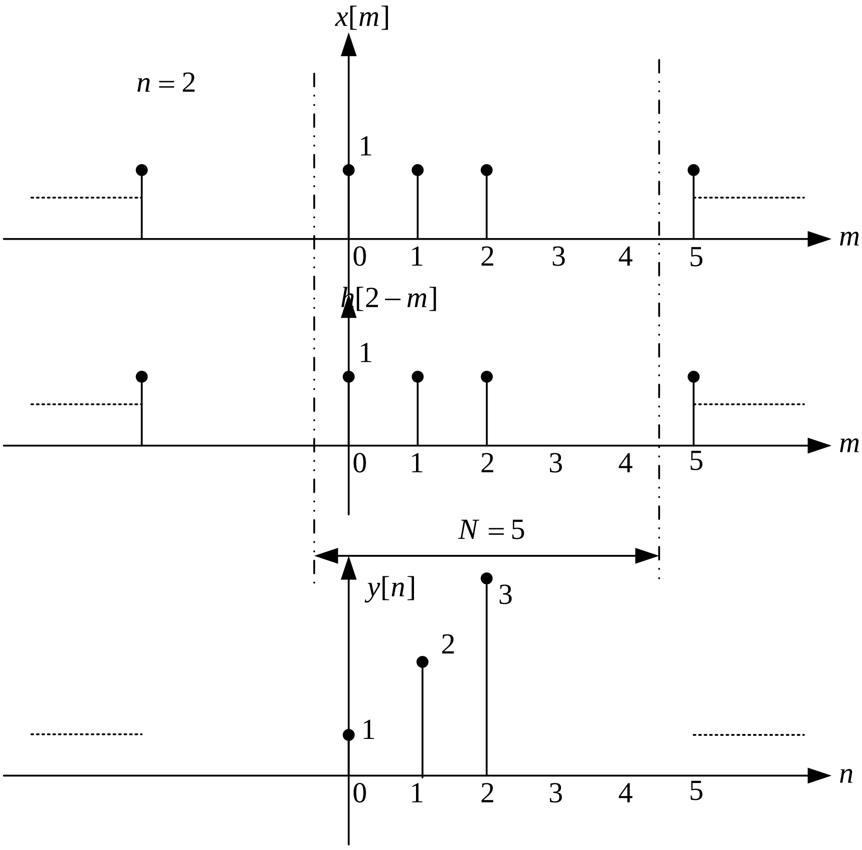
(2)



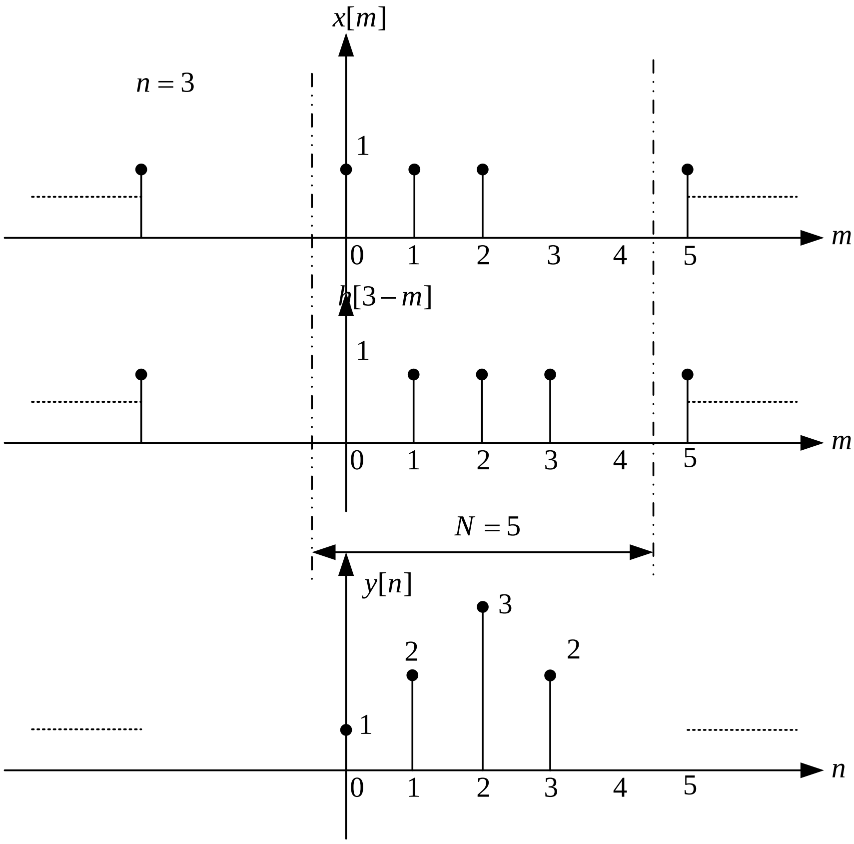
$n = 0 \Rightarrow y[0] = 1$



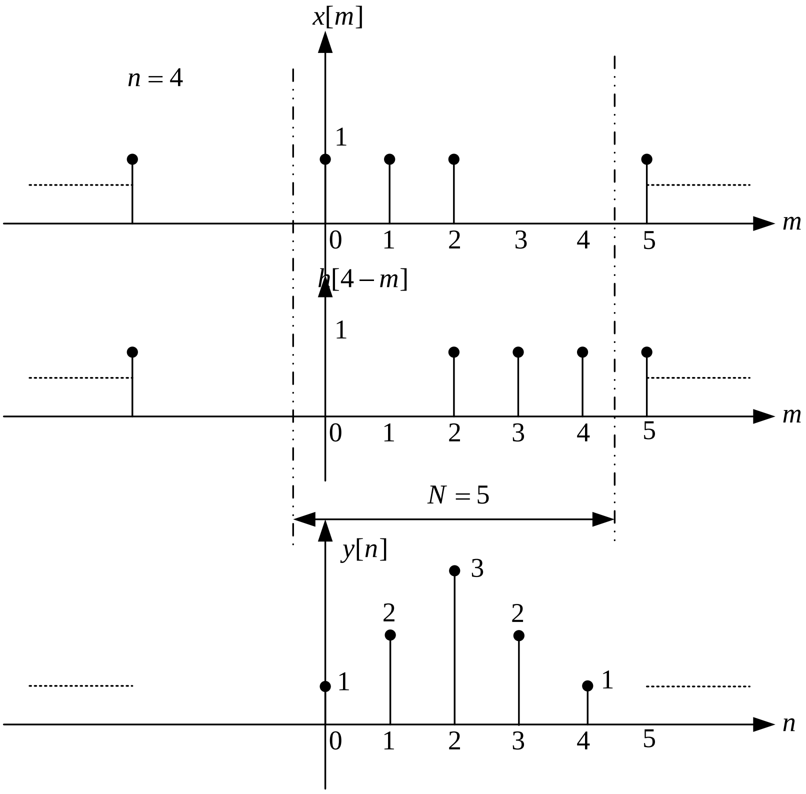
$n = 1 \Rightarrow y[1] = 2$



$n = 2 \Rightarrow y[2] = 3$



$n = 3 \Rightarrow y[3] = 2$



$n=4 \Rightarrow y[4]=1$

(3) There are three steps to find $y[n]$

a. Calculate the 5-point DFTs $X[k]$ and $H[k]$ of $x[n]$ and $h[n]$.
b. Multiply these 5-point DFTs together to obtain $Y[k]=X[k]H[k]$
c. Calculate the inverse DFT of $Y[k]$ to get $y[n]$.

10.

(1) (3%)

$$H_1(e^{j\Omega})=H_{lp}(e^{j(\Omega-\pi)})$$

$$H_1(e^{j\Omega})=\begin{cases}0, & |\Omega|<0.8\pi,\\ 1, & 0.8\pi\le|\Omega|\le\pi.\end{cases}\Rightarrow \text{HPF}.$$

(2) (4%)

$$H_2(e^{j\Omega})=H_{lp}(e^{j\Omega})*[\delta(\Omega-0.5\pi)+\delta(\Omega+0.5\pi)]$$

$$H_2(e^{j\Omega})=\begin{cases}0, & |\Omega|<0.3\pi,\\ 1, & 0.3\pi\le|\Omega|\le0.7\pi,\\ 0, & 0.7\pi<|\Omega|\le\pi.\end{cases}\Rightarrow \text{BPF}.$$

(3) NO! The reasons are infinite length and non-causal property of $h_{lp}[n]$. (3%)