Midterm Exam I (Make Up) Reference Solutions

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1. x(t)=u(t)-2u(t-1)+u(t-3),

   h(t)=t · [u(t+1)-u(t-1)];

   y(t)=x(t)∗h(t)

   for $t < -1,\ y(t) = 0,$

   for $-1 \le t < 0,\ y(t) = \int_0^{t+1} (t-\tau)d\tau = \frac{t^2}{2} - \frac{1}{2},$

   for $0 \le t < 1,\ y(t) = \int_0^{1} (t-\tau)d\tau - \int_1^{t+1} (t-\tau)d\tau = -\frac{t^2}{2} + 2t - \frac{1}{2},$

   for $1 \le t < 2,\ y(t) = \int_{t-1}^{1} (t-\tau)d\tau - \int_1^{t+1} (t-\tau)d\tau = -t^2 + 2t,$

   for $2 \le t < 4,\ y(t) = \int_{t-1}^{3} (t-\tau)d\tau = \frac{t^2}{2} - 3t + 4,$

   for $t > 4,\ y(t) = 0.$

2.

(1) For a discrete-time LTI system:

$$\underbrace{13}_{\text{length of output}} = \underbrace{7}_{\text{length of input}} + (\text{length of impulse response}) - 1$$

$\therefore$ length of impulse response = 7

(2)

$$\because\ y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

$$\therefore\ y[0] = 2 = x[0]h[0] = 2\times h[0] \Rightarrow h[0] = 1$$

$$y[1] = 7 = x[0]h[1] + x[1]h[0]$$
$$= 2\times h[1] + 3\times 1 \qquad \Rightarrow h[1] = 2$$

$$y[2] = 5 = x[0]h[2] + x[1]h[1] + x[2]h[0]$$
$$= 2\times h[2] + 3\times 2 + 1\times 1 \qquad \Rightarrow h[2] = -1$$

(3)

z[n]=x1[n]∗y1[n],

z[0]=4, z[1]=20, z[2]=33, z[3]=24, z[4]=12, z[5]=5;

z[n]=0, for otherwise n.

3. x1[n]=x[n-1]+x[n-2]+x[n-3]

y1[n]=y[n-1]+y[n-2]+y[n-3]

=-1 δ [n-1]-3 δ [n-2]-3 δ [n-3]+3 δ [n-5]+3 δ [n-6]+ δ [n-7].

4.

(1) Find $y^{(h)}[n]$

$$r^2-\frac{1}{9}=0,\ r=\frac{1}{3},\ -\frac{1}{3} \quad \therefore y^{(h)}[n]=c_1\left(\frac{1}{3}\right)^n+c_2\left(-\frac{1}{3}\right)^n$$

(2) Find $y^{(p)}[n]$

$$\because x[n]=\left(\frac{1}{3}\right)^n u[n] \quad \Rightarrow y^{(p)}[n]=cn\left(\frac{1}{3}\right)^n u[n]$$

Use $y^{(p)}[n]-\frac{1}{9}y^{(p)}[n-2]=2x[n]$

$$\Rightarrow cn\left(\frac{1}{3}\right)^n-\frac{1}{9}c(n-2)\left(\frac{1}{3}\right)^{n-2}=2\left(\frac{1}{3}\right)^n,\ n\ge 0$$

$$\Rightarrow c=1 \quad \therefore y^{(p)}[n]=n\left(\frac{1}{3}\right)^n u[n]$$

(3) Find $y[n]=y^{(h)}[n]+y^{(p)}[n]$

$$y[n]=c_1\left(\frac{1}{3}\right)^n+c_2\left(-\frac{1}{3}\right)^n+n\left(\frac{1}{3}\right)^n u[n],\ y[-2]=-9,\ y[-1]=12$$

$$y[0]=2x[0]+\frac{1}{9}y[-2]=1$$
$$y[1]=2x[1]+\frac{1}{9}y[-1]=2$$

$$\Rightarrow \begin{cases} c_1+c_2=1 \\ c_1-c_2=5 \end{cases} \Rightarrow \begin{cases} c_1=3 \\ c_2=-2 \end{cases}$$

$$\therefore y[n]=3\left(\frac{1}{3}\right)^n-2\left(-\frac{1}{3}\right)^n+n\left(\frac{1}{3}\right)^n u[n]$$

(4) Find $y^{(n)}[n]$

Use $y^{(n)}[n] = A\left(\frac{1}{3}\right)^n + B\left(-\frac{1}{3}\right)^n$, $y[-2] = -9,\ y[-1] = 12$

$$\begin{cases} 9A + 9B = -9 = y[-2] \\ 3A - 3B = 12 = y[-1] \end{cases} \Rightarrow \begin{cases} A = 3/2 \\ B = -5/2 \end{cases} \quad \therefore y^{(n)}[n] = \frac{3}{2}\left(\frac{1}{3}\right)^n - \frac{5}{2}\left(-\frac{1}{3}\right)^n$$

(5) Find $y^{(f)}[n]$

Use $y^{(f)}[n] = \left\{k_1\left(\frac{1}{3}\right)^n + k_2\left(-\frac{1}{3}\right)^n + n\left(\frac{1}{3}\right)^n\right\}u[n]$, $y[-2] = y[-1] = 0$

$$\begin{cases} y[0] = 2x[0] + \frac{1}{9}y[-2] = 2 \\ y[1] = 2x[1] + \frac{1}{9}y[-1] = \frac{2}{3} \end{cases} \quad => \begin{cases} k_1 + k_2 = 2 \\ k_1 - k_2 = 1 \end{cases} \Rightarrow \begin{cases} k_1 = 3/2 \\ k_2 = 1/2 \end{cases}$$

$$\therefore y^{(f)}[n] = \left\{\frac{3}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(-\frac{1}{3}\right)^n + n\left(\frac{1}{3}\right)^n\right\}u[n]$$

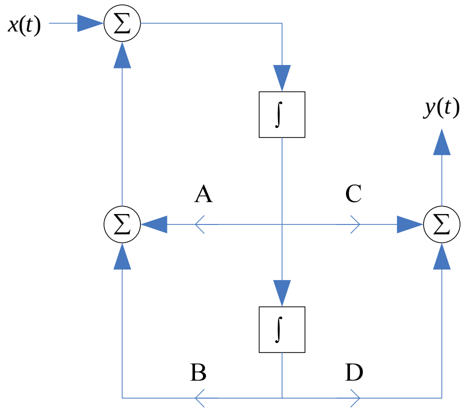
5.

(1)

$$a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_0x(t) + b_1\frac{dx(t)}{dt},\ a_2 \neq 0$$

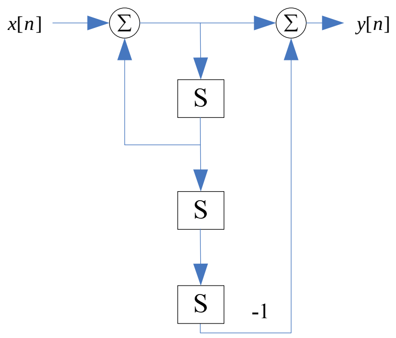
$$=> y(t) = \frac{-a_1}{a_2}\int_{-\infty}^{t} y(\tau)d\tau + \frac{-a_0}{a_2}\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} y(\sigma)d\sigma\right)d\tau + \frac{b_1}{a_2}\int_{-\infty}^{t} x(\tau)d\tau + \frac{b_0}{a_2}\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x(\sigma)d\sigma\right)d\tau$$

$$=> A = \frac{-a_1}{a_2},\ B = \frac{-a_0}{a_2},\ C = \frac{b_1}{a_2},\ D = \frac{b_0}{a_2}$$



(2)

(a) $y[n] = x[n] - x[n-3] + y[n-1]$



(b)

$$y[n] - y[n-1] = x[n] - x[n-3] \;\; => \;\; 0 = x[n] - x[n-3] + y[n-1] - y[n]$$

$$\therefore\; y[n] = x[n] - \cancel{x[n-3]} + \cancel{y[n-1]}$$

$$0 = x[n-1] - \cancel{x[n-4]} + \cancel{y[n-2]} - \cancel{y[n-1]}$$

$$0 = x[n-2] - \cancel{x[n-5]} + \cancel{y[n-3]} - \cancel{y[n-2]}$$

$$0 = \cancel{x[n-3]} - \cancel{x[n-6]} + \cancel{y[n-4]} - \cancel{y[n-3]}$$

$$0 = \cancel{x[n-4]} - \cancel{x[n-7]} + \cancel{y[n-5]} - \cancel{y[n-4]}$$

$$0 = \cancel{x[n-5]} - \cancel{x[n-8]} + \cancel{y[n-6]} - \cancel{y[n-5]}$$

$+)\;\; \vdots$

$$y[n] = x[n] + x[n-1] + x[n-2]$$

6.

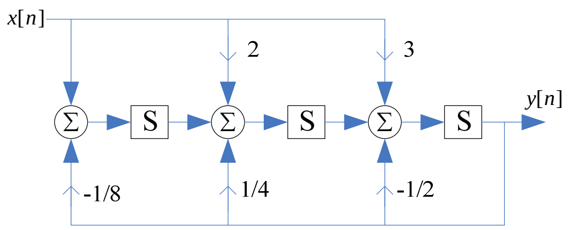
(1)

$$\frac{dq_1(t)}{dt} = 3q_1(t) + q_2(t) + x(t), \quad \frac{dq_2(t)}{dt} = q_1(t) + 3q_2(t), \text{ and}$$

$$y(t) = 2q_1(t) + q_2(t)$$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \text{ and } \; D = 0.$$

(2)



(3)

$$\mathbf{T} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathrm{T}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}' = \mathbf{TA}\mathrm{T}^{-1} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{b}' = \mathbf{Tb} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\mathbf{c}' = \mathbf{cT}^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \end{bmatrix}$$

$$D' = D = 0$$