Chapter 7 The *z*-Transform

7-1 Definition of the *z*-Transform

1. For a complex variable *z*, $x[n]=z^n$ is an eigenfunction of discrete-time LTI systems:

$$\begin{aligned} y[n] &= h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} \\ &= z^n H(z) = H(z)z^n \end{aligned} \tag{7.1}$$

$$H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k} \; \cdots\cdots \text{ eigenvalue} \tag{7.2}$$

2. The *z*-transform or the bilateral *z*-transform of a sequence *x*[*n*] is defined as

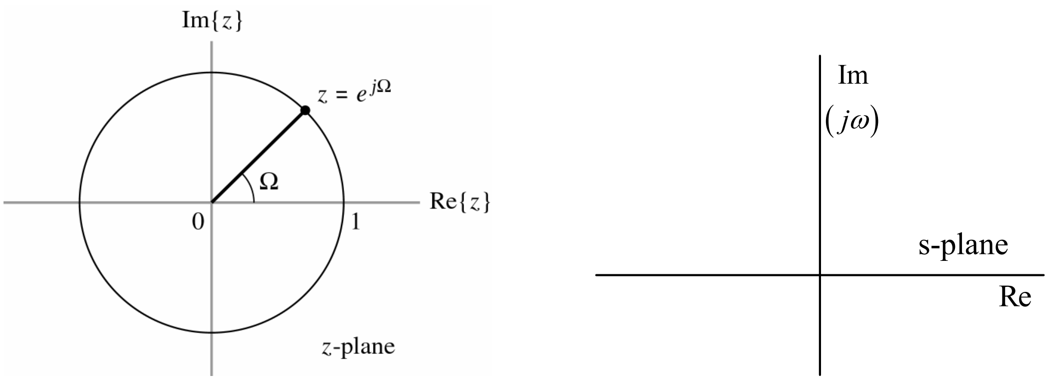
$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \begin{cases} x[n] \xleftrightarrow{z} X(z) \\ Z\{x[n]\} = X(z) \end{cases} \tag{7.3}$$

3. The *z*-transform of *x*[*n*] can be interpreted as the Fourier transform of *x*[*n*] after multiplication by a real exponential r^{-n}. $\left(z=re^{j\Omega}\right)$

$$\begin{aligned} X(z) = X\left(re^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\Omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{x[n]r^{-n}\right\} e^{-j\Omega n} \\ &= \mathcal{F}\left\{x[n]r^{-n}\right\} \end{aligned} \tag{7.4}$$

Note:

- The *z*-transform reduces to the Fourier transform when the magnitude of transform variable *z* is unity (i.e., for $z=e^{j\Omega}$).
- The Laplace-transform reduces to the Fourier transform when the real part of the transform variable is zero (i.e., for $s=j\omega$).



(a) (b)

Figure 7.1 The *z*-plane and *s*-plane: (a) *z*-plane; (b) *s*-plane.

- The unit circle of z-plane and the $j\omega$-axis of s-plane play a similar role.
- For convergence of the z-transform, we require that the Fourier transform of $x[n]r^{-n}$ converges.
- The range of values for which the z-transform exists is referred to as the region of convergence (ROC) of the z-transform.
- If the ROC includes the unit circle, then the Fourier transform also converges.
- The unilateral z-transform of a sequence $x[n]$ is defined as $X(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$.

Example 7.1: Determine the z-transform of the following sequence:

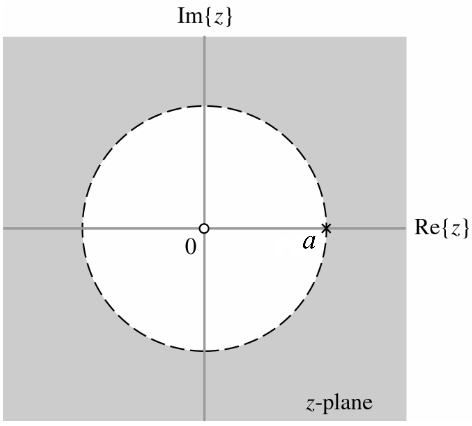
$$x[n]=a^n u[n],\ a>0.$$

$$X(z)=\sum_{n=-\infty}^{\infty}a^n u[n]z^{-n}=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

For convergence of $X(z)$, we have $\left|az^{-1}\right|<1 \quad (|z|>|a|)$

$$\Rightarrow X(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a},\quad |z|>|a|.$$

The pole-zero plot (or diagram) and ROC:



The Fourier transform of $x[n]$ converges only if $|a|<1$. ■

Example 7.2: Determine the z-transform of the following sequence:

$$x[n]=-a^n u[-n-1],\ a>0.$$

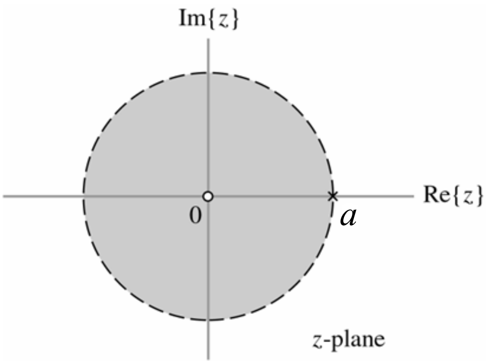
$$X(z)=-\sum_{n=-\infty}^{\infty}a^n u[-n-1]z^{-n}=-\sum_{n=-\infty}^{-1}a^n z^{-n}$$

$$=-\sum_{n=1}^{\infty}a^{-n}z^{n}=1-\sum_{n=0}^{\infty}\left(a^{-1}z\right)^n$$

$X(z)$ converges if $\left|a^{-1}z\right|<1 \quad (|z|<a)$

$$\Rightarrow X(z)=1-\frac{1}{1-a^{-1}z}=\frac{1}{1-az^{-1}}=\frac{z}{z-a}, \quad |z|<a\,.$$

The pole-zero plot and ROC:



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Example 7.3: Determine the z-transform of the following sequence:

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(\frac{1}{3}\right)^n u[n].$$

$$X(z)=\sum_{n=-\infty}^{\infty}\left\{\left(\frac{1}{2}\right)^n u[n]+\left(\frac{1}{3}\right)^n u[n]\right\}z^{-n}$$

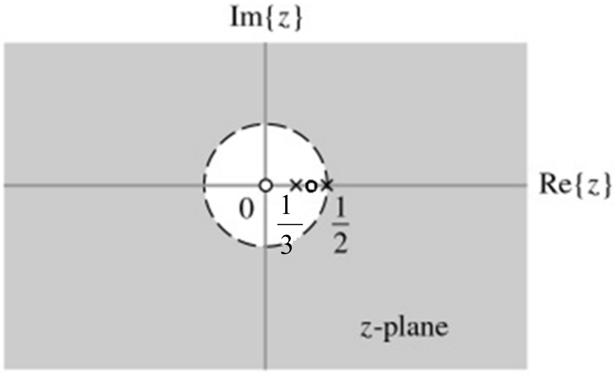
$$=\underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n}_{\substack{\text{converges}\\ \text{if } \left|\frac{1}{2}z^{-1}\right|<1 \\ \left(\text{i.e. } |z|>\frac{1}{2}\right)}}+\underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n}_{\substack{\text{converges}\\ \text{if } \left|\frac{1}{3}z^{-1}\right|<1 \\ \left(\text{i.e. } |z|>\frac{1}{3}\right)}}$$

$$\Rightarrow X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1-\frac{1}{3}z^{-1}}$$

$$=\frac{2-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$=\frac{z\left(2z-\frac{5}{6}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \qquad |z|>\frac{1}{2}\,.$$

The pole-zero plot and ROC:



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Example 7.4: Determine the *z*-transform of the following sequence:

$$h[n] = \frac{r^n \sin[(n+1)\Omega]}{\sin\Omega} u[n], \quad 0 < r < 1.$$

$$H(z) = \sum_{n=0}^{\infty} \frac{r^n \sin[(n+1)\Omega]}{\sin\Omega} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{r^n z^{-n}}{2j\sin\Omega}\left(e^{j(n+1)\Omega} - e^{-j(n+1)\Omega}\right)$$

$$= \sum_{n=0}^{\infty}\left[\left(rz^{-1}e^{j\Omega}\right)^n \frac{e^{j\Omega}}{2j\sin\Omega}\right] - \sum_{n=0}^{\infty}\left[\left(rz^{-1}e^{-j\Omega}\right)^n \frac{e^{-j\Omega}}{2j\sin\Omega}\right]$$

Both series converge if

$$\left|rz^{-1}e^{j\Omega}\right| = \left|rz^{-1}e^{-j\Omega}\right| < 1. \quad \text{(i.e. } |z| > r\text{)} \quad \Rightarrow \text{The ROC is } |z| > r.$$

$$\Rightarrow H(z) = \frac{1}{2j\sin\Omega}\left(\frac{e^{j\Omega}}{1 - rz^{-1}e^{j\Omega}} - \frac{e^{-j\Omega}}{1 - rz^{-1}e^{-j\Omega}}\right)$$

$$= \frac{1}{\left(1 - rz^{-1}e^{j\Omega}\right)\left(1 - rz^{-1}e^{-j\Omega}\right)}$$

$$= \frac{z^2}{z^2 - 2r(\cos\Omega)z + r^2}$$

poles: $z = re^{j\Omega}$, $z = re^{-j\Omega}$ zeros: $z = 0$, $z = 0$

■

Note: Some common *z*-transform pairs are listed in Table 7.1.

7-2 The Region of Convergence for the *z*-Transform

Properties of the ROC for the *z*-transform:

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

$$X\left(re^{j\Omega}\right)=\mathcal{F}\left\{x[n]r^{-n}\right\}$$

The convergence of $X(z)$ is dependent only on $r=|z|$ but not on Ω.

2. The ROC does not contain any poles.

3. If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z=0$ and/or $z=\infty$.

$$X(z)=\sum_{n=N_1}^{N_2}x[n]z^{-n}$$

$\Rightarrow$ The z-transform is the sum of a finite number of terms.

$\Rightarrow X(z)$ will converge for z not equal to zero or infinity.

(1) $N_1\geq 0$ (only negative powers of z) $\Rightarrow z=\infty$ is included in the ROC, and $z=0$ is not included in the ROC.

(2) $N_1<0$ and $N_2>0 \Rightarrow z=0$ and $z=\infty$ are not included in the ROC.

(3) $N_2\leq 0$ (only positive powers of z) $\Rightarrow z=0$ is included in the ROC, and $z=\infty$ is not included in the ROC.

4. If $x[n]$ is a right-sided sequence and if the circle $|z|=r_0$ is in the ROC, then all finite values of z for which $|z|\geq r_0$ will also be in the ROC.

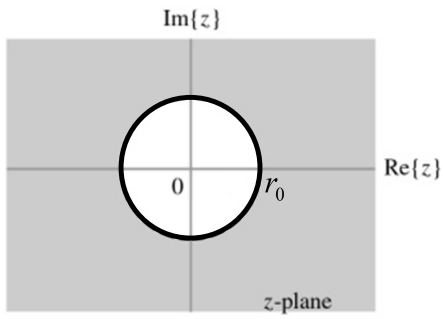


Figure 7.2 The ROC for a right-sided sequence.

$$X(z)=\sum_{n=N_1}^{\infty}x[n]z^{-n}$$

When $N_1<0$, the summation includes terms with positive powers of z which become unbounded as $|z|\to\infty$. Consequently, for right-sided sequences, in general, the ROC will not include infinity.

Suppose that the z-transform of $x[n]$ converges for some value of $r = r_0$, i.e., $|z| = r_0$ is in the ROC. Then

$$\sum_{n=-\infty}^{\infty} |x[n]| r_0^{-n} < \infty \Rightarrow \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty . \tag{7.5}$$

For $r_1 \geq r_0$,

$$\Rightarrow \sum_{n=N_1}^{\infty} |x[n]| r_1^{-n} = \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} \left(\frac{r_1}{r_0}\right)^{-n} \leq \left(\frac{r_1}{r_0}\right)^{-N_1} \cdot \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty. \tag{7.6}$$

$$\left(\because \text{ The maximum value of } \left(\frac{r_1}{r_0}\right)^{-n} \text{ in the summation is } \left(\frac{r_1}{r_0}\right)^{-N_1} \right)$$

$\Rightarrow$ The z-plane for $|z| \geq r_0$ is in the ROC.

5. If $x[n]$ is a left-sided sequence and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| \leq r_0$ will also be in the ROC.

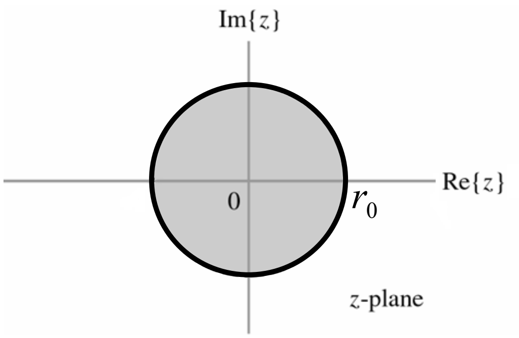


Figure 7.3 The ROC for a left-sided sequence.

$$X(z) = \sum_{n=-\infty}^{N_2} x[n] z^{-n}$$

When $N_2 > 0$, the summation includes terms with negative powers of z which become unbound when $|z| \to 0$. Consequently, for left-sided sequences, in general, the ROC will not include $z = 0$.

Note: When $N_2 < 0$, the ROC will include $z = 0$.
Suppose $|z| = r_0$ is in the ROC. Then

$$\sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n} < \infty . \tag{7.7}$$

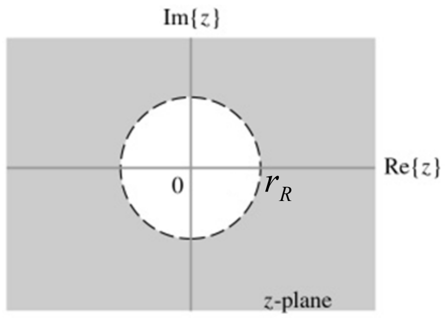
For $r_1 \le r_0$,

$$\sum_{n=-\infty}^{N_2} \left|x[n]\right| r_1^{-n} = \sum_{n=-\infty}^{N_2} \left|x[n]\right| r_0^{-n} \left(\frac{r_1}{r_0}\right)^{-n} \le \left(\frac{r_1}{r_0}\right)^{-N_2} \sum_{n=-\infty}^{N_2} \left|x[n]\right| r_0^{-n} < \infty . \quad (7.8)$$

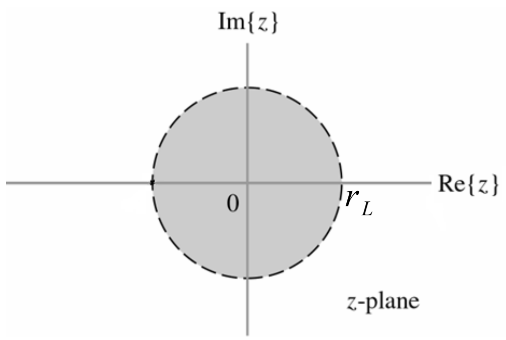
$\left(\because \text{ The maximum value of } \left(\frac{r_1}{r_0}\right)^{-n} \text{ in the summation is } \left(\frac{r_1}{r_0}\right)^{-N_2} \right)$

6. If $x[n]$ is a two-sided and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane which includes the circle $|z| = r_0$.

   Note: A two-sided sequence can be expressed as a sum of a right-sided sequence and a left-sided sequence.



(ROC for the right-sided sequence)



(ROC for the left-sided sequence)

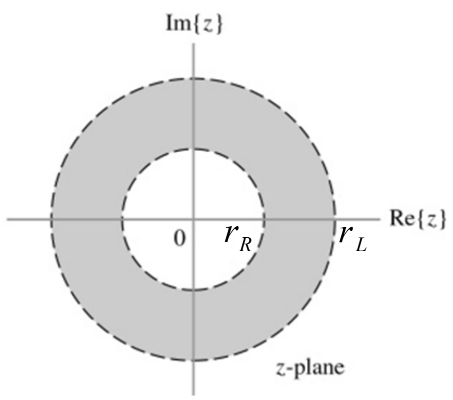
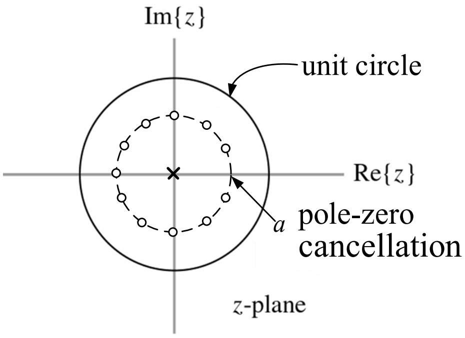


Figure 7.4 The ROC for a two-sided sequence. r_L must be greater than r_R; otherwise the ROC for $X(z)$ does not exist.

Example 7.5: Determine the z-transform of the following sequence:

$$x[n]=\begin{cases} a^n \ , & 0\le n\le N-1 \ , \quad a>0 \\ 0 \ , & \text{otherwise.} \end{cases}$$

$$X(z)=\sum_{n=0}^{N-1} a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n$$
$$=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}=\frac{1}{z^{N-1}}\cdot\frac{z^N-a^N}{z-a}$$



The ROC includes the entire z-plane except the origin.

$$\begin{aligned} &\text{pole: } z=0 && \left.\begin{aligned} z&=a \\ z&=a \end{aligned}\right\}\text{pole-zero cancellation} \\ &\text{zero: } z^N-a^N=0 && \\ &\quad z=ae^{j(2\pi k/N)} \ , && k=1,2,\cdots,N-1 \end{aligned}$$

■

Example 7.6: Determine the z-transform of the following sequence:

$$x[n]=b^{|n|} \ , \quad b>0.$$
$$\Rightarrow x[n]=b^n u[n]+b^{-n}u[-n-1]$$

$$x_1[n]=b^n u[n]\stackrel{z}{\longleftrightarrow}X_1[z]=\frac{1}{1-bz^{-1}} \ , \quad |z|>b \ .$$

$$x_2[n]=b^{-n}u[-n-1]\stackrel{z}{\longleftrightarrow}X_2[z]=\frac{-1}{1-b^{-1}z^{-1}} \ , \quad |z|<\frac{1}{b} \ .$$

For $b<1$,

$$X(z)=\frac{1}{1-bz^{-1}}+\frac{-1}{1-b^{-1}z^{-1}}$$
$$=\frac{b^2-1}{b}\cdot\frac{z}{(z-b)\left(z-b^{-1}\right)}, \quad b<|z|<\frac{1}{b} \ .$$

For $b>1$, there is no common ROC, and thus the z-transform does not exist.

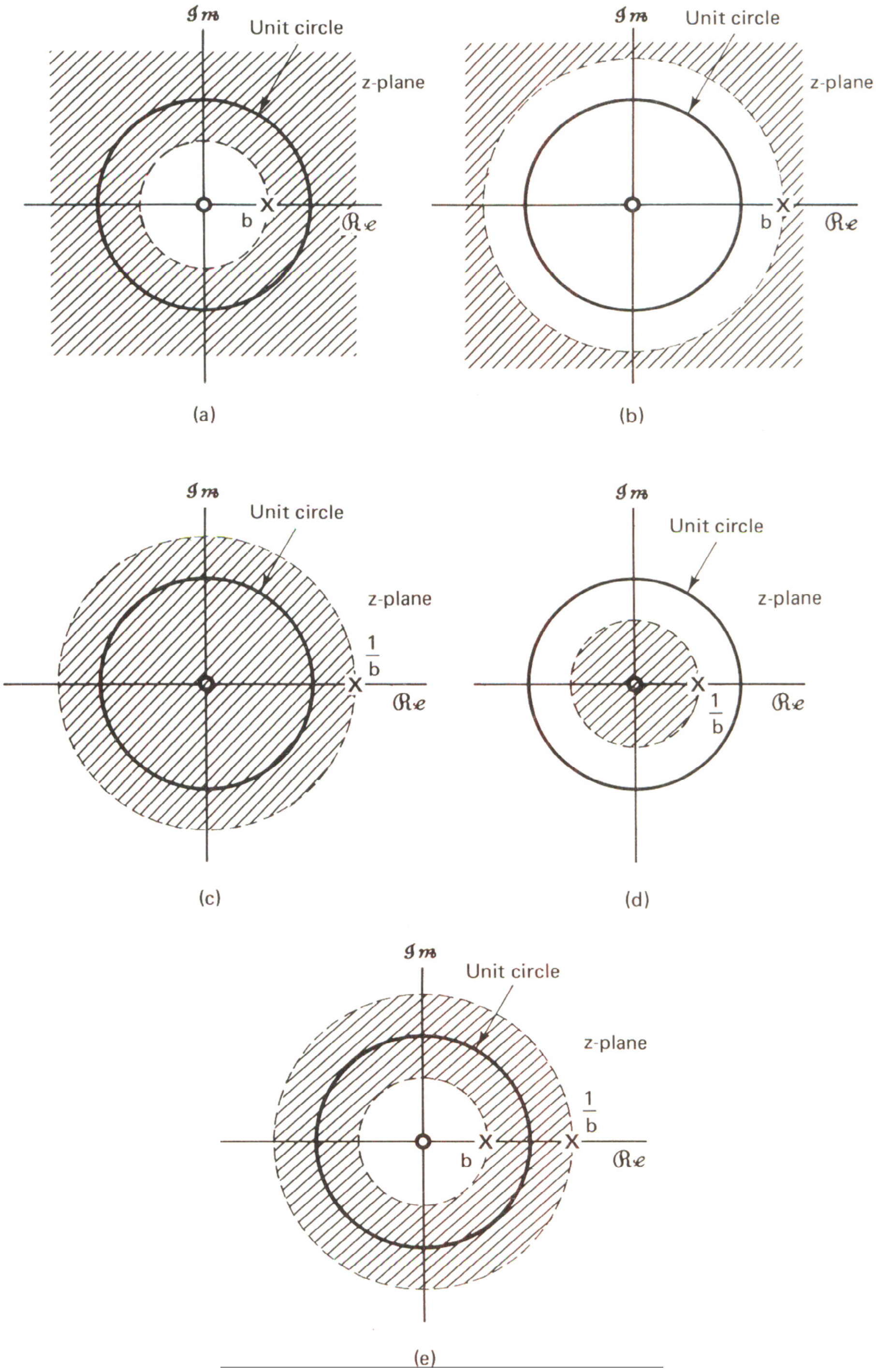


Figure 7.5 The pole-zero plots and ROCs for Example 7.6: (a) $X_1[z]$ for $|b| < 1$; (b) $X_1[z]$ for $|b| > 1$; (c) $X_2[z]$ for $|b| < 1$; (d) $X_2[z]$ for $|b| > 1$; (e) $X[z]$ for $|b| < 1$. ■

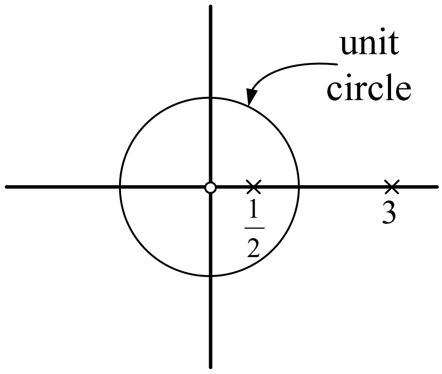
Note: For any rational z-transform

- The ROC will be bounded by poles or will extend to infinity.
- For a right-sided sequence, the ROC is bounded on the inside by the pole with the largest magnitude and on the outside by infinity.
- For a left-sided sequence, the ROC is bounded on the outside by the pole with the smallest magnitude and on the inside by zero.

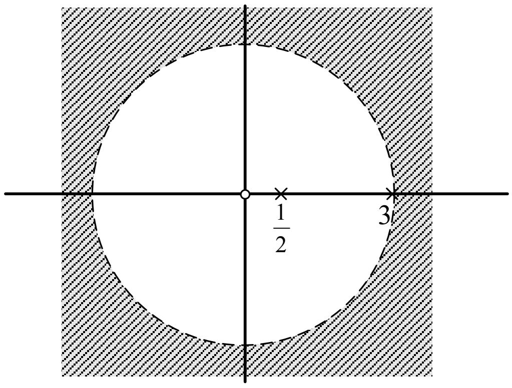
Example 7.7: Determine the pole-zero plot and ROC for

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}.$$

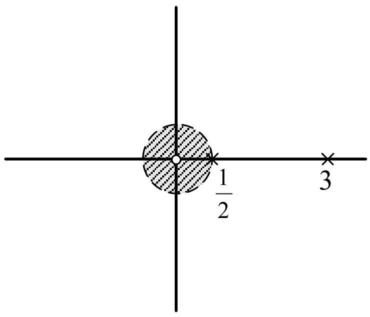
The pole-zero plot:



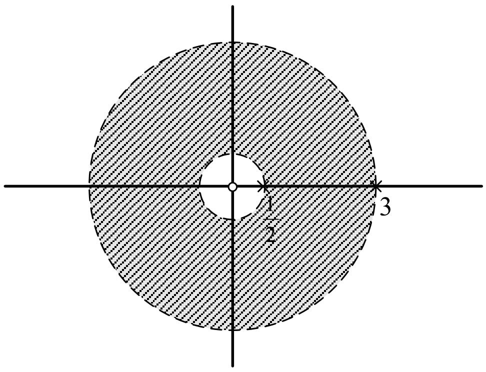
The ROC when $x[n]$ is right-sided:



The ROC when $x[n]$ is left-sided:



The ROC when $x[n]$ is two-sided:



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7-3 The Inverse *z*-Transform

1. Formulation of the inverse *z*-transform

$$x[n] = Z^{-1}\{X(z)\}$$

$$\Rightarrow \quad \begin{aligned} X\left(re^{j\Omega}\right) &= \mathcal{F}\left\{x[n]r^{-n}\right\} \\ x[n]r^{-n} &= \mathcal{F}^{-1}\left\{X\left(re^{j\Omega}\right)\right\} \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x[n] &= r^n \cdot \mathcal{F}^{-1}\left\{X\left(re^{j\Omega}\right)\right\} \\ &= r^n \cdot \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\Omega}\right)e^{j\Omega n}d\Omega \\ &= \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\Omega}\right)\left(re^{j\Omega}\right)^n d\Omega \\ &= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \end{aligned} \tag{7.9}$$

$$\left(\begin{aligned} & z = re^{j\Omega} \ \text{ and } \ r \ \text{ fixed} \\ & \Rightarrow dz = jre^{j\Omega}d\Omega = jzd\Omega \\ & \Rightarrow d\Omega = \frac{1}{j}z^{-1}dz \end{aligned} \right)$$

$\oint$: a counterclockwise closed circular contour centered at the origin and with radius r that can be chosen as any value for which $X(z)$ converges.

2. Another derivation of the inverse *z*-transform

The Cauchy Integral Theorem from the theory of complex variables states that

$$\frac{1}{2\pi j}\oint_{\Gamma} z^{k-1}dz = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{7.10}$$

where Γ is a counterclockwise contour of integration enclosing the origin. Multiplying both sides of Eq. (7.3) by $z^{k-1}/(2\pi j)$ and doing integration along a suitable Γ enclosing the origin in the ROC of $X(z)$, we obtain

$$\begin{aligned} \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{k-1}dz &= \frac{1}{2\pi j}\oint_{\Gamma}\sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}dz \\ &= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_{\Gamma} z^{-n+k-1}dz = x[k]. \end{aligned}$$

Note that a suitable Γ can always be found for the integration since the ROC is an annular ring centered on the origin. Thus, the inverse *z*-transform can be expressed by

$$x[n] = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz\,. \tag{7.11}$$

3. Various methods for computing the inverse z-transform

 (1) Cauchy integration

 In the usual case where $X(z)$ is a rational function of z, the Cauchy Residue Theorem states that Eq. (7.11) can be evaluated by

$$x[n] = \sum_i \rho_i \tag{7.12}$$

where all ρ_i are the corresponding residues of $X(z)z^{n-1}$ at the poles inside Γ. To show the case of k multiple poles at $z = p_i$ explicitly, we write

$$X(z)z^{n-1} = \frac{\Phi_i(z)}{(z-p_i)^k} \tag{7.13}$$

and the residue at the pole $z = p_i$ is then given by

$$\rho_i = \frac{1}{(k-1)!}\frac{d^{k-1}\Phi_i(z)}{dz^{k-1}}\bigg|_{z=p_i}. \tag{7.14}$$

For the case of $k=1$, Eq. (7.14) becomes simply

$$\rho_i = \Phi_i(p_i). \tag{7.15}$$

Example 7.8: Determine the inverse z-transform of

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|.$$

The function $X(z)z^{n-1} = z^n/(z-a)$ has only one pole at $z=a$ for $n \geq 0$, and has poles at $z=a$ and $z=0$ for $n<0$. Any Γ in the ROC $|z|>|a|$ will enclose all of these poles. Thus, for $n \geq 0$, we have only one residue $\rho_1 = z^n\big|_{z=a} = a^n$. For $n=-1$, there are two residues at $z=a$ and $z=0$ given by

$$\rho_1 = z^{-1}\big|_{z=a} = a^{-1}; \quad \rho_2 = \frac{1}{z-a}\bigg|_{z=0} = -a^{-1}.$$

Thus $x[-1] = \rho_1 + \rho_2 = 0$. For all $n < -1$, we can calculate the residue at $z=a$ using Eq. (7.15) and the residue at $z=0$ using (7.14). It can be checked that $x[n]=0$ for $n \leq -1$. As a consequence,

$$x[n] = a^n u[n].$$

The result agrees with the z-transform pair described in Example 7.1.

■

(2) Long division

$$X(z)=\frac{N(z)}{D(z)}=\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- If the ROC is the exterior of a circle, the corresponding inverse z-transform is a right-sided sequence. In this case, we can express $N(z)$ and $D(z)$ in polynomial form of z^{-1} (starting with the lowest powers of z^{-1}), and perform long division of $N(z)$ and $D(z)$ to expand $X(z)$ in a power series of z^{-1}. Then, the inverse z-transform can be obtained from the power series coefficients of $X(z)$.
- If the ROC is inside a circle, the corresponding inverse z-transform is a left-sided sequence. In this case, we can express $N(z)$ and $D(z)$ in polynomial form of z (starting with the lowest powers of z), and perform long division of $N(z)$ and $D(z)$ to expand $X(z)$ in a power series of z to obtain the inverse z-transform.

Example 7.9:

$$X(z)=\frac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}},\ |z|>r$$

$$\begin{array}{r l}
 & 1-4z^{-2}+8z^{-3}-32z^{-5} \\
1+2z^{-1}+4z^{-2}\,\big) & 1+2z^{-1} \\
 & \underline{1+2z^{-1}+4z^{-2}} \\
 & \qquad -4z^{-2} \\
 & \underline{\qquad -4z^{-2}-8z^{-3}-16z^{-4}} \\
 & \qquad\qquad 8z^{-3}+16z^{-4} \\
 & \underline{\qquad\qquad 8z^{-3}+16z^{-4}+32z^{-5}} \\
 & \qquad\qquad\qquad -32z^{-5} \\
 & \qquad\qquad\qquad \vdots
\end{array}$$

$$\Rightarrow\ x[0]=1,\ x[1]=0,\ x[2]=-4,\ x[3]=8,\ x[4]=0,\ x[5]=-32,\cdots$$ ■

(3) The Cauchy product and a recurrence relation

$$X(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_Mz^{-M}}=\sum_{n=0}^{\infty}x[n]z^{-n}$$

Consider that the numerator and denominator have the same degree. If this is not the case, we merely have some coefficients that are zero for the numerator polynomial or the denominator polynomial.

Thus we have

$$\left(\sum_{n=0}^{\infty} b_n z^{-n}\right) = \left(\sum_{n=0}^{\infty} a_n z^{-n}\right)\left(\sum_{n=0}^{\infty} x[n] z^{-n}\right)$$

where $a_n = b_n = 0$ for $n > M$. Applying the Cauchy product to the right-hand side results in

$$\sum_{n=0}^{\infty} b_n z^{-n} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} x[k] a_{n-k}\right] z^{-n}$$

$$\Rightarrow\ b_n = \sum_{k=0}^{n} x[k] a_{n-k} = x[n] a_0 + \sum_{k=0}^{n-1} x[k] a_{n-k}$$

Assume $a_0 \neq 0$, we have

$$\begin{cases} x[0] = \dfrac{b_0}{a_0}, & n = 0\ , \\ x[n] = \dfrac{1}{a_0}\left[b_n - \sum_{k=0}^{n-1} x[k] a_{n-k}\right], & n > 0\ . \end{cases}$$

Example 7.10:

$$X(z) = \frac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}},\ |z| > r$$

$$\begin{cases} a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_n = 0,\ \text{for}\ n \geq 3 \\ b_0 = 1,\ b_1 = 2,\ b_2 = 0,\ b_n = 0,\ \text{for}\ n \geq 3 \end{cases}$$

$$\Rightarrow \begin{cases} x[0] = \dfrac{b_0}{a_0} = 1 \\ x[1] = b_1 - x[0]a_1 = 2 - 1\times 2 = 0 \\ x[2] = b_2 - x[0]a_2 - x[1]a_1 = 0 - 1\times 4 - 0\times 2 = -4 \\ x[3] = b_3 - x[0]a_3 - x[1]a_2 - x[2]a_1 = -(-4)\times 2 = 8 \\ \vdots \\ \vdots \end{cases}$$

■

(4) Partial-fraction expansion

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- If $M < N$ and $X(z)$ has no multiple poles, it may be expanded in a partial fraction of the form

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}, \quad |z| > r \quad \text{(right-sided)} \tag{7.16}$$

with p_k being poles of $X(z)$. Note that each term in the summation is just the z-transform of an exponential sequence, and thus the inverse z-transform of $X(z)$ is given by

$$x[n] = \sum_{k=1}^{N} A_k p_k^n u[n]. \tag{7.17}$$

- If $M \ge N$, we divide $N(z)$ and $D(z)$ starting with the highest powers of z^{-1} to produce

$$a_N z^{-N} + \cdots + a_0 \overset{C_{M-N} z^{-M+N} + \cdots + C_1 z^{-1} + C_0}{\overline{\big)\, b_M z^{-M} + \cdots + b_1 z^{-1} + b_0}} + \frac{R(z)}{D(z)} \tag{7.18}$$

where the remainder polynomial $R(z)$ is of order $M' = N - 1$ or less. Then, $R(z)/D(z)$ can be expanded in a partial-fraction expansion as before and $x[n]$ is given by

$$x[n] = \sum_{i=0}^{M-N} C_i \delta[n-i] + \sum_{k=1}^{N} A_k' p_k^n u[n]. \tag{7.19}$$

- For the case of multiple poles, e.g., K multiple poles of p_1, $X(z)$ should be expanded as

$$\begin{aligned} X(z) = {} & \frac{A_{11}}{1 - p_1 z^{-1}} + \frac{A_{12}}{\left(1 - p_1 z^{-1}\right)^2} + \cdots \\ & + \frac{A_{1K}}{\left(1 - p_1 z^{-1}\right)^K} + \frac{A_{21}}{1 - p_2 z^{-1}} + \cdots + \frac{A_{N1}}{1 - p_N z^{-1}} . \end{aligned} \tag{7.20}$$

Example 7.11: Consider

$$X(z) = \frac{z^{-2} + 2z^{-1} + 2}{z^{-1} + 1}, \quad |z| > 1 .$$

By the long division method, we obtain

$$X(z) = 2 + z^{-2} - z^{-3} + z^{-4} - z^{-5} + \cdots$$

and thus

$$x[n]=\begin{cases}0, & n<0\\ 2, & n=0\\ 0, & n=1\\ (-1)^n, & n\geq 2.\end{cases}$$

By the partial-fraction expansion method, we have

$$X(z)=z^{-1}+1+\frac{1}{z^{-1}+1},\quad |z|>1$$

and thus

$$x[n]=\delta[n-1]+\delta[n]+(-1)^n u[n].$$

It can be checked that the results derived from different methods are the same.

■

Example 7.12:

$$X(z)=\frac{z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$
$$=\frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}=\frac{4}{1-\frac{1}{2}z^{-1}}-\frac{4}{1-\frac{1}{4}z^{-1}}$$

ROC: $|z|>\frac{1}{2}\Rightarrow\ x[n]=4\left[\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\right]u[n]$

ROC: $\frac{1}{4}<|z|<\frac{1}{2}\Rightarrow\ x[n]=-4\left(\frac{1}{2}\right)^n u[-n-1]-4\left(\frac{1}{4}\right)^n u[n]$

ROC: $|z|<\frac{1}{4}\Rightarrow\ x[n]=4\left[-\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n\right]u[-n-1]$

■

Example 7.13:

$$X(z)=\frac{3+\frac{11}{2}z^{-1}+7z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+2z^{-1}+4z^{-2}\right)}$$
$$=\frac{A}{1-\frac{1}{2}z^{-1}}+\frac{B+Cz^{-1}}{1+2z^{-1}+4z^{-2}}$$

$$A = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\bigg|_{z^{-1}=2} = \frac{3 + \frac{1}{2}z^{-1} + 7z^{-2}}{1 + 2z^{-1} + 4z^{-2}}\Bigg|_{z^{-1}=2} = 2$$

$$\Rightarrow \; X(z) - \frac{2}{1 - \frac{1}{2}z^{-1}} = \frac{B + Cz^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\Rightarrow \; \frac{1 + \frac{3}{2}z^{-1} - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1} + 4z^{-2}\right)} = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1} + 4z^{-2}\right)}$$

$$= \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}} = \frac{B + Cz^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\Rightarrow \; X(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\Rightarrow \; x[n] = \left[2\left(\frac{1}{2}\right)^n + 2^n\left(\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{3}n\right)\right)\right]u[n]$$

An alternative solution:

$$X(z) = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - pz^{-1}} + \frac{C}{1 - p^* z^{-1}}, \quad \begin{array}{l} p = -1 + \sqrt{3}i \\ p^* = -1 - \sqrt{3}i \end{array}$$

■

Example 7.14: Consider a right-sided sequence with z-transform

$$X(z) = \frac{1}{\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)} \quad , \quad a \neq b.$$

$$\Rightarrow X(z) = \frac{z^2}{(z-a)(z-b)} = \frac{a^{-1}b^{-1}}{\left(z^{-1} - a^{-1}\right)\left(z^{-1} - b^{-1}\right)}$$

$$= \frac{1}{b-a} \cdot \frac{1}{z^{-1} - a^{-1}} + \frac{1}{a-b} \cdot \frac{1}{z^{-1} - b^{-1}}$$

$$= \frac{a}{a-b} \cdot \frac{1}{1 - az^{-1}} + \frac{b}{b-a} \cdot \frac{1}{1 - bz^{-1}}$$

$$\Rightarrow x[n] = \frac{a}{a-b}a^n u[n] + \frac{b}{b-a}b^n u[n]$$

■

Example 7.15:

$$X(z)=\frac{4-8z^{-1}+6z^{-2}}{\left(1-2z^{-1}\right)^2\left(1+z^{-1}\right)}$$
$$=\frac{A}{1-2z^{-1}}+\frac{B}{\left(1-2z^{-1}\right)^2}+\frac{C}{1+z^{-1}}$$

$$B=\left(1-2z^{-1}\right)^2 X(z)\Big|_{z^{-1}=\frac{1}{2}}=1$$
$$C=\left(1+z^{-1}\right)X(z)\Big|_{z^{-1}=-1}=2$$
$$X(z)\left(1-2z^{-1}\right)^2=\frac{4-8z^{-1}+6z^{-2}}{1+z^{-1}}=A\left(1-2z^{-1}\right)+B+\frac{C}{1+z^{-1}}\left(1-2z^{-1}\right)^2$$

Applying differentiation to both sides, we have

$$\frac{d\left[X(z)\left(1-2z^{-1}\right)^2\right]}{dz^{-1}}=-2A+C\frac{d}{dz^{-1}}\left[\frac{\left(1-2z^{-1}\right)^2}{1+z^{-1}}\right]$$
$$\Rightarrow A=-\frac{1}{2}\left\{\frac{d\left[\frac{4-8z^{-1}+6z^{-2}}{1+z^{-1}}\right]}{dz^{-1}}\right\}\Bigg|_{z^{-1}=\frac{1}{2}}-\frac{1}{2}C\frac{d}{dz^{-1}}\left[\frac{\left(1-2z^{-1}\right)^2}{1+z^{-1}}\right]\Bigg|_{z^{-1}=\frac{1}{2}}$$
$$=-\frac{1}{2}\left\{\frac{\left(-8+12z^{-1}\right)\left(1+z^{-1}\right)-\left(4-8z^{-1}+6z^{-2}\right)\cdot 1}{\left(1+z^{-1}\right)^2}\right\}\Bigg|_{z^{-1}=\frac{1}{2}}-0$$
$$=-\frac{1}{2}\frac{\left(-3\cdot\frac{3}{2}\right)-\left(4-4+\frac{3}{2}\right)}{\frac{9}{4}}=1$$
$$\Rightarrow X(z)=\frac{1}{1-2z^{-1}}+\frac{1}{\left(1-2z^{-1}\right)^2}+\frac{2}{1+z^{-1}}$$

or

$$X(z)=\frac{1}{1-2z^{-1}}+\frac{\left(1-2z^{-1}\right)+2z^{-1}}{\left(1-2z^{-1}\right)^2}+\frac{2}{1+z^{-1}}$$
$$=\frac{2}{1-2z^{-1}}+\frac{2z^{-1}}{\left(1-2z^{-1}\right)^2}+\frac{2}{1+z^{-1}}$$
$$\Rightarrow x[n]=\left[2\left(2^n\right)+n\left(2^n\right)+2\left(-1\right)^n\right]u[n]$$ ■

(5) Power-series expansion

If $X(z)$ is given as a closed-form expression, its inverse z-transform $x[n]$ can be obtained by deriving an appropriate power series or using a previously derived power-series expansion.

Example 7.16: Assume a z-transform of the form

$$X(z)=e^{a/z}, \qquad |z|>0 .$$

Since the ROC contains $z=\infty$, the sequence $x[n]$ must be causal. The power (Maclaurin) series for $X(z)$ is given by

$$X(z)=\sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n} .$$

From this, we have immediately that

$$x[n]=\frac{a^n}{n!}u[n] .$$

■

Example 7.17: Consider

$$X(z)=\log\left(1-az^{-1}\right) , \qquad |z|>|a| .$$

The power series expansion for $\log(1-y)$ is of the form

$$\log(1-y)=\sum_{n=1}^{\infty}\frac{-1}{n}y^n .$$

From this, we have

$$X(z)=\sum_{n=1}^{\infty}\frac{-1}{n}a^n z^{-n} .$$

Hence

$$x[n]=-\frac{a^n}{n}u[n-1] .$$

■

Example 7.18: Determine the inverse z-transform of

$$X(z)=\frac{1}{1-az^{-1}} , \qquad |z|>a .$$

$$\Rightarrow X(z)=1+az^{-1}+a^2z^{-2}+\cdots\cdots=\sum_{n=0}^{\infty}a^n z^{-n}=\sum_{n=-\infty}^{\infty}a^n u[n]z^{-n}$$

$$\Rightarrow x[n]=a^n u[n]$$

■

7-4 Properties of the *z*-Transform

1. Linearity

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z) \quad , \quad \text{ROC} = R_1$$
$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z) \quad , \quad \text{ROC} = R_2$$

$$\Rightarrow a_1x_1[n] + a_2x_2[n] \overset{z}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z), \text{ ROC containing } R_1 \cap R_2 \qquad (7.21)$$

Note: If pole-zero cancellation occurs, the ROC may be larger than $R_1 \cap R_2$.

2. Time shifting

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad , \quad \text{ROC} = R_x$$

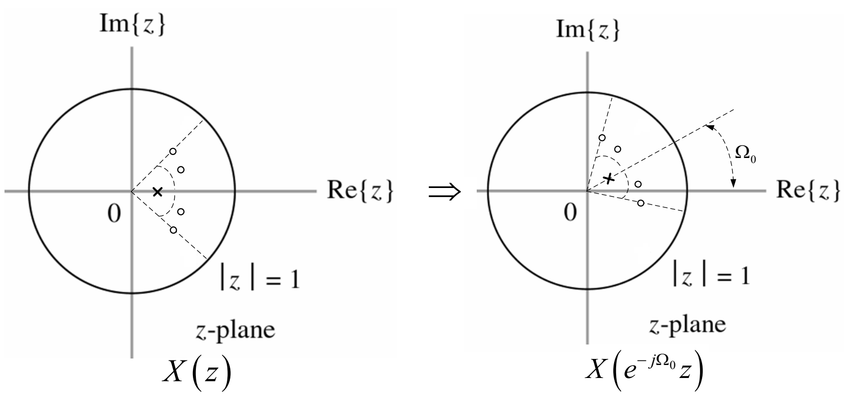
$$\Rightarrow x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z) \quad , \quad \text{ROC} = R_x \text{ (except for possible addition or deletion of the origin or infinity)} \qquad (7.22)$$

$$n_0 > 0 \Rightarrow z^{-n_0} \to \infty \text{ as } z = 0$$
$$\therefore \text{ introducing a pole } z = 0$$
$$\text{a zero } z = \infty$$
$$n_0 < 0 \Rightarrow z^{-n_0} \to \infty \text{ as } z = \infty$$
$$\therefore \text{ introducing a pole } z = \infty$$
$$\text{a zero } z = 0$$

3. Frequency shifting

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad , \quad \text{ROC} = R_x$$

$$\Rightarrow e^{j\Omega_0 n}x[n] \overset{z}{\longleftrightarrow} X\left(e^{-j\Omega_0}z\right) \quad , \quad \text{ROC} = R_x \qquad (7.23)$$



$$\Rightarrow z_0^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right) \quad , \quad \mathrm{ROC} = z_0 R_x \tag{7.24}$$

$|z_0| = 1 \Rightarrow z_0 = e^{j\Omega_0}$, reduce to the above

$z_0 = re^{j\Omega_0} \Rightarrow$ the pole and zero locations are rotated in the z-plane by an angle of Ω_0 and scaled in position radially by a factor of *r*.

4. Time reversal

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad , \quad \mathrm{ROC} = R_x$$

$$\begin{aligned} &\Rightarrow x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right) \quad , \quad \mathrm{ROC} = \frac{1}{R_x} \\ &\frac{1}{z} = z_p \Rightarrow z = \frac{1}{z_p} \end{aligned} \tag{7.25}$$

5. Convolution property

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad \mathrm{ROC} = R_1$$
$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad \mathrm{ROC} = R_2$$

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z) X_2(z), \quad \text{ROC contains } R_1 \cap R_2 \tag{7.26}$$

Proof:

$$\begin{aligned} y[n] &= x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \\ \therefore Y(z) &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{k=-\infty}^{\infty} x_2[n-k] z^{-n} \right\} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \cdot X_2(z) \\ &= X_1(z) X_2(z) \end{aligned}$$

■

Note:

- The ROC may be larger than $R_1 \cap R_2$ if pole-zero cancellation occurs in $X_1(z) X_2(z)$.
- When two polynomials or power series $X_1(z)$ and $X_2(z)$ are multiplied, the coefficients in the polynomial representing the product are the convolution of the coefficients in the polynomials $X_1(z)$ and $X_2(z)$.

$$X_1(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N} \rightarrow (N+1) \text{ points}$$
$$X_2(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N} \rightarrow (N+1) \text{ points}$$
$$\text{if } X_3(z) = X_1(z) X_2(z) \text{ then}$$
$$X_3(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{2N} z^{-2N} \rightarrow (2N+1) \text{ points}$$
$$c_n = \sum_{k=0}^{N} a_k b_{n-k} = a_n * b_n \ , \quad n = 0,1,2,\cdots,2N$$

6. Differentiation in the z-domain

$$x[n] \xleftrightarrow{z} X(z) \ , \quad \text{ROC} = R_x$$
$$\Rightarrow nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z) \ , \quad \text{ROC} = R_x \qquad (7.27)$$

If there is a pole at $z=0$ originally, then an extra pole at $z=0$ will occur after differentiating and that will be cancelled with the new zero $z=0$.

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} -nx[n] z^{-n+1}$$
$$\Rightarrow -z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n] z^{-n}$$

■

Example 7.19:

$$X(z) = \ln\left(1 + az^{-1}\right) \ , \quad |z| > a \ .$$

Find $x[n] = ?$

$$nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z) = \frac{az^{-1}}{1+az^{-1}} \ , \quad |z| > |a|$$
$$\because (-a)^n u[n] \xleftrightarrow{z} \frac{1}{1+az^{-1}} \ , \quad |z| > |a|$$
$$\therefore a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}} \ , \quad |z| > |a|$$
$$\Rightarrow a(-a)^{n-1} u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{1+az^{-1}} \ , \quad |z| > |a|$$
$$\Rightarrow nx[n] = -(-a)^n u[n-1]$$
$$\Rightarrow x[n] = \frac{-(-a)^n}{n} u[n-1]$$

■

Example 7.20:

$$X(z)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad , \quad |z|>|a|$$

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad , \quad |z|>|a|$$

$$\Rightarrow na^n u[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad , \quad |z|>|a|$$

■

7. The initial value theorem

If $x[n]=0$ for $n<0$, then

$$x[0]=\lim_{z\to\infty} X(z). \tag{7.28}$$

Proof:

$$\begin{aligned}\lim_{z\to\infty} X(z) &= \lim_{z\to\infty}\sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \lim_{z\to\infty}\sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \lim_{z\to\infty}\left[x[0]z^0+x[1]z^{-1}+x[2]z^{-2}+\cdots\right] \\ &= x[0]\end{aligned}$$

■

8. The final value theorem

If $x[n]$ is causal and stable with z-transform $X(z)$, then

$$\lim_{n\to\infty} x[n]=\lim_{z\to 1}\left(1-z^{-1}\right)X(z) \quad (X(z) \text{ has poles inside the unit circle.}) \tag{7.29}$$

Proof:

First we know $Z\{x[k]-x[k-1]\}=X(z)-z^{-1}X(z)=\left(1-z^{-1}\right)X(z)$.

Then take limit as $z\to 1$ on both sides:

$$\begin{aligned}&\lim_{z\to 1}\left(1-z^{-1}\right)X(z) \\ &=\lim_{z\to 1}\left\{\sum_{k=0}^{\infty}\left(x[k]-x[k-1]\right)z^{-k}\right\}=\sum_{k=0}^{\infty}\left(x[k]-x[k-1]\right) \\ &=x[0]+\left(x[1]-x[0]\right)+\left(x[2]-x[1]\right)+\ldots x[\infty] \\ &=x[\infty]=\lim_{n\to\infty} x[n]\end{aligned}$$

■

7-5 The System Function for LTI Systems

1. The system is stable $\Leftrightarrow$ The impulse response $h[n]$ is absolutely summable.
   $\Leftrightarrow$ The Fourier transform of the impulse response $h[n]$ converges.
   $\Leftrightarrow$ The ROC of $H(z)$ must include the unite circle

2. If the system is both causal and stable, the ROC of the *z*-transform of the impulse response must include the unit circle and be outside the outermost pole.

3. For a causal and stable system, all the poles of the system function must be inside the unit circle.

Example 7.21:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Assume the system is causal and stable. Then the ROC is $|z|>0.5$.

$$\Rightarrow h[n]=z^{-1}\{H(z)\}$$
$$=\left(\frac{1}{2}\right)^{n}u[n]+\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

■

4. Invertible systems:

   If an LTI system *h*[*n*] is invertible, there must exist an inverse system with impulse response $h_I[n]$ such that

$$h[n]*h_I[n]=\delta[n]. \tag{7.30}$$

   Expressing this relationship in terms of the *z*-transforms of *h*[*n*], $h_I[n]$, and $\delta[n]$, we have

$$H(z)H_I(z)=1 \quad \text{or} \quad H_I(z)=\frac{1}{H(z)}. \tag{7.31}$$

   If *H*(*z*) is the rational fraction *B*(*z*)/*A*(*z*), then $H_I(z)$ is the rational faction *A*(*z*)/*B*(*z*); the poles of *H*(*z*) are the zeros of $H_I(z)$, and vice versa. In general, the inverse system $H_I(z)$

for a given $H(z)$ is not unique because multiple ROCs can be defined for a rational z-transform $A(z)/B(z)$ having at least poles at other than $z=0$ or $z=\infty$. However, if we set the requirements of stability and/or causality on $H_I(z)$, it will be unique.

Example 7.22: Consider the accumulator system function

$$H(z)=\frac{1}{1-z^{-1}},\quad |z|>1\ .$$

The associated inverse system is $H_I(z)=1-z^{-1}$, $|z|>0$, and the corresponding impulse response is $h_I[n]=\delta[n]-\delta[n-1]$. This system is known as a *first-difference operator* and is unique because $H_I(z)$ has only a pole at $z=0$. It can be checked that

$$\begin{aligned}h[n]*h_I[n]&=u[n]*\{\delta[n]-\delta[n-1]\}\\&=u[n]-u[n-1]\\&=\delta[n]\ .\end{aligned}$$

■

Example 7.23: Consider a stable and causal system given by

$$H(z)=\frac{1+0.8z^{-1}}{1-0.5z^{-1}},\quad |z|>0.5\ .$$

We can identify two different inverse systems as follows:

$$H_{I1}(z)=\frac{1-0.5z^{-1}}{1+0.8z^{-1}},\quad |z|>0.8$$

and

$$H_{I2}(z)=\frac{1-0.5z^{-1}}{1+0.8z^{-1}},\quad |z|<0.8\ .$$

In most practical applications, however, only $H_{I1}(z)$ is useful because it is both stable and causal. On the other hand, for the stable and causal system

$$H(z)=\frac{1-2z^{-1}}{1-0.5z^{-1}},\quad |z|>0.5$$

the two possible inverse system are

$$H_{I3}(z)=\frac{1-0.5z^{-1}}{1-2z^{-1}},\quad |z|>2$$

and

$$H_{I4}(z)=\frac{1-0.5z^{-1}}{1-2z^{-1}},\quad |z|<2\ .$$

Hence, in this case, we must choose between stability and causality for the inverse system because $H_{I3}(z)$ is causal but not stable, while $H_{I4}(z)$ is stable but not causal. ■

5. Systems described by linear constant-coefficient difference equations:

   Consider a linear constant-coefficient difference equation given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

   Taking z-transform on both sides, we have

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z).$$

   The corresponding system function can be expressed by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{7.32}$$

Example 7.24: Consider the first-order linear difference equation

$$y[n] - ay[n-1] = x[n].$$

We have

$$(1 - az^{-1})Y(z) = X(z).$$

The actual system function can be either

$$H_1(z) = \frac{1}{1 - az^{-1}}, \quad |z| > a$$

or

$$H_2(z) = \frac{1}{1 - az^{-1}}, \quad |z| < a$$

corresponding to the causal and anticausal impulse responses

$$h_1[n] = a^n u[n]$$

and

$$h_2[n] = -a^n u[-n-1]$$

respectively. Since $h_1[n]$ and $h_2[n]$ are both nonzero for an infinite time duration, they are classified as *infinite-impulse-response* (IIR) filters. Clearly, any filter with at least one nonzero, finite pole (i.e., a pole at other than $z = 0$ or $z = \infty$) that is not canceled by a zero, will be IIR because such poles imply exponential components in $h[n]$.

■

Example 7.25: Consider the first-difference operator defined by the system function

$$H(z) = 1 - z^{-1}, \quad |z| > 0 .$$

With $H(z) = Y(z)/X(z)$ for Eq. (7.32), we can derive the corresponding difference equation as follows:

$$y[n] = x[n] - x[n-1] .$$

Since the associated impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

is nonzero for only a finite time duration, this filter is classified as a *finite-impulse response* (FIR) filter. Note that, in contrast to the IIR case, this FIR filter has only a pole at $z = 0$.

■

Example 7.26: *Linear-Phase FIR Filters*

Consider an FIR filter described by the finite-order nonrecursive difference equation

$$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

It is easy to see that the corresponding impulse response is

$$h[n] = \begin{cases} b_n, & n = 0, 1, \ldots, M \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $h[n]$ is real with either even or odd symmetry about the midpoint of $h[n]$, i.e.,

$$b_n = b_{M-n} \quad \text{or} \quad b_n = -b_{M-n} .$$

For even and odd values of *M*, some examples are given in Fig. 7.6 to demonstrate such symmetric properties. The system function $H(z)$ of this FIR filter can be expressed as

$$\begin{aligned} H(z) &= \sum_{n=0}^{M} b_n z^{-n} \\ &= b_c z^{-M/2} + \sum_{n=0}^{L} (b_n z^{-n} + b_{M-n} z^{-(M-n)}) \end{aligned}$$

where L is the integer part of $(M-1)/2$ and b_c is the central coefficient (if there exists) given by

$$b_c = \begin{cases} b_{M/2}, & M \text{ even} \\ 0, & M \text{ odd.} \end{cases}$$

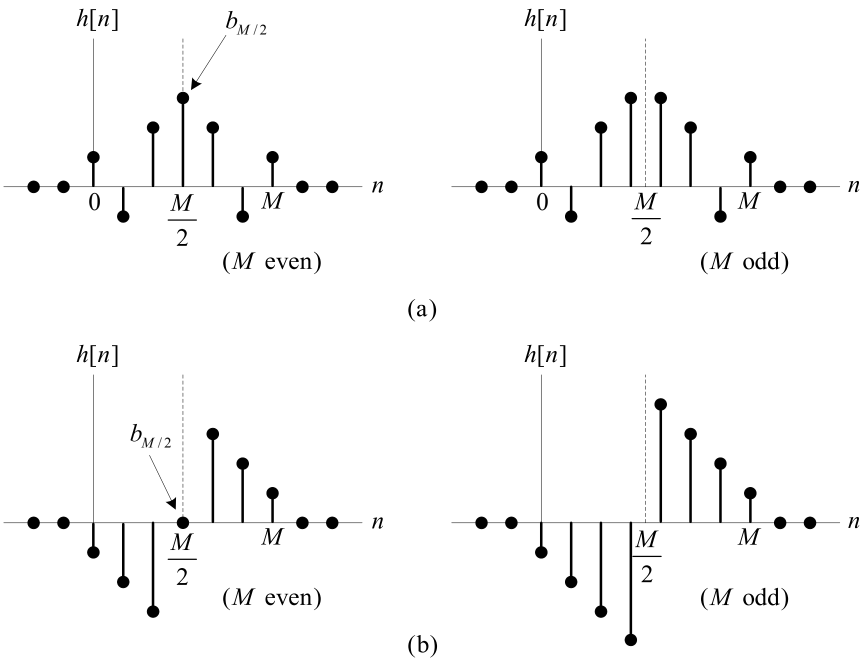


Figure 7.6 Four cases of symmetry for an FIR filter: (a) even symmetry and (b) odd symmetry.

For the even-symmetry case $(b_n = b_{M-n})$, the frequency response can be rewritten as follows:

$$\begin{aligned}
H(e^{j\Omega}) &= b_c e^{-j\Omega M/2} + \sum_{n=0}^{L} b_n (e^{-j\Omega n} + e^{-j\Omega(M-n)}) \\
&= e^{-j\Omega M/2}\left\{b_c + \sum_{n=0}^{L} b_n (e^{j\Omega(M/2-n)} + e^{-j\Omega(M/2-n)})\right\} \\
&= e^{-j\Omega M/2}\left\{b_c + \sum_{n=0}^{L} 2b_n \cos[\Omega(\frac{M}{2}-n)]\right\} \\
&= e^{-j\Omega M/2} R(\Omega)
\end{aligned}$$

where $R(\Omega)$ is a purely real function of Ω. Thus, the magnitude and phase responses are

$$\left|H(e^{j\Omega})\right| = \left|R(\Omega)\right|$$

and

$$\angle H(e^{j\Omega}) = \frac{-\Omega M}{2} + \angle R(\Omega)$$

where $\angle R(\Omega) = 0$ if $R(\Omega) > 0$, and $\angle R(\Omega) = \pm\pi$ if $R(\Omega) < 0$. The following example is a lowpass filter with $H(z) = 1 + z^{-1}$ and frequency response described by

$$H(e^{j\Omega}) = 2e^{-j\Omega/2} \cos\frac{\Omega}{2}.$$

The corresponding magnitude and phase responses are

$$\left|H(e^{j\Omega})\right| = 2\cos\frac{\Omega}{2}$$

and

$$\angle H(e^{j\Omega}) = \frac{-\Omega}{2}, \quad -\pi < \Omega < \pi$$

as shown in Fig. 7.7.

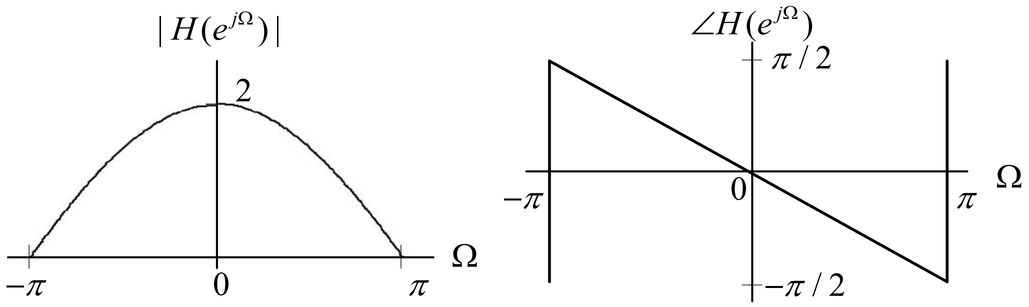


Figure 7.7 Magnitude and phase responses for $H(z) = 1 + z^{-1}$.

Note:

- Because the real function $R(\Omega) = 2\cos\Omega/2$ changes sign at $\Omega = (2k \pm 1)\pi$ for $k = 0,1,2,...$, the phase discontinuities of π radians occur at these frequencies.
- Except these discontinuities, the phase $\angle H(e^{j\Omega})$ is a linear function of Ω. So such FIR filters have (piecewise) linear-phase responses.

For the odd-symmetry case ($b_n = -b_{M-n}$), we have

$$\begin{aligned} H(e^{j\Omega}) &= je^{-j\Omega M/2}\sum_{n=0}^{L} 2b_n \sin\left[\Omega(\frac{M}{2} - n)\right] \\ &= je^{-j\Omega M/2}R(\Omega) \\ &= e^{j(\pi/2 - \Omega M/2)}R(\Omega)\ . \end{aligned}$$

Therefore, the magnitude and phase responses are

$$|H(e^{j\Omega})| = |R(\Omega)|$$

and

$$\angle H(e^{j\Omega}) = \frac{\pi}{2} - \frac{\Omega M}{2} + \angle R(\Omega)\,.$$

The following example is a highpass with $H(z) = 1 - z^{-1}$ and frequency response described by

$$H(e^{j\Omega}) = 2je^{-j\Omega/2}\sin\frac{\Omega}{2}.$$

The corresponding magnitude and phase responses are

$$|H(e^{j\Omega})| = 2\left|\sin\frac{\Omega}{2}\right|$$

and

$$\angle H(e^{j\Omega}) = \begin{cases} \pi/2 - \Omega/2, & 0 < \Omega \le \pi \\ -\pi/2 - \Omega/2, & -\pi \le \Omega < 0 \end{cases}.$$

They are depicted as follows:

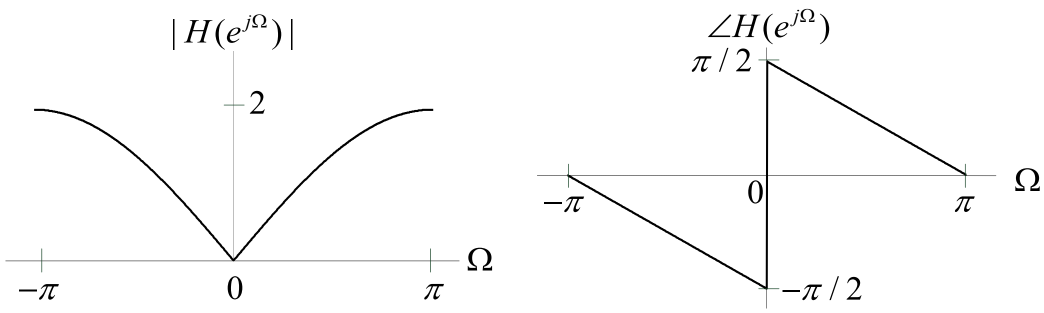


Figure 7.8 Magnitude and phase responses for $H(z) = 1 - z^{-1}$.

Note:

- Because the real function $R(\Omega) = 2\sin\Omega/2$ changes sign at $\Omega = 2k\pi$ for $k = 0,1,2,\ldots$, the phase discontinuities of π radians occur at these frequencies.
- Except these discontinuities, the phase $\angle H(e^{j\Omega})$ is a linear function of Ω. Again, such FIR filters have (piecewise) linear-phase responses.

■

7-7 The Unilateral *z*-Transform

The unilateral *z*-transform of a sequence $x[n]$ is defined by

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{7.33}$$

It is a useful tool for finding the response of a causal system to a causal input when the system is described by a linear constant-coefficient difference equation with nonzero initial conditions. Note that such responses cannot be obtained through the relationship $Y(z) = H(z)X(z)$ using the bilateral *z*-transform, which is applicable to LTI systems only. The basic properties of the unilateral z-transform that are useful for practical applications are related to the transforms of delayed signals $x[n-k]$. For the unit delay case, i.e., $x[n-1]$, the unilateral z-transform can be written by

$$\begin{aligned}\sum_{n=0}^{\infty} x[n-1]z^{-n} &= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} \\ &= x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m} \\ &= x[-1] + z^{-1}X(z) \ .\end{aligned} \tag{7.34}$$

Similarly, the unilateral z-transform of $x[n-2]$ can be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty} x[n-2]z^{-n} &= x[-2] + \sum_{n=1}^{\infty} x[n-2]z^{-n} \\ &= x[-2] + z^{-1}\sum_{m=0}^{\infty} x[m-1]z^{-m} \\ &= x[-2] + z^{-1}x[-1] + z^{-2}X(z)\end{aligned} \tag{7.35}$$

and so forth. The properties for some other delay cases are listed in Table 7.2.

***Example 7.27*:**

Consider a discrete-time system described by the linear difference equation

$$y[n] - ay[n-1] = x[n] = b^n u[n], \quad y[-1] = Y_I \ .$$

Applying the unilateral z-transform to both sides of this equation, we obtain

$$\begin{aligned} & Y(z) - az^{-1}Y(z) - ay[-1] = \frac{1}{1-bz^{-1}} \\ \Rightarrow \ & (1-az^{-1})Y(z) - aY_I = \frac{1}{1-bz^{-1}} \\ \Rightarrow \ & Y(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} + \frac{aY_I}{1-az^{-1}} \ .\end{aligned}$$

The inverse z-transform of $Y(z)$ is given by

$$y[n] = \frac{b^{n+1} - a^{n+1}}{b-a} + Y_I a^{n+1}, \qquad n \geq 0 \ .$$

Thus, the zero-input response $y_{zs}[n]$ and the zero-state response $y_{zi}[n]$ are as follows:

$$y_{zs}[n] = \frac{b^{n+1} - a^{n+1}}{b-a}, \quad y_{zi}[n] = Y_I a^{n+1}.$$

■

Example 7.28:

Consider the second-order difference equation

$$y[n]+a_1y[n-1]+a_2y[n-2]=x[n]$$

with $x[n]=0$ for $n\geq 0$ and initial conditions $y[-1]=Y_{I1}$ and $y[-2]=Y_{I2}$. Applying the unilateral z-transform to both sides of the difference equation, we obtain

$$Y(z)+a_1\{z^{-1}Y(z)+y[-1]\}+a_2\{z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\}=0$$
$$\Rightarrow Y(z)[1+a_1z^{-1}+a_2z^{-2}]=-[a_1Y_{I1}+a_2Y_{I2}]-a_2Y_{I1}z^{-1}$$
$$\Rightarrow Y(z)=\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}=(b_0+b_1z^{-1})\frac{1}{1+a_1z^{-1}+a_2z^{-2}}=X(z)H(z)$$

where $b_0=-[a_1Y_{I1}+a_2Y_{I2}]$ and $b_1=-a_2Y_{I1}$. The implies that the system can be described by an LTI system with input $X(z)=(b_0+b_1z^{-1})$ and system function

$$H(z)=\frac{1}{1+a_1z^{-1}+a_2z^{-2}}\ .$$

Note:

- The response of an all-pole discrete-time system with zero input for $n\geq 0$ and nonzero initial conditions can be modeled as the impulse response of a pole-zero system at initial rest.

■

References

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, Signals and Systems, 2nd Ed., Pearson Education Limited, 2014 (or Prentice-Hall, 1997).

[2] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd Ed., Hoboken, NJ: John Wiley & Sons, 2003.

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Table 7.1 Common z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	all z
$\delta[n-m], \quad m>0$	z^{-m}	$\lvert z\rvert > 0$
$\delta[n+m], \quad m>0$	z^{m}	$\lvert z\rvert < \infty$
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos[n\theta]u[n]$	$\dfrac{1-z^{-1}\cos\theta}{1-2z^{-1}\cos\theta+z^{-2}}$	$\lvert z\rvert > 1$
$\sin[n\theta]u[n]$	$\dfrac{1-z^{-1}\sin\theta}{1-2z^{-1}\cos\theta+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos[n\theta]u[n]$	$\dfrac{1-az^{-1}\cos\theta}{1-2az^{-1}\cos\theta+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n\sin[n\theta]u[n]$	$\dfrac{1-az^{-1}\sin\theta}{1-2az^{-1}\cos\theta+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 7.2 *Unilateral z-Transforms of Delayed Signals*

Delayed Signal	Unilateral z-Transform
$x[n-1]$	$z^{-1}X(z)+x[-1]$
$x[n-2]$	$z^{-2}X(z)+x[-1]z^{-1}+x[-2]$
$x[n-3]$	$z^{-3}X(z)+x[-1]z^{-2}+x[-2]z^{-1}+x[-3]$
$x[n-k]$	$z^{-k}X(z)+x[-1]z^{-(k-1)}+\cdots+x[-(k-1)]z^{-1}+x[-k]$