Midterm Exam II Reference Solutions

Dec. 15, 2011

Instructor: *Chin-Liang Wang*

1. By utilizing the concept of eigenfunction: (10%)

$$y(t)=\sum_{k=0}^{3}(-1)^k\frac{e^{j2\pi kt}H(2\pi k)-e^{-j2\pi kt}H(-2\pi k)}{2j}=\sum_{k=0}^{3}(-1)^k\frac{e^{j2\pi k(t-1)}-e^{-j2\pi k(t-1)}}{2j}$$
$$=-\sin(2\pi(t-1))+\sin(2\pi(2t-2))-\sin(2\pi(3t-3)).$$

2. $T=6$, $\omega_0=\dfrac{\pi}{3}$, and

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt=\frac{1}{6}\int_{-3}^{3}x(t)e^{-jk\omega_0 t}dt=\frac{1}{6}\int_{-2}^{-1}1\cdot e^{-jk\omega_0 t}dt+\frac{1}{6}\int_{1}^{2}(-1)\cdot e^{-jk\omega_0 t}dt$$
$$=\frac{1}{jk\pi}\left[\cos(\frac{2\pi}{3}k)-\cos(\frac{\pi}{3}k)\right]$$

$a_0=0$ and for $k=\text{even}$, $a_k=0$. (10%)

3.

(1) $$X(j\omega)=\frac{1}{1+j\omega}+\frac{1}{3+j\omega}$$

$$Y(j\omega)=\frac{2}{1+j\omega}-\frac{2}{4+j\omega}$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}=\frac{9+3j\omega}{(j\omega)^2+6j\omega+8}$$ (5%)

(2) $$H(j\omega)=\frac{9+3j\omega}{(4+j\omega)(2+j\omega)}=\frac{3/2}{(4+j\omega)}+\frac{3/2}{(2+j\omega)}$$

$$h(t)=\frac{3}{2}(e^{-4t}+e^{-2t})u(t)$$ (6%)

4.

(1) $\int_{-\infty}^{\infty}x(t)dt=X(0)=2$ (2%)

(2) $\int_{-\infty}^{\infty}|x(t)|^2dt=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega=\dfrac{10}{\pi}$ (3%)

(3) $\int_{-\infty}^{\infty}x(t)e^{j2t}dt=X(-2)=1$ (2%)

(4) $x(0)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{-j\omega\cdot 0}d\omega=\dfrac{6}{\pi}$ (2%)

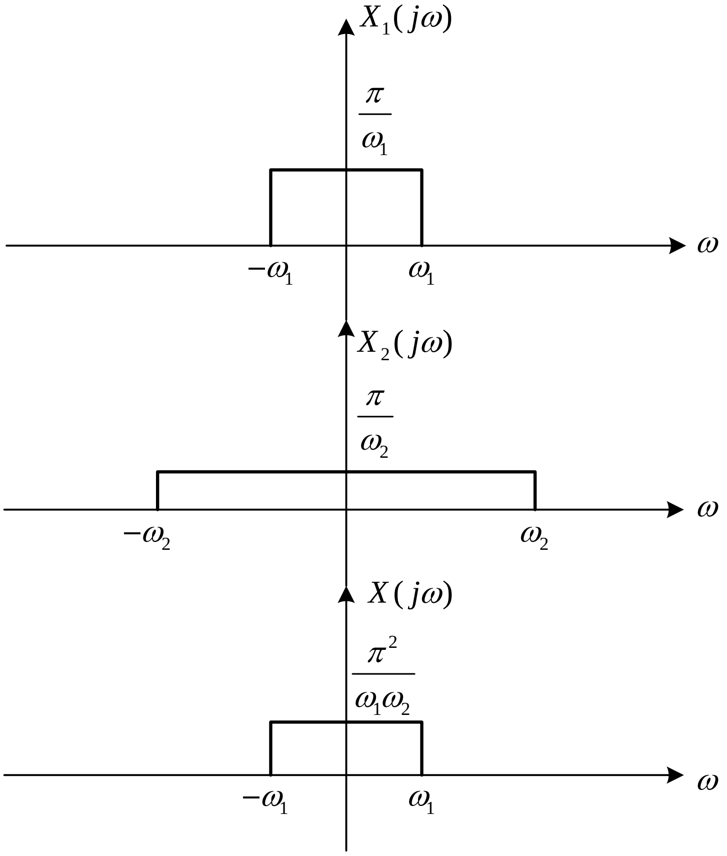
(5) $\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X\left(j\omega\right)e^{-j\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega}_{\text{odd function}} \cdot \underbrace{X\left(j\omega\right)}_{\text{even function}} d\omega = 0$ (3%)

5.

(1) $x(t) = x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = X_1(j\omega)X_2(j\omega)$ (4%)

$$x_1(t) = \frac{\pi}{\omega_1}\cdot\frac{\sin(\omega_1 t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = \begin{cases} \frac{\pi}{\omega_1}, & |\omega| < \omega_1 \\ 0, & |\omega| > \omega_1 \end{cases}$$

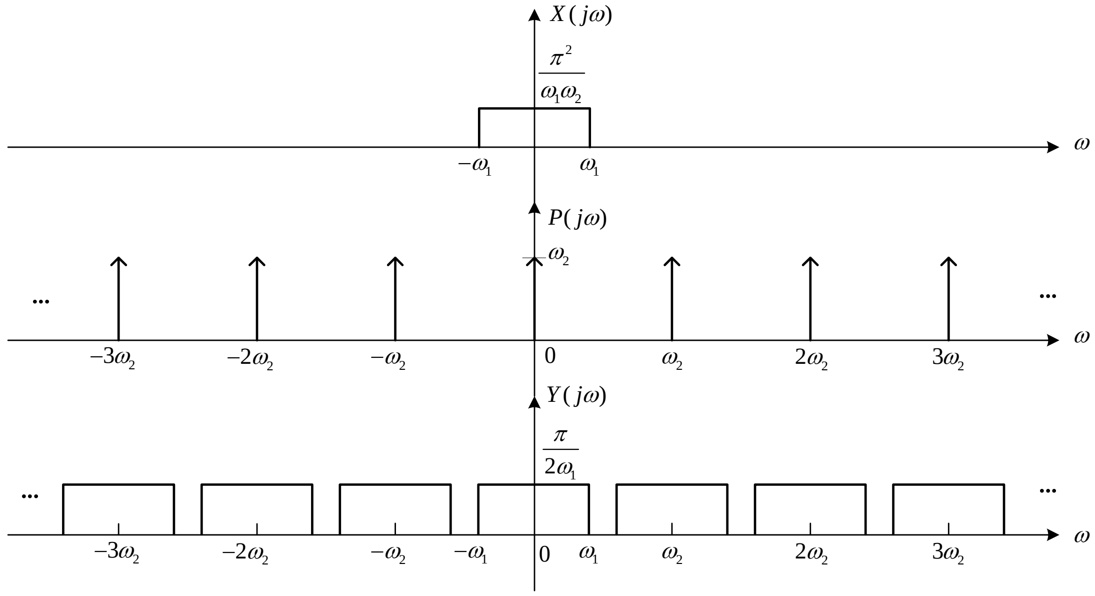
$$x_2(t) = \frac{\pi}{\omega_2}\cdot\frac{\sin(\omega_2 t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} \frac{\pi}{\omega_2}, & |\omega| < \omega_2 \\ 0, & |\omega| > \omega_2 \end{cases}$$



(2) $y\left(t\right) = x\left(t\right)p\left(t\right) \overset{\mathcal{F}}{\longleftrightarrow} Y\left(j\omega\right) = \frac{1}{2\pi}\left[X\left(j\omega\right) * P\left(j\omega\right)\right]$ (4%)

$p\left(t\right) = \sum_{k=-\infty}^{\infty}\delta\left(t - kT\right) \Rightarrow P\left(j\omega\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi}{T}k\right)$, where

$\omega_s = \frac{2\pi}{T} = \omega_2$.



(3) Yes. Since the sampling frequency $\omega_s = \omega_2 > 2\omega_1$, no aliasing in $Y(j\omega)$. $x(t)$ can be reconstructed from $y(t)$. (4%)

6.

(1) (5%)

The F.S. coefficients of $x[n]$ are $a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi kn/4} = \frac{1}{4}$ for all k. Thus, the DTFS representation of $x[n]$ is $x[n] = \sum_{k=<4>} \frac{1}{4} e^{j2\pi kn/4}$.

(2) (6%)

$$H(e^{j\Omega}) = 1 + e^{-j\Omega} + e^{-2j\Omega} - e^{j\Omega} - e^{2j\Omega}$$

$$Y(e^{j\Omega}) = \frac{1}{4}(1 + e^{-j\Omega} + e^{-2j\Omega} - e^{j\Omega} - e^{2j\Omega})$$

$$\Rightarrow b_k = \frac{1}{4}(1 + e^{jk\pi/2} + e^{-jk\pi/2})$$

7.

(1) (5%)

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{j\Omega}}$$

$$Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega}) = \frac{1}{1-\frac{1}{9}e^{-j2\Omega}}$$

$$= \frac{1/2}{1+\frac{1}{3}e^{-j\Omega}} + \frac{1/2}{1-\frac{1}{3}e^{-j\Omega}}.$$

$$\therefore y[n] = \frac{1}{2}[(\frac{1}{3})^n + (-\frac{1}{3})^n]u[n].$$

(2) (5%)

$$x[n] = \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}) \text{ is an eigenfunction.}$$

$$H(e^{j\frac{\pi}{2}n}) = \frac{9}{10}(1+\frac{1}{2}j),\ H(e^{-j\frac{\pi}{2}n}) = \frac{9}{10}(1-\frac{1}{2}j).$$

$$y[n] = \frac{1}{2}(\frac{9}{10}(1+\frac{1}{2}j)e^{j\frac{\pi}{2}n} + \frac{9}{10}(1-\frac{1}{2}j)e^{-j\frac{\pi}{2}n})$$

$$= \frac{9}{10}(\cos(\frac{\pi}{2}n) - \frac{1}{2}\sin(\frac{\pi}{2}n)).$$

8.

(1) (6%)

By considering 3-points DFT,

$$X[k] = 1 + e^{-j\frac{2\pi}{3}k},\ H[k] = 1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{2\pi}{3}2\cdot k},\ k = 0 \sim 2.$$

$$Y[k] = X[k]H[k] = 2 + 2e^{-j\frac{2\pi}{3}k} + 2e^{-j\frac{2\pi}{3}2\cdot k},\ k = 0 \sim 2.$$

$$\therefore \tilde{y}[n] = \begin{cases} 2, & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

(2) (8%)

$N_{\min} = 2 + 3 - 1 = 4$. Taking 4-point DFT,

$$X[k] = 1 + e^{-j\frac{\pi}{2}k},\ H[k] = 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k},\ k = 0 \sim 3.$$

$$Y[k] = H[k]X[k] = 1 + 2e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + e^{-j\frac{2\pi}{4}\cdot 3k},\ k = 0 \sim 3.$$

$$\therefore y[n] = \begin{cases} 1, & n = 0,\ 3 \\ 2, & n = 1,\ 2 \\ 0, & \text{otherwise.} \end{cases}$$

9.

(1) (4%)

$$H_1(e^{j\Omega}) = \left|A(e^{j\Omega})\right| e^{-j\alpha\Omega},\ H_2(e^{j\Omega}) = \left|B(e^{j\Omega})\right| e^{-j\beta\Omega}.$$

$$H_C(e^{j\Omega}) = \left|A(e^{j\Omega})\right|\left|B(e^{j\Omega})\right| e^{-j(\alpha+\beta)\Omega}.$$

$$\angle H_C(e^{j\Omega}) = -(\alpha+\beta)\Omega.$$

The $\angle H_C(e^{j\Omega})$ is linearly propotional to Ω.

A cascade system of two linear phase systems is linear phase.

(2) (4%)

$$H_1(e^{j\Omega}) = \left|A(e^{j\Omega})\right| e^{-j\alpha\Omega},\ H_2(e^{j\Omega}) = \left|B(e^{j\Omega})\right| e^{-j\beta\Omega}.$$

$$H_P(e^{j\Omega}) = \left|A(e^{j\Omega})\right| e^{-j\alpha\Omega} + \left|B(e^{j\Omega})\right| e^{-j\beta\Omega}$$

$$\angle H_P(e^{j\Omega}) \neq -(\alpha+\beta)\Omega.$$

In general, a parallel system of two linear phase systems is not linear phase.

10. (12%)

Refer to lecture handout of Chapter 4 in Page 146.

Frequency response (4%):

$$H(e^{j\Omega}) = \frac{\dfrac{e^{j\theta}}{2j\sin\theta}}{1-(0.2e^{j\theta})e^{-j\Omega}} + \frac{\dfrac{e^{-j\theta}}{2j\sin\theta}}{1-(0.2e^{-j\theta})e^{-j\Omega}}.$$

Impulse response (8%) :

$$h[n] = (0.2)^n \frac{\sin[(n+1)\theta]}{\sin\theta} u[n].$$